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Isometric-path numbers of block graphs ☆

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Abstract

An isometric path between two vertices in a graph G is a shortest path joining them. The isometric-path number of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of G. In this paper, we determine exact values of isometric-path numbers of block graphs. We also give a linear-time algorithm for finding the corresponding paths.

Keywords: Isometric path; Block graph; Cut-vertex; Algorithms

1. Introduction

An *isometric path* between two vertices in a graph G is a shortest path joining them. The *isometric-path number* of G, denoted by $\mathrm{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of G. This concept has a close relationship with the game of cops and robbers described as follows. The game is played by two players, the *cop* and the *robber*, on a graph. The two players move alternatively, starting with the cop. Each player's first move consists of choosing a vertex at which to start. At each subsequent move, a player may choose either to stay at the same vertex or to move to an adjacent vertex. The object for the cop is to catch the robber, and for the robber is to prevent this from happening. Nowakowski and Winkler [7] and Quilliot [8] independently proved that the cop wins if and only if the graph can be reduced to a single vertex by successively removing pitfalls, where a *pitfall* is a vertex whose closed neighborhood is a subset of the closed neighborhood of another vertex. As not all graphs are cop-win graphs, Aigner and Fromme [1] introduced the concept of the *cop-number* of a general graph G, denoted by $c(G)$, which is the minimum number of

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cops needed to put into the graph in order to catch the robber. On the way to giving an upper bound for the cop-numbers of planar graphs, they showed that a single cop moving on an isometric path P guarantee that after a finite number of moves the robber will be immediately caught if he moves onto P. Observing this fact, Fitzpatrick [4] then introduced the concept of isometric-path cover and pointed out that $c(G) \leqslant \mathrm{ip}(G)$.

The isometric-path number of the Cartesian product $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_d}$ has been studied in the literature. Fitzpatrick [5] gave bounds for the case when $n_1 = n_2 = \cdots = n_d$. Fisher and Fitzpatrick [3] gave exact values for the case $d = 2$. Fitzpatrick et al. [6] gave a lower bound, which is in fact the exact value if $d + 1$ is a power of 2, for the case when $n_1 = n_2 = \cdots = n_d = 2$.

The purpose of this paper is to give exact values of isometric-path numbers of block graphs. We also give a linear-time algorithm to find the corresponding paths. For technical reasons, we consider a slightly more general problem as follows. Suppose every vertex v in the graph G is associated with a non-negative integer $f(v)$. We call such function f a *vertex labeling* of G. An *f-isometric-path cover* of G is a family $\mathcal{C}$ of isometric paths such that the following conditions hold.

(C1) If $f(v) = 0$, then v is in an isometric path in $\mathcal{C}$.
(C2) If $f(v) \geqslant 1$, then v is an end vertex of at least $f(v)$ isometric paths in $\mathcal{C}$, while the counting is twice if v itself is a path in $\mathcal{C}$.

The *f-isometric-path number* of G, denoted by $\mathrm{ip}_f(G)$, is the minimum cardinality of an f-isometric-path cover of G. It is clear that when $f(v) = 0$ for all vertices v in G, we have $\mathrm{ip}(G) = \mathrm{ip}_f(G)$. The attempt of is paper is to determine the f-isometric-path number of a block graph. Recall that a *block graph* is a graph in which every block is a complete graph. A *cut-vertex* of a graph is a vertex whose removal results in a graph with more components than the original graph. It is well known that in a block graph all internal vertices of an isometric path are cut-vertices.

2. Block graphs

In this section, we determine the f-isometric-path numbers for block graphs G. Without loss of generality, we may assume that G is connected. First, a useful lemma.

Lemma 1. *Suppose x is a non-cut-vertex of a block graph G with a vertex labeling f. If vertex labeling f' is the same as f except that $f'(x) = \max\{1, f(x)\}$, then $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G)$.*

Proof. As any internal vertex of an isometric path in a block graph is a cut-vertex but x not a cut-vertex, x must be an end vertex of any isometric path. It follows that a collection $\mathcal{C}$ is an f-isometric-path cover if and only if it is an f'-isometric-path cover. The lemma then follows. □

So, now we may assume that $f(v) \geqslant 1$ for all non-cut-vertices v of G, and call such a vertex labeling *regular*. Now, we have the following theorem for the inductive step.

Theorem 2. *Suppose G is a block graph with a regular labeling f, and x is a non-cut-vertex in a block B with exactly one cut-vertex y or with no cut-vertex in which case let y be any vertex of $B - \{x\}$. When $f(x) = 1$, let $G' = G - x$ with a regular vertex labeling f' which is the same as f except $f'(y) = f(y) + 1$. When $f(x) \geqslant 2$, let $G' = G$ with a regular vertex labeling f' which is the same as f except $f'(x) = f(x) - 1$ and $f'(y) = f(y) + 1$. Then $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G')$.*

Proof. We first prove that $\mathrm{ip}_f(G) \geqslant \mathrm{ip}_{f'}(G')$. Suppose $\mathcal{C}$ is an optimal f-isometric-path cover of G. Choose a path P in $\mathcal{C}$ having x as an end vertex. We consider four cases.

Case 1.1. $P = x$ and $f(x) = 1$ (i.e., $G' = G - x$). In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{y\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geqslant |\mathcal{C}'| \geqslant \mathrm{ip}_{f'}(G')$.

Case 1.2. $P = x$ and $f(x) \geqslant 2$ (i.e., $G' = G$). In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{xy\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geqslant |\mathcal{C}'| \geqslant \mathrm{ip}_{f'}(G')$.

Case 1.3. $P = xz$ for some vertex z in $B - \{x, y\}$. In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{xy\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geqslant |\mathcal{C}'| \geqslant \mathrm{ip}_{f'}(G')$.

Case 1.4. $P = xyQ$, where Q contains no vertices in B. In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{yQ\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geqslant |\mathcal{C}'| \geqslant \mathrm{ip}_{f'}(G')$.

Next, we prove that $\mathrm{ip}_f(G) \leqslant \mathrm{ip}_{f'}(G')$. Suppose $\mathcal{C}'$ is an optimal f'-isometric-path cover of G'. Choose a path P' in $\mathcal{C}'$ having y as an end vertex. We consider three cases.

Case 2.1. $P' = yx$. In this case, $G' = G$ and $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{x\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leqslant |\mathcal{C}| \leqslant |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Case 2.2. $P' = yz$ for some z in $B - \{x, y\}$. In this case, $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{xz\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leqslant |\mathcal{C}| \leqslant |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Case 2.3. $P' = yQ$, where Q contains no vertex in B. In this case, $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{xyQ\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leqslant |\mathcal{C}| \leqslant |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Consequently, we have the following result for f-isometric-path numbers of connected block graphs.

Theorem 3. *If G is a connected block graph with a regular vertex labeling f, then $\mathrm{ip}_f(G) = \lceil s(G)/2 \rceil$, where $s(G) = \sum_{v \in V(G)} f(v)$.*

Proof. The theorem is obvious when G has only one vertex. For the case when G has more than one vertex, we apply Theorem 2 repeatedly until the graph becomes trivial. Notice that $s(G') = s(G)$ when Theorem 2 is applied. □

For the isometric-path-cover problem, we have

Corollary 4. *If G is a connected block graph, then $\mathrm{ip}(G) = \lceil \mathrm{nc}(G)/2 \rceil$, where $\mathrm{nc}(G)$ is the number of non-cut-vertices of G.*

Proof. The corollary follows from Theorem 3 and the fact that $\mathrm{ip}(G) = \mathrm{ip}_f(G)$ for the regular vertex labeling f with $f(v) = 1$ if v is a non-cut-vertex and $f(n) = 0$ otherwise. □

3. Algorithm

Based on Theorem 2, we are able to design an algorithm for the isometric-path-cover problem in block graphs. Notice that we may only consider connected block graphs with regular vertex labelings. To speed up the algorithm, we may modify Theorem 2 a little bit so that each time a non-cut-vertex is handled.

Theorem 5. *Suppose G is a block graph with a regular labeling f, and x is a non-cut-vertex in a block B with exactly one cut-vertex y or with no cut-vertex in which let y be any vertex in $B - \{x\}$. Let $G' = G - x$ with a regular vertex labeling f' which is the same as f except $f'(y) = f(y) + f(x)$. Then $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G')$.*

Proof. The theorem follows from repeatedly applying Theorem 2. □

Now, we are ready to give the algorithm.

Algorithm PG. Find the f-isometric-path number $\mathrm{ip}_f(G)$ of a connected block graph.

Input. A connected block graph G and a regular vertex labeling f.

Output. An optimal f-isometric-path cover $\mathcal{C}$ of G and $\mathrm{ip}_f(G)$.

Method.

```
construct a stack S which is empty at the beginning;
let G' ← G;
while (G' has more than one vertex) do
    choose a block B with exactly one cut-vertex y or
    with no cut-vertex in which case choose any y ∈ B;
    for (all vertices x in B − {y}) do
        f(y) ← f(y) + f(x);
        push (x, y, f(x)) into S;
        G' ← G' − x;
    end for;
end while;
ip_f(G) ← ⌈f(r)/2⌉, where r is the only vertex of G';
let C be the family of isometric paths containing ip(G)
    copies of the path r;
while (S is not empty) do
    pop (x, y, i) from S;
    choose i copies of path P in C using y as an end
        vertex;
    if (P = yx) then
        replace the i copies of P by i copies of x in C;
```

if ($P = yz$ for some vertex z in the block of G containing x) **then**
  replace the i copies of P by i copies of xz in $\mathcal{C}$;
if ($P = yQ$ where Q has no vertices in the block of G containing x) **then**
  replace the i copies of P by the i copies of xyQ in $\mathcal{C}$;
end while.

Algorithm PG can be implemented in time linear to the number of vertices and edges. Notice that we can use the depth-first search to find all blocks and cut-vertices of a graph in linear time, see [2].

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References

[1] M. Aigner, M. Fromme, A game of cops and robbers, Discrete Appl. Math. 8 (1984) 1–12.

[2] T.H. Cormen, C.E. Leiserson, R.L. Rivest, Introduction to Algorithms, MIT Press, Cambridge, MA, 1990.

[3] D.C. Fisher, S.L. Fitzpatrick, The isometric path number of a graph, J. Combin. Math. Combin. Comput. 38 (2001) 97–110.

[4] S.L. Fitzpatrick, Aspects of domination and dynamic domination, Ph.D. Thesis, Dalhousie University, Nova Scotia, Canada, 1997.

[5] S.L. Fitzpatrick, The isometric path number of the Cartesian product of paths, Congr. Numer. 137 (1999) 109–119.

[6] S.L. Fitzpatrick, R.J. Nowakowski, D. Holton, I. Caines, Covering hypercubes by isometric paths, Discrete Math. 240 (2001) 253–260.

[7] R. Nowakowski, P. Winkler, Vertex-to-vertex pursuit in a graph, Discrete Math. 43 (1983) 235–239.

[8] A. Quilliot, Thèse de 3e cycle, Université de Paris VI, 1978.