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Integral Equations and Operator Theory

Numerical Radii for Tensor Products of Operators

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Abstract. For bounded linear operators A and B on Hilbert spaces H and K, respectively, it is known that the numerical radii of A, B and $A \otimes B$ are related by the inequalities $w(A)w(B) \leq w(A \otimes B) \leq \min\{\|A\|w(B), w(A)\|B\|\}$. In this paper, we show that (1) if $w(A \otimes B) = w(A)w(B)$, then $w(A) = \rho(A)$ or $w(B) = \rho(B)$, where $\rho(\cdot)$ denotes the spectral radius of an operator, and (2) if A is hyponormal, then $w(A \otimes B) = w(A)w(B) = \|A\|w(B)$. Here (2) confirms a conjecture of Shiu's and is proven via dilating the hyponormal A to a normal operator N with the spectrum of N contained in that of A. The latter is obtained from the Sz.-Nagy–Foiaş dilation theory.

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1. Introduction

For any bounded linear operator A on a complex Hilbert space H, its *numerical range* $W(A)$ is, by definition, the subset $\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ of the complex plane $\mathbb{C}$, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and its associated norm in H, respectively. The *numerical radius* $w(A)$ of A is $\sup\{|z| : z \in W(A)\}$. It is known that $W(A)$ is a nonempty bounded convex subset of $\mathbb{C}$, and $w(A)$ satisfies $\|A\|/2 \leq w(A) \leq \|A\|$, where $\|A\|$ denotes the usual operator norm of A. For other properties of the numerical range and numerical radius, the reader may consult [5, Chapter 22] or [4].

The *tensor product* $H \otimes K$ of Hilbert spaces H and K is the completion of the inner product space consisting of elements of the form $\sum_{j=1}^{n} x_j \otimes y_j$ with x_j in H and y_j in K for any $n \geq 1$ under the inner product $\langle x \otimes y, u \otimes v\rangle = \langle x, u\rangle\langle y, v\rangle$. Here $x \otimes y$ is defined algebraically so as to be bilinear in the two

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arguments x and y. The *tensor product* $A \otimes B$ of operators A on H and B on K is the operator defined on $H \otimes K$ via $(A \otimes B)(x \otimes y) = Ax \otimes By$. In particular, if A and B are represented as matrices $[a_{ij}]_{i,j}$ and $[b_{ij}]_{i,j}$, respectively, then $A \otimes B$ can be represented by $[a_{ij}B]_{i,j}$. A nice account of tensor products of operators on Hilbert spaces or, for that matter, of C^*-algebras is in [9, Section 6.3].

The numerical ranges of tensor products have been studied in [11]. It is easily seen that $W(A \otimes B)$ always contains the product $W(A) \cdot W(B) = \{z_1 z_2 : z_1 \in W(A), z_2 \in W(B)\}$, and if A or B is normal, then $\overline{W(A \otimes B)} = (\overline{W(A)} \cdot \overline{W(B)})^\wedge$ holds (cf. [11, Theorem 3′]), where $\triangle^\wedge$ denotes the convex hull of a subset $\triangle$ of $\mathbb{C}$. Thus, in particular, we have the inequality $w(A \otimes B) \geq w(A)w(B)$, and the equality holds if A or B is normal. On the other hand, we also have $w(A \otimes B) \leq \|A\| w(B)$, which can be proven either using the unitary dilation of a contraction and then invoking the above-mentioned equality for normal operators (cf. [3, Proposition 1.1]), or appealing to [7, Theorem 3.4] directly since $A \otimes B$ is the product of $A \otimes I_K$ and $I_H \otimes B$ (I_K and I_H are the identity operators on K and H, respectively) and the latter two operators doubly commute, that is, $A \otimes I_K$ commutes with both $I_H \otimes B$ and its adjoint $I_H \otimes B^*$. In [3], we obtained various necessary/sufficient conditions on finite matrices A and B in order that $w(A \otimes B)$ be equal to $\|A\| w(B)$. The present one is more concerned with when the equality $w(A \otimes B) = w(A)w(B)$ holds.

In Section 2 below, we prove that if $w(A \otimes B) = w(A)w(B)$, then either $w(A) = \rho(A)$ or $w(B) = \rho(B)$, where $\rho(\cdot)$ denotes the *spectral radius* of an operator: $\rho(A) = \sup\{|z| : z \in \sigma(A)\}$ ($\sigma(A)$ is the spectrum of A). Unfortunately, this necessary condition is not sufficient. An extension of it to a complete characterization is given in Proposition 2.5, which, however, is not very useful. Then, in Section 3, we confirm a conjecture of Shiu [11] by proving that if A is a hyponormal operator, then $\overline{W(A \otimes B)} = (\overline{W(A)} \cdot \overline{W(B)})^\wedge$ for any operator B. Thus, in particular, we have $w(A \otimes B) = \|A\| w(B) = w(A)w(B)$ for a hyponormal A and an arbitrary B. This is proven by showing, via the Sz.-Nagy–Foiaş dilation theory [12], that every hyponormal operator A can be dilated to a normal operator N with $\sigma(N)$ contained in $\sigma(A)$.

For any Hilbert space H, let I_H denote the identity operator on H, and $\mathcal{B}(H)$ the C^*-algebra of all operators on H. An operator A on H is *dilated* to operator B on K if there is an operator V from H to K such that $V^*V = I_H$ and $A = V^*BV$. This is equivalent to saying that B is unitarily equivalent to an operator of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$. We say that an operator A *attains its numerical radius* if there is a λ in $W(A)$ such that $|\lambda| = w(A)$.

2. $w(A \otimes B) = w(A)w(B)$

The main result of this section is the following.

Theorem 2.1. *Let A and B be operators on H and K, respectively. If $w(A \otimes B) = w(A)w(B)$, then either $w(A) = \rho(A)$ or $w(B) = \rho(B)$.*

For the proof, we need the next lemma.

Lemma 2.2. *Let A be an operator on H. If λ in $W(A)$ is such that $|\lambda| = w(A)$, then*

$$A = \begin{bmatrix} \lambda & B \\ -e^{2i\theta}B^* & * \end{bmatrix},$$

where θ in $\mathbb{R}$ is the argument of $\lambda : \lambda = |\lambda|e^{i\theta}$.

Proof. Let x be a unit vector in H such that $\langle Ax, x\rangle = \lambda$, and let L be the one-dimensional subspace of H generated by x. Then $A = \begin{bmatrix} \lambda & B \\ C & * \end{bmatrix}$ on $H = L \oplus L^\perp$. Since

$$\langle(\mathrm{Re}\,(e^{-i\theta}A))y, y\rangle = \mathrm{Re}\,(e^{-i\theta}\langle Ay, y\rangle) \le |\lambda| = \mathrm{Re}\,(e^{-i\theta}\lambda) = \langle \mathrm{Re}\,(e^{-i\theta}\lambda)y, y\rangle$$

for any unit vector y in H, we have

$$\mathrm{Re}\,(e^{-i\theta}(\lambda I_H - A)) = \begin{bmatrix} 0 & -(e^{-i\theta}B + e^{i\theta}C^*)/2 \\ -(e^{i\theta}B^* + e^{-i\theta}C)/2 & * \end{bmatrix} \ge 0.$$

From this, we infer that $e^{i\theta}B^* + e^{-i\theta}C = 0$ or $C = -e^{2i\theta}B^*$ as asserted. □

Another tool we need for the proof of Theorem 2.1 is the Berberian representation for operators [1,2].

Lemma 2.3. *For any Hilbert space H, there is another Hilbert space H' which contains H and a unital $*$-isomorphism α from $\mathcal{B}(H)$ to $\mathcal{B}(H')$ such that the following conditions hold for all A in $\mathcal{B}(H)$:*

(a) *$\alpha(A)$ maps H' to H', and the restriction of $\alpha(A)$ to H equals A,*
(b) $\|\alpha(A)\| = \|A\|$,
(c) $\sigma(\alpha(A)) = \sigma(A)$, *and*
(d) $W(\alpha(A)) = \overline{W(A)}$.

We remark that the construction of the asserted H' and α is analogous to the Gelfand–Naimark–Segal construction for the cyclic representation of a C^*-algebra associated with a positive linear functional.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. First assume that both A and B attain their numerical radii, and let a in $W(A)$ and b in $W(B)$ be such that $|a| = w(A)$ and $|b| = w(B)$. We may assume that both a and b are nonzero. Replacing A by $(1/a)A$ and B by $(1/b)B$, we may further assume that $a = b = 1$. If x is a unit vector in H such that $\langle Ax, x\rangle = 1$ and L is the one-dimensional subspace of H generated by x, then, by Lemma 2.2, A can be represented as $\begin{bmatrix} 1 & C \\ -C^* & * \end{bmatrix}$ on $H = L \oplus L^\perp$. Similarly, B is of the form $\begin{bmatrix} 1 & D \\ -D^* & * \end{bmatrix}$. Then $A \otimes B = \begin{bmatrix} 1 & C \otimes D \\ C^* \otimes D^* & * \end{bmatrix}$. As $w(A \otimes B) = w(A)w(B) = 1$, we obtain from Lemma 2.2 again that $C^* \otimes D^* = -(C \otimes D)^*$ or $C^* \otimes D^* = 0$. Thus $C = 0$ or $D = 0$. This shows that 1 is an eigenvalue of A or B. Therefore, either $w(A) = \rho(A) = 1$ or $w(B) = \rho(B) = 1$.

For the general case, let H' and K' be the Hilbert spaces which contain H and K, respectively, and let $\alpha : \mathcal{B}(H) \to \mathcal{B}(H')$ and $\beta : \mathcal{B}(K) \to \mathcal{B}(K')$ be the unital $*$-isomorphisms as given in Lemma 2.3. Then $\alpha \otimes \beta$:

$\overline{\mathcal{B}(H)\otimes\mathcal{B}(K)}\to\overline{\mathcal{B}(H'\otimes K')}$ is also a unital $*$-isomorphism, which satisfies $\overline{W(\alpha(A)\otimes\beta(B))}=\overline{W(A\otimes B)}$ (cf. [2, Theorem 2]). Hence

$$w(\alpha(A)\otimes\beta(B))=w(A\otimes B)=w(A)w(B)=w(\alpha(A))w(\beta(B)).$$

Since $W(\alpha(A))$ and $W(\beta(B))$ are closed, the first part of the proof yields that $w(\alpha(A))=\rho(\alpha(A))$ or $w(\beta(B))=\rho(\beta(B))$. It follows that $w(A)=\rho(A)$ or $w(B)=\rho(B)$. □

Note that the necessary condition in the preceding theorem is far from sufficient. For example, if $A=[\lambda]\oplus\begin{bmatrix}0&1\\0&0\end{bmatrix}$, where $1/2\le|\lambda|<\sqrt{2}/2$, then $w(A)=\rho(A)=|\lambda|$ and

$$\begin{aligned}w(A\otimes A)&=w\left([\lambda^2]\oplus\begin{bmatrix}0&\lambda\\0&0\end{bmatrix}\oplus\begin{bmatrix}0&\lambda\\0&0\end{bmatrix}\oplus\begin{bmatrix}0&0\\0&0\end{bmatrix}\oplus\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\\&=\frac{1}{2}>|\lambda|^2=w(A)^2.\end{aligned}$$

One obvious sufficient condition for $w(A\otimes B)=w(A)w(B)$ is for $w(A)$ to be equal to $\|A\|$ or $w(B)$ equal to $\|B\|$.

Proposition 2.4. *If $w(A)=\|A\|$ or $w(B)=\|B\|$, then $w(A\otimes B)=w(A)w(B)$.*

Proof. Assume that $w(A)=\|A\|$. Then $w(A\otimes B)\le\|A\|w(B)=w(A)w(B)$ by [3, Proposition 1.1]. Since the reversed inequality $w(A\otimes B)\ge w(A)w(B)$ was already noted in Section 1, we thus have $w(A\otimes B)=w(A)w(B)$. □

In particular, this proposition is applicable when A or B is a hyponormal or a Toeplitz operator (cf. [5, Problem 205 and Corollaries 1 and 4 to Problem 245]).

The next proposition expands Theorem 2.1 to a necessary and sufficient condition for $w(A\otimes B)=w(A)w(B)$ when both A and B attain their numerical radii.

Proposition 2.5. *Let A and B be operators on H and K, respectively, such that their numerical radii are attained. Then $w(A\otimes B)=w(A)w(B)$ if and only if either $A=[a]\oplus A'$, where $|a|=w(A)=\rho(A)$ and $w(A'\otimes B)\le|a|w(B)$, or $B=[b]\oplus B'$, where $|b|=w(B)=\rho(B)$ and $w(A\otimes B')\le|b|w(A)$.*

Proof. If $w(A\otimes B)=w(A)w(B)$, then, by (the proof of) Theorem 2.1, we may assume that $A=[a]\oplus A'$ with $|a|=w(A)=\rho(A)$. In this case, we have

$$|a|w(B)=w(A\otimes B)=w(aB\oplus(A'\otimes B))=\max\{|a|w(B),w(A'\otimes B)\}$$

and thus $w(A'\otimes B)\le|a|w(B)$. The converse follows easily from the above equalities. □

Admittedly, the above conditions are not easily applicable. In one special case, they reduce to a simple one.

Corollary 2.6. *An operator A on a two-dimensional space satisfies $w(A\otimes A)=w(A)^2$ if and only if it is normal.*

Proof. The necessity follows easily from Proposition 2.5, and the sufficiency is true for any normal operator A. □

Unfortunately, the preceding corollary is no longer true even for operators on a three-dimensional space. For example, if $A = [\sqrt{2}/2] \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $w(A) = \sqrt{2}/2$, $w(A \otimes A) = 1/2$, and hence $w(A \otimes A) = w(A)^2$, but A is not normal.

We conclude this section with a strengthening of the conditions in Proposition 2.5. In the next proposition, let $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Proposition 2.7. *The following conditions are equivalent for an operator A with attaining numerical radius:*
(a) $w(A \otimes B) = w(A)w(B)$ *for all operators B,*
(b) $w(A \otimes J_2) = w(A)w(J_2)$*, and*
(c) $A = [a] \oplus A'$*, where* $|a| = w(A) \geq \|A'\|$.

Proof. The implication (a) $\Rightarrow$ (b) is trivial. To prove (b) $\Rightarrow$ (c), assume that $w(A \otimes J_2) = w(A)w(J_2)$. Then $A = [a] \oplus A'$ with $|a| = w(A) = \rho(A)$ and $w(A' \otimes J_2) \leq |a|w(J_2) = |a|/2$ by Proposition 2.5. Since $A' \otimes J_2$ is unitarily equivalent to $J_2 \otimes A' = \begin{bmatrix} 0 & A' \\ 0 & 0 \end{bmatrix}$ and the closure of the numerical range of the latter is $\{z \in \mathbb{C} : |z| \leq \|A'\|/2\}$ (cf. [13, Theorem 2.1]), we have $w(A' \otimes J_2) = \|A'\|/2$. We then infer from $w(A' \otimes J_2) \leq |a|/2$ that $\|A'\| \leq |a|$. This proves (c). Finally, if (c) holds, then, for any operator B, we have $A \otimes B = aB \oplus (A' \otimes B)$. Since $w(A \otimes B) = \max\{|a|w(B), w(A' \otimes B)\}$ and $w(A' \otimes B) \leq \|A'\|w(B) \leq |a|w(B)$ by [3, Proposition 1.1] and our assumption, we conclude that $w(A \otimes B) = |a|w(B) = w(A)w(B)$. This proves (a). □

3. Hyponormal Operators

Recall that an operator A is *hyponormal* if $A^*A \geq AA^*$. Basic properties of hyponormal operators can be found in [5, pp. 108–111]. The main result of this section is the following theorem.

Theorem 3.1. *If A is a hyponormal operator, then* $\overline{W(A \otimes B)} = (\overline{W(A)} \cdot \overline{W(B)})^\wedge$ *and* $w(A \otimes B) = w(A)w(B) = \|A\|w(B)$ *for any operator B.*

This confirms Shiu's conjecture in [11, p. 260]. It is a consequence of the next two results, the first of which is proven via the Sz.-Nagy–Foiaş dilation theory [12].

Theorem 3.2. *Any hyponormal operator A on H can be dilated to a normal operator N with* $\sigma(N) \subseteq \sigma_\ell(A) \cap \sigma_r(A)$.

Here $\sigma_\ell(A)$ (resp., $\sigma_r(A)$) denotes the *left spectrum* (resp., *right spectrum*) of A, that is, $\sigma_\ell(A)$ (resp., $\sigma_r(A)$) $= \{\lambda \in \mathbb{C} : A - \lambda I_H$ is not left invertible (resp., not right invertible)$\}$. Since $\sigma_\ell(A) \cap \sigma_r(A)$ contains the boundary of $\sigma(A)$ (cf. [5, Problem 78]), it is always nonempty.

It is well known that every operator A can be dilated to a normal operator N (cf. [5, Corollary to Problem 222]), but the dilation N may not satisfy $\sigma(N) \subseteq \sigma(A)$. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then it has no normal dilation N with $\sigma(N) \subseteq \sigma(A) = \{0\}$.

Proof of Theorem 3.2. Let λ be any point in $\sigma_\ell(A) \cap \sigma_r(A)$. Then $A - \lambda I_H$ is also hyponormal. [12, Theorem] implies that there is a normal M on some Hilbert space K and a contraction X ($\|X\| \leq 1$) from H to K such that $X^*MX = A - \lambda I_H$ and $\sigma(M) \subseteq \sigma_\ell(A - \lambda I_H) \cap \sigma_r(A - \lambda I_H)$. Let

$$Y = \begin{bmatrix} X \\ (I_H - X^*X)^{1/2} \end{bmatrix} : H \to K \oplus H$$

and $N_1 = M \oplus 0$ on $K \oplus H$. Then $Y^*Y = I_H$ and

$$Y^*N_1Y = [X^* \quad *] \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ * \end{bmatrix} = X^*MX = A - \lambda I_H.$$

This shows that $A - \lambda I_H$ dilates to N_1. Hence A dilates to the normal $N \equiv N_1 + \lambda I_{K \oplus H}$ on $K \oplus H$ with

$$\begin{aligned} \sigma(N) &= \sigma((M + \lambda I_K) \oplus \lambda I_H) = \sigma(M + \lambda I_K) \cup \{\lambda\} \\ &\subseteq (\sigma_\ell(A) \cap \sigma_r(A)) \cup \{\lambda\} = \sigma_\ell(A) \cap \sigma_r(A) \end{aligned}$$

as required. □

Another result needed for the proof of Theorem 3.1 is the next proposition.

Proposition 3.3. *Assume that the operator A on H has a normal dilation N on K such that $\sigma(N) \subseteq \sigma(A)$. Then the following hold:*

(a) $\|A\| = \|N\| = w(N) = \rho(N) = \rho(A) = w(A)$.
(b) $\|A^n\| = \|A\|^n$ *for all* $n \geq 1$.
(c) $\overline{W(A)} = \overline{W(N)} = \sigma(N)^\wedge = \sigma(A)^\wedge$.
(d) *A is Hermitian if and only if* $\sigma(A) \subseteq \mathbb{R}$.
(e) *A is quasinilpotent* ($\sigma(A) = \{0\}$) *if and only if* $A = 0$.
(f) $\overline{W(A \otimes B)} = (\overline{W(A)} \cdot \overline{W(B)})^\wedge$ *for any operator B.*
(g) $w(A \otimes B) = w(A)w(B) = \|A\|w(B)$ *for any operator B.*

Examples of operators satisfying the above normal dilation property are subnormal operators (cf. [5, Problem 200]), Toeplitz operators (cf. [5, Problem 245]), and any operator with numerical range a triangular region (cf. [10, Theorem 2]). Our Theorem 3.2 adds the class of hyponormal operators to this list. Note that it is well known that subnormal operators are hyponormal (cf. [5, p. 109]), but not the other two classes of operators. Back in 1976/77, Halmos asked what the underlying reasons are for the subnormal and Toeplitz operators to share so many properties in common (cf. [6]). Some efforts have been made to extrapolate this analogy (cf. [8]). The preceding proposition provides instead a list of consequences of the normal dilation property.

Note also that the equality $w(A \otimes B) = \|A\|w(B)$ does not imply the one with A and B switched. For example, if A is any nonzero operator satisfying the normal dilation property in Proposition 3.3 and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$w(A \otimes B) = \|A\|w(B) = w(A)w(B) = w(A)/2 < w(A)\|B\|$$

by Proposition 3.3 (g).

Proof of Proposition 3.3. (a) We obviously have $\|A\| \leq \|N\|$ and $\rho(A) \leq w(A) \leq \|A\|$. Also, by the spectral theorem, $\|N\| = w(N) = \rho(N)$ for the normal N. Since our assumption $\sigma(N) \subseteq \sigma(A)$ implies that $\rho(N) \leq \rho(A)$, the asserted equalities follow.

(b) The assertion is equivalent to $\|A\| = \rho(A)$ (cf. [5, p. 110]).

(c) As N is a dilation of A, we have $W(A) \subseteq W(N)$. But $\overline{W(N)} = \sigma(N)^\wedge$ for the normal N (cf. [5, Problem 216]). Thus the containments

$$\sigma(A)^\wedge \subseteq \overline{W(A)} \subseteq \overline{W(N)} = \sigma(N)^\wedge \subseteq \sigma(A)^\wedge$$

yield the equalities of these sets.

(d) If A is Hermitian, then, obviously, $\sigma(A) \subseteq \mathbb{R}$. Conversely, if $\sigma(A) \subseteq \mathbb{R}$, then (c) implies that $W(A) \subseteq \mathbb{R}$, from which follows the Hermitianness of A.

(e) If A is quasinilpotent, then $\overline{W(A)} = \sigma(A)^\wedge = \{0\}$ by (c). This yields that $A = 0$.

(f) Since $A \otimes B$ dilates to $N \otimes B$ for any operator B, we have $W(A \otimes B) \subseteq W(N \otimes B)$. For the normal N, the equality $\overline{W(N \otimes B)} = (\overline{W(N)} \cdot \overline{W(B)})^\wedge$ holds by [11, Theorem 3′]. On the other hand, (c) gives $\overline{W(N)} = \overline{W(A)}$. These together yield that $\overline{W(A \otimes B)} \subseteq (\overline{W(A)} \cdot \overline{W(B)})^\wedge$. Since the converse containment is always true, this proves (f).

(g) The asserted equalities follow easily from (f) and (a).

□

Obviously, Theorem 3.1 is an immediate consequence of Theorem 3.2 and Proposition 3.3.

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