Optim Lett (2014) 8:569–579
DOI 10.1007/s11590-012-0598-x

ORIGINAL PAPER

A note on "Reducing the number of binary variables in cutting stock problems"

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Received: 17 February 2011 / Accepted: 30 November 2012 / Published online: 22 December 2012

Abstract This study proposes a deterministic model to solve the two-dimensional cutting stock problem (2DCSP) using a much smaller number of binary variables and thereby reducing the complexity of 2DCSP. Expressing a 2DCSP with m stocks and n cutting rectangles requires $2n^2 + n(m + 1)$ binary variables in the traditional model. In contrast, the proposed model uses $n^2 + n\lceil\log_2 m\rceil$ binary variables to express the 2DCSP. Experimental results showed that the proposed model is more efficient than the existing model.

Keywords Deterministic model · Cutting stock problem · Binary variables

1 Introduction

This study considers the two-dimensional cutting stock problems (2DCSPs) in real-world applications, such as cutting steel tubes, paper tubes, carpet, and glass. The 2DCSP seeks optimal cutting patterns to minimize the total number of stocks required to fulfill orders and reduce the total amount of scrap for each stock in a schedule. For example, in paper mills, paper tubes (i.e., raw materials) are cut into different

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products with different sizes [6,13] according to customer requirements, and the trim loss of the tubes needs to be minimized. Examples of the cutting stock problems in other fields include placing all devices into a system-on-a-chip circuit [24], container loading and shipping problems in the transport industry [28], and cutting Thin Film Transistor-Liquid Crystal Display (TFT-LCD) plates from glass substrate [31]. An optimal production scheme minimizes the number of stock sheets required to complete customer orders, thereby reducing manufacturing costs and increasing company competitiveness.

Research on optimal solutions for trim-loss problems dates back to several decades, as shown by the classic papers of Gilmore and Gomory [11], Chambers and Dyson [3], and Hinxman [14]. Their methods attempted to minimize the stock wastage subject to customer demands, setup costs, processing times, and characteristics of cutting patterns. More recently, Holthaus [15] has proposed an integer decomposition method using different types of patterns of standard length, whereas Umetani et al. [32] utilized meta-heuristics and adaptive pattern generation techniques to minimize the number of patterns. Gradisar and Trkman [12] developed a mixed hybrid approach that combined sequential heuristic procedures to improve the performance of the branch-and-bound algorithm. In some works, the trim-loss problem is called the "(strip) packing problem" or the "loading problem" [8,9,17,26,27,30].

The 2DCSP concept was first proposed by Brooks et al. [2]. Since then, various methods based on different algorithms have been developed. Generally, these algorithms can be divided into two classes.

(i) Deterministic algorithms: deterministic algorithms are based on mathematical programs, which utilize the branch-and-bound algorithm to derive an optimal solution. For example, Chen et al. [4] proposed a mixed integer programming model for a class of assortment problems, and their model minimized trim-loss in only one rectangular area. Li and Chang [20] and Li et al. [21,22] reformulated the mathematical model to improve the approximate solution and speed up the computation time. However, the above models were not suitable for treating 2DCSP. If the Li's original model for solving the 2DCSP is directly extended, a large number of binary variables may be required, resulting in high computational complexity [10].
(ii) Heuristic algorithms: numerous heuristic approaches are available in the literatures, and their main advantage lies in the ease in solving 2DCSP within an acceptable time. For example, Jakobs [17] developed an application of genetic algorithms to solve the two-dimensional packing problems. Leung et al. [18,19] also proposed a mixed simulated annealing-genetic algorithm for the two-dimensional packing problems, and Lin [25] designed a genetic algorithm that incorporated a novel random packing process and an encoding scheme to solve a special 2DCSP within one stock. In real-world cases, minimum trim loss in the 2DCSP is an important issue within an acceptable solving time, such as cutting rectangular steel bars in manufacturing, and guillotine-cutting problem in paper industry [1,12,27,29]. Column generation algorithms are also widely used to solve the 2DCSPs [5,33]. Cui et al. [7] developed a recursive version of the branch-and-bound algorithm to obtain an approximate solution, whereas Tsai

et al. [31] proposed a cutting stock algorithm for the TFT-LCD industry. The latter algorithm sought a feasible fixed-size cutting pattern for raw materials to minimize the number of stocks required to satisfy customer requirements. All the above algorithms can find easily feasible solutions; however, the solution quality cannot be guaranteed.

Based on comparisons of the above works, we propose a novel deterministic model to solve the 2DCSP. The advantages of the proposed model are as follows:

(i) It solves 2DCSPs effectively using a much smaller number of binary variables.
(ii) It guarantees that an optimal solution is achievable.

The rest of this paper is organized as follows. In Sect. 2, we discuss existing reference models for solving 2DCSPs, and in Sect. 3, we introduce the proposed model, which uses a much smaller number of 0–1 variables. The results of the numerical tests on practical examples are presented in Sect. 4. Section 5 presents some concluding remarks.

2 Reference model

The parameters and decision variables used in this paper are listed below [4,20]:

Parameters

m	The number of the stock sheets.
S	The set of cutting rectangles, $S = \{1, 2, \ldots, n\}$.
(p_i, q_i)	The length and width of the cutting rectangle i, $i \in S$, (p_i and q_i are constants).
$(Width, Length)$	The length and width of the stock sheet.

Decision variables

(X, Y)	The top right-hand corner coordinates of the enveloping rectangle.
(x_i, y_i)	The bottom-left coordinates of the cutting rectangle i, $i \in S$(x_i and y_i are variables).
s_i	An orientation indicator for the given cutting rectangle $i, i \in S$. $s_i = 1$ if p_i is parallel to the x-axis; otherwise, $s_i = 0$ if p_i is parallel to the y-axis (s_i denotes a binary variable).
$(a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j})$	The non-overlapping condition for a pair of cutting rectangles (i, j).

Chen et al. [4] proposed mixed-integer program as a basic model for 2DCSP using only one stock sheet, aimed at minimizing the size of the stock sheet, also called an assortment problem. The concept of the basic model proposed by Chen et al. [4] is introduced as follows:

P1 (basic model)

$$\begin{aligned}&\text{Min } XY\\&\text{s.t. } (i) \text{ all the rectangles are non-overlapping,}\\&\quad (ii) \text{ all the rectangles are within the range of } X \text{ and } Y.\end{aligned}$$

Li and Chang [20] proposed a method that employed a much smaller number of binary variables to reformulate the non-overlapping constraints, and Li et al. [21] tried to linearize approximately the cross term (i.e., XY) in the objective function by using a piecewise linearization technique. The accuracy of the approximative solution depends on the number of break points in piecewise linearization [23]. As the cross term is a nonlinear programming problem that is difficult to solve when searching for an optimal solution and the original model is only suitable for assortment problem, reformulating P1 (basic model) is necessary to solve 2DCSP. We modify the objective function in P1 as Min Y, fix the width of the stock sheets as a given value, and give the number of stocks as m. P1 can be extended to another model for general 2DCSPs. The specific 2DCSP linear program is reformulated as follows:

P2 (modified 2DCSP model)

$$\text{Min } \; Y$$

$$\text{s.t.} \quad (x_i - x_j) + M(1 - a_{i,j}) \geq p_j s_j + q_j(1 - s_j), \quad i, j \in S, \; i < j, \tag{1}$$

$$(x_j - x_i) + M(1 - b_{i,j}) \geq p_i s_i + q_i(1 - s_i), \quad i, j \in S, \; i < j, \tag{2}$$

$$(y_i - y_j) + M(1 - c_{i,j}) \geq q_j s_j + p_j(1 - s_j), \quad i, j \in S, \; i < j, \tag{3}$$

$$(y_j - y_i) + M(1 - d_{i,j}) \geq q_i s_i + p_i(1 - s_i), \quad i, j \in S, \; i < j, \tag{4}$$

$$a_{i,j} + b_{i,j} + c_{i,j} + d_{i,j} = 1, \quad i, j \in S, \; i < j, \tag{5}$$

$$x_i + p_i s_i + q_i(1 - s_i) \leq Width, \quad i \in S, \tag{6}$$

$$y_i + q_i s_i + p_i(1 - s_i) \leq Length \cdot \sum_{k=1}^{m} (Q_{i,k} \cdot k), \quad i \in S, \tag{7}$$

$$y_i \geq Length \cdot \sum_{k=1}^{m} (Q_{i,k} \cdot (k - 1)), \quad i \in S, \tag{8}$$

$$Y \geq y_i + q_i s_i + p_i(1 - s_i), \quad i \in S, \tag{9}$$

where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, Q_{i,k}, s_i \in \{0, 1\}$, and M is a sufficiently large constant. Here, Constraints (1)–(5) ensure that the rectangles are non-overlapping. Constraints (6)–(8) indicate that each rectangle is fitly packed into only one of the stock sheets. The decision variable Y in Constraint (9) denotes the length of the accumulated stock sheets.

Remark 1 In the P2 model, the numbers of binary variables and constraints are $2n^2 + n(m + 1)$ and $(5n^2 + 3n)/2$, respectively.

To reduce the complexity of 2DCSP (i.e., $2n^2 + n(m+1)$ binary variables), we propose a novel model that uses a much smaller number of binary variables in 2DCSP (i.e., fixed-width stocks). The model is described in detail in the next section.

3 Proposed model

We first introduce the binary vector $\mathbf{w}_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,\theta})$, where i denotes a small rectangle cut from the kth stock sheet for $k = 1, \ldots, m$, and $\theta = \lceil \log_2 m \rceil$. We then have the following expressions:

$$k = 1 + \sum_{r=1}^{\theta} 2^{r-1} w_{i,r}, \quad \theta = \lceil \log_2 m \rceil, \; w_{i,r} \in \{0, 1\}. \tag{10}$$

Let $S(k) \subseteq \{1, \ldots, \theta\}$ be a subset of indexes such that

$$k = 1 + \sum_{r \in S(k)} 2^{r-1}. \tag{11}$$

In addition, let $|S(k)|$ be the number of elements in $S(k)$; for example, $S(1) = \phi$ and $|S(1)| = 0$, $S(2) = \{1\}$ and $|S(2)| = 1$, $S(4) = \{1, 2\}$ and $|S(4)| = 2$, and so on. We then introduce the following propositions:

Proposition 1 *Define m equations to represent a binary vector $\mathbf{w}_i$ based on $(w_{i,1}, \ldots, w_{i,\theta})$ as follows:*

$$F_k(\mathbf{w}_i) = |S(k)| - \sum_{r \in S(k)} w_{i,r} + \sum_{r \notin S(k)} w_{i,r} \quad \text{for } k = 1, \ldots, m. \tag{12}$$

Proof (i) If $k = 1 + \sum_{r=1}^{\theta} 2^{r-1} w_{i,r}$, then $|S(k)| = \sum_{r \in S(k)} w_{i,r}$ and $\sum_{r \notin S(k)} w_{i,r} = 0$, which ensures that $F_k(\mathbf{w}_i) = 0$.

(ii) If $k \neq 1 + \sum_{r=1}^{\theta} 2^{r-1} w_{i,r}$, then $|S(k)| > \sum_{r \in S(k)} w_{i,r}$ and $\sum_{r \notin S(k)} w_{i,r} \geq 0$ or $|S(k)| = \sum_{r \in S(k)} w_{i,r}$ and $\sum_{r \notin S(k)} w_{i,r} \geq 1$, which ensures that $F_k(\mathbf{w}_i) \geq 1$.

(iii) We then prove that $F_k(\mathbf{w}_i) = 0$ if and only if $k = 1 + \sum_{r=1}^{\theta} 2^{r-1} w_{i,r}$; otherwise, $F_k(\mathbf{w}_i) \geq 1$. □

Remark 2 Only $\lceil \log_2 m \rceil$ binary variables are used in the Proposition 1.

We utilize the Proposition 1 to express m equations using only $\lceil \log_2 m \rceil$ binary variables. As it is straightforward to introduce Proposition 1, we can derive the Proposition 2 for 2DCSP afterward.

Proposition 2 *n small rectangles need to be cut from m stock sheets. Referring to the Proposition 1, we introduce $\theta = \lceil \log_2 m \rceil$ binary variables (i.e., $w_{i,r}$ for $r = 1, \ldots, \theta$) to express m functions (i.e., $F_k(\mathbf{w}_i)$) for each cutting rectangle i ($i \in S$) cut exactly from one of the stock sheets as follows:*

$$1+\sum_{r=1}^{\theta} 2^{r-1} w_{i,r} \le m, \quad i \in S, \tag{13}$$

$$y_i + (k-1)\cdot Length \cdot F_k(\mathbf{w}_i) \ge (k-1)\cdot Length,$$
$$i \in S,\ k=1,\ldots,m, \tag{14}$$

$$y_i + q_i s_i + p_i(1-s_i) - (m-k)\cdot Length \cdot F_k(\mathbf{w}_i) \le k \cdot Length,$$
$$i \in S,\ k=1,\ldots,m, \tag{15}$$

where the $Length$ means the limited length of a stock sheet and the function $F_k(\mathbf{w}_i)$ is the same as Eq. (12).

Proof Based on the Proposition 1, if $F_{k^*}(\mathbf{w}_i)=0$ (k^* is an arbitrary integer, and $k^* \in \{1,\ldots,m\}$), then the other $F_k(\mathbf{w}_i) \ge 1$ ($k=1,\ldots,m,$ and $k \ne k^*$). Constraints (14) and (15) will only be active if

(i) $y_i \ge (k^*-1)\cdot Length,$
(ii) $y_i + q_i s_i + p_i(1-s_i) \le k^* \cdot Length.$

On the other hand, Constraints (14) and (15) will be inactive, and $F_k(\mathbf{w}_i) \ge 1$. The conditions will ensure that rectangle i is cut from the stock sheet k^*. The Proposition 2 is therefore proven. □

The P2 model refers to Chen's method (1994), whose model used four binary variables ($a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$) to handle the non-overlapping issue of the pair of rectangular items (i, j). By referring to Li's model, the non-overlapping issue can be expressed using only two binary variables $(u_{i,j}, v_{i,j})$ to reduce the complexity of 2DCSP.

Based on Li's model, P2 , and the Proposition 2, a novel model of 2DCSP can be formulated using a much smaller number of binary variables as follows:

P3 (proposed 2DCSP model)

Min Y

s.t. (13)–(15),

$$(x_i - x_j) + Width(u_{i,j} + v_{i,j}) \ge p_j s_j + q_j(1-s_j), \quad i, j \in S,\ i<j, \tag{16}$$

$$(x_j - x_i) + Width(1 - u_{i,j} + v_{i,j}) \ge p_i s_i + q_i(1-s_i), \quad i, j \in S,\ i<j, \tag{17}$$

$$(y_i - y_j) + m\cdot Length\cdot(1 + u_{i,j} - v_{i,j}) \ge q_j s_j + p_j(1-s_j), \quad i, j \in S,\ i<j, \tag{18}$$

$$(y_j - y_i) + m\cdot Length\cdot(2 - u_{i,j} - v_{i,j}) \ge q_i s_i + p_i(1-s_i), \quad i, j \in S,\ i<j, \tag{19}$$

$$x_i + p_i s_i + q_i(1-s_i) \le Width, \quad i \in S, \tag{20}$$

$$y_i + q_i s_i + p_i(1-s_i) \le Y, \quad i \in S, \tag{21}$$

$$x_i \ge 0,\ y_i \ge 0, \quad i \in S, \tag{22}$$

where $u_{i,j}, v_{i,j}, w_{i,r}, s_i \in \{0, 1\}$.

Remark 3 P3 requires $n^2 + n\lceil \log_2 m \rceil$ binary variables and $2n^2 + n(2m+1)$ constraints.

By comparing Remark 3 with Remark 1, the complexity of the proposed 2DCSP model is much less than that of the original model. The numerical experiments conducted to evaluate the performance of the proposed model are discussed in the next section.

4 Numerical experiments

Two numerical examples are presented to demonstrate the effectiveness of the proposed model and to compare its performance with that of the original model. The first example is the sound box design assembly problem, and the second is the TFT-LCD cutting stock problem. In both cases, the objective is to minimize the number of stocks required to satisfy customer's requirements. The numerical examples were coded in IBM ILOG CPLEX [16] environment, and run on a PC with an Intel Pentium(D) 2.8 GHz CPU and 2 GB RAM.

Example 1 This problem arises in the sound box design, which requires cutting of rectangular items from a standard size piece of wood (60 cm × 110 cm), i.e., $Width = 60$ and $Length = 110$ for each stock. The parameters of the required plates are shown in Table 1. The given number of stock sheets is $m = 4$. The proposed model utilizes two binary variables, $w_{i,1}$ and $w_{i,2}$, to construct $F_k(\mathbf{w}_i)$ for each plate because $\lceil \log_2 4 \rceil = 2$. The problem is formulated as follows:

$$
\begin{aligned}
\text{Min} \quad & Y \\
\text{s.t.} \quad & (16)-(22), \\
& 1 + w_{i,1} + 2w_{i,2} \le 4, \quad i = 1, \dots, 7, \\
& y_i + 440(w_{i,1} + w_{i,2}) \ge 0, \quad i = 1, \dots, 7, \\
& y_i + q_i s_i + p_i(1 - s_i) + 440(w_{i,1} + w_{i,2}) \le 110, \quad i = 1, \dots, 7, \\
& y_i + 440(1 - w_{i,1} + w_{i,2}) \ge 110, \quad i = 1, \dots, 7, \\
& y_i + q_i s_i + p_i(1 - s_i) + 440(1 - w_{i,1} + w_{i,2}) \le 220, \quad i = 1, \dots, 7, \\
& y_i + 440(1 + w_{i,1} - w_{i,2}) \ge 220, \quad i = 1, \dots, 7, \\
& y_i + q_i s_i + p_i(1 - s_i) + 440(1 + w_{i,1} - w_{i,2}) \le 330, \quad i = 1, \dots, 7, \\
& y_i + 440(2 - w_{i,1} - w_{i,2}) \ge 330, \quad i = 1, \dots, 7, \\
& y_i + q_i s_i + p_i(1 - s_i) + 440(2 - w_{i,1} - w_{i,2}) \le 440, \quad i = 1, \dots, 7.
\end{aligned}
$$

This problem is solved using CPLEX. The maximal number of binary variables and linear constraints required under the proposed model is 63 and 161, respectively. Moreover, the optimal solution $Y = 277$ is obtained in a feasible number of iterations and at a reasonable time (i.e., iterations = 1,066,902 and CPU time = 274.36 seconds), corresponding to three pieces of stock. Table 2 lists the results of the original and the proposed models, and Fig. 1 shows the solution in graphical form.

Table 1 The parameters of the required plates **in** Example 1

#1 (82,60)	#2 (90,30)	#3 (85,27)	#4 (60,25)	#5 (60,20)
#6 (55,29)	#7 (57,30)			

Plate #(p_i, q_i)

Table 2 Experiment results of Example 1

Items	0–1 variables	# of constraints	Iterations	CPU Time (s)
P2 (reference model)	196	448	n/a	n/a
P3 (proposed model)	63	105	1,066,902	274.36

$m = 4$. The CPU time in P1 is outside the limit (time >2,000)

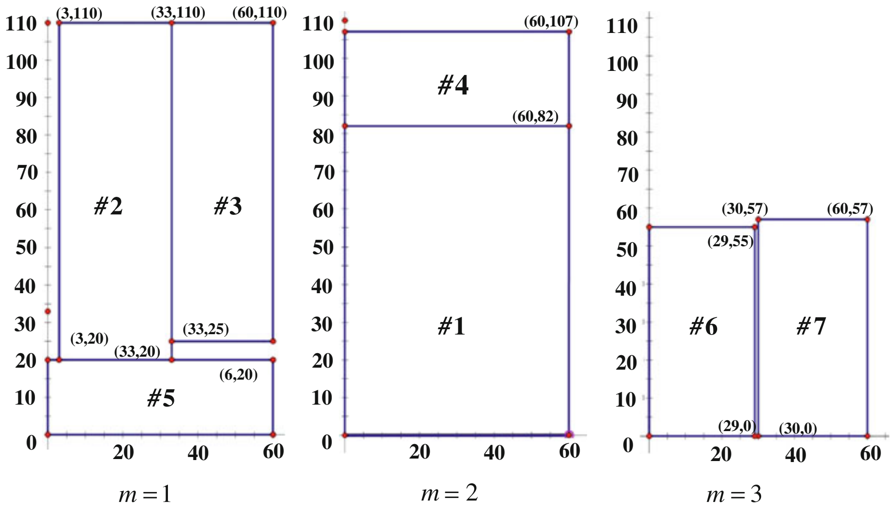

Fig. 1 Graphic solution of Example 1

Example 2 In the TFT-LCD industry example [30], the dimensions of each glass substrate (i.e., stock) are fixed at (150 cm × 180 cm), where the number of stocks is four. Assuming that a certain production line needs 18 different-sized products, as shown in Table 3, the problem is formulated as follows:

$$
\begin{aligned}
\text{Min} \quad & Y \\
\text{s.t.} \quad & (16)\text{–}(22), \\
& 1 + w_{i,1} + 2w_{i,2} \leq 4, \quad i = 1, \ldots, 18, \\
& y_i + 720F_k(\mathbf{w}_i) \geq 180(k-1), \quad i = 1, \ldots, 18, \; k = 1, \ldots, 4, \\
& y_i + q_i s_i + p_i(1 - s_i) - 720F_k(\mathbf{w}_i) \leq 180k, \quad i = 1, \ldots, 18, \; k = 1, \ldots, 4.
\end{aligned}
$$

This problem is also solved using CPLEX. The maximal number of binary variables and linear constraints required under the proposed model is 360 and 810, respectively. Moreover, the optimal solution $Y = 525$ is obtained in a feasible number of iterations and at a reasonable time (i.e., approximately 24 million iterations and 2,000 s), corresponding to three pieces of stocks. The result of Example 2 under P2 is not available due to the limited solution time (i.e., insufficient memory). Table 2 lists the results of

Table 3 Eighteen kinds of products in Example 2

#1 (130,30)	#2 (130,10)	#3 (120,25)	#4 (100,100)	#5 (95,95)
#6 (90,90)	#7 (95,85)	#8 (80,80)	#9 (80,75)	#10 (70,70)
#11 (60,60)	#12 (55,50)	#13 (40,40)	#14 (50,40)	#15 (100,30)
#16 (45,20)	#17 (20,15)	#18 (25,10)		

Plate #(p_i, q_i)

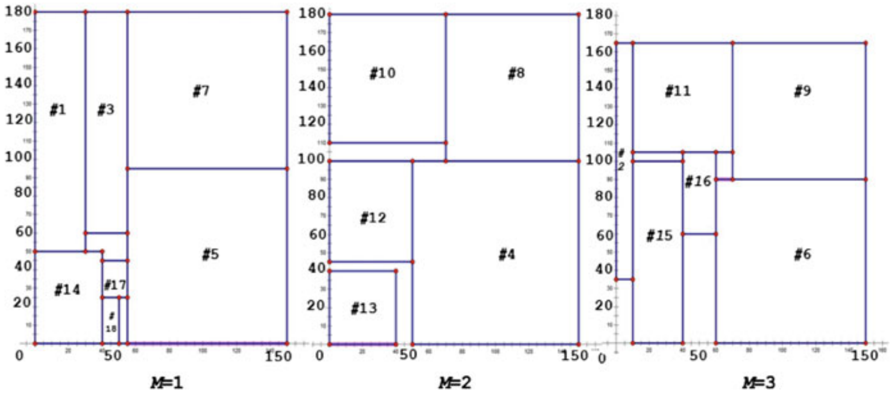

Fig. 2 Graphic result of the TFT-LCD example

Table 4 Experiment results of Example 2

Items	0–1 variables	# of constraints	Iterations	CPU time (s)
P2 (reference model)	1,296	2,736	n/a	n/a
P3 (proposed model)	360	655	24,879,402	2,023.94

$m = 4$. The CPU time in P1 is outside the limit (time >5,000)

the original and the proposed models, and Fig. 2 shows the solution in graphical form (Table 4).

The results of Examples 1 and 2 demonstrate that, compared with the reference model, the proposed model requires a logarithmic number of binary variables to formulate a model of 2DCSP, and the binary variables are used to ensure that each assigned rectangles is exactly cut from one of the stocks. Thus, it is computationally more efficient due to the reduction of the complexity of binary variables. From this point of view, Example 2 is considered as an illustration which randomly generates different-sized products with various numbers of 0–1 variables, and the stock size is also fixed at (150 cm × 180 cm). After solving ten tests, we investigate the tendency of P2 and P3 using different m and n with various 0–1 variables, as shown in Fig. 3. Here, we mark the running time for each test. Examining the results of these tests, we have the following observations:

(i) Both models were able to find the same optimal solution for each of the first five tests (tests 1–5).

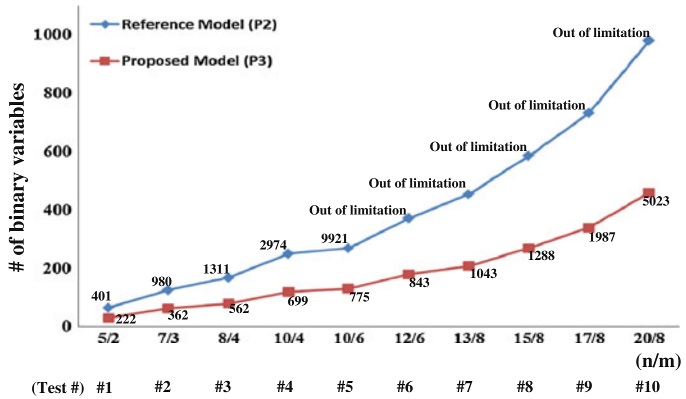

Fig. 3 Trend of CPU time in the ten tests

(ii) Owing to limitation problem (CPLEX default setting) caused by the large number of 0–1 variables and constraints, P2 failed to provide solution in 3 h in our experiments using $n \geq 12$ and $m \geq 6$.
(iii) P3 successfully solved all ten tests within the default limitation of the CPLEX software.

5 Conclusions

We have proposed a deterministic model that only requires logarithmic binary variables and additional constraints to solve 2DCSPs. Compared with the current model, the proposed model can solve the same problem with larger scale size. On the other hand, to obtain a feasible solution within a reasonable time, merging the column generation techniques, distributed algorithms, or heuristic methods (i.e., genetic algorithms, simulated annealing, and tabu-search) is a sensible practice direction to enhance the computational efficiency in future research.

Acknowledgments The authors would like to thank the editor and anonymous referee for providing most valuable comments for us to improve the quality of this manuscript. This research was supported by the project granted by ROC NSC 99-2221-E-030-005- and NSC 100-2221-E-030 -009-.

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