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The optimal consecutive-k-out-of-n:G line for $n \leqslant 2k$

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Abstract

Two proofs for the problem in the title have been published but both are incomplete. In this note we observe the subtle errors in these proofs and give a new proof by a different approach.

Keywords: Reliability; Consecutive systems; Optimal assignment

1. Introduction

A linear consecutive-k-out-of-n:G system, abbreviated as $\text{con}_{\text{L}}(k, n{:}G)$, is a line of n components which works if and only if some k consecutive components all work. Similarly, we can define a circular consecutive-k-out-of-n:G system, abbreviated as $\text{con}_{\text{C}}(k, n{:}G)$. Suppose we have n components with reliabilities $p_{[1]} \leqslant p_{[2]} \leqslant \cdots \leqslant p_{[n]}$. For easier presentation, we assume that the inequalities are strict from now on. The problem is to assign them to the n positions on the line or the cycle to maximize the reliability of the system, i.e., the probability that the system works. An optimal assignment is called *invariant* if it depends only on the ordering of the p_i, but not their actual values. For $n > 2k$ in the linear case and $n > 2k + 1$ in the circular case, Kuo et al. (1990) gave an invariant assignment for $k = 2$ and showed that no invariant assignments exist for $k \geqslant 3$.

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For $n \leqslant 2k$, Kuo, Zhang and Zuo gave a proof that

$$\alpha = (p_{[1]}, p_{[3]}, p_{[5]}, \ldots, p_{[6]}, p_{[4]}, p_{[2]})$$

is an invariant assignment for the line. For $n \leqslant 2k+1$, Zuo and Kuo (1990) proved that the cyclic version of α yields an invariant assignment for the cycle. Note that the second result implies the first result by setting $p_{[1]} = 0$.

Unfortunately, both proofs are incomplete. The same kind of logical slip occurs in both proofs. In each case a claim of necessary conditions for optimal assignment is stated. To be specific, the claim for the linear case is that for the $\mathrm{con_L}(k, 2k{:}G)$ system

$$(p_i - p_j)(p_{i-1} - p_{j+1}) > 0,$$

$$(p_i - p_j)(p_i - p_{j+1}) < 0 \quad \text{for } 1 < i \leqslant k,\ \ j = n - i + 1.$$

The claim for the circular case is

$$(p_i - p_j)(p_{i-1} - p_{j+1}) > 0 \quad \text{for all } 1 < i < j < n.$$

For a given assignment L, L_{ij} is defined as the assignment obtained from L by interchanging two components i and j. The purported proof of the claim shows that if an assignment satisfies the necessary conditions, then the system reliability cannot be improved through the operation of interchanging any pair. However, such a proof merely shows that α cannot be improved through the local operation of pairwise interchange but does not compare α with assignments which cannot be obtained from α in this way.

In this note we give a proof for the linear case.

2. A proof for the linear system

Let p_i denote the reliability of the ith component of the line and, as usual, let $q_i = 1 - p_i$ be the probability of failure. Consider the case $n = 2k$ and let L^* denote an optimal linear consecutive-k-out-of-$2k{:}G$ line. Tong (1985) proved

Lemma 1. *$p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$; $p_n \leqslant p_{n-1} \leqslant \cdots \leqslant p_{k+1}$ in L^*.*

Under the proposed conditions all these may be replaced by strict inequalities. Since the reliability of a line remains unchanged if the line is reversed, we may assume $p_k < p_{k+1}$ without loss of generality. Tong noted that, for $n \leqslant 2k$, the reliability of the system L can be calculated from

$$R(L) = \prod_{i=1}^{k} p_i + \sum_{j=1}^{n-k} q_i \prod_{i=j+1}^{j+k} p_i = \sum_{j=1}^{n-k+1} \prod_{i=j}^{j+k-1} p_i - \sum_{j=1}^{n-k} \prod_{i=j}^{j+k} p_i, \tag{2.1}$$

because there can be at most one run of length at least k consecutive working components, so the system will work if all of the first k components work or the jth component fails and the next k all work, for any j from 1 to $n-k$.

All of the following results can be obtained directly from (2.1). However, some of the arguments are simplified if, instead, we make use of the following Lemma 2.

Let L_{ij} denote the line obtained from L by interchanging the ith and jth components. Let s be a complete specification of the states of all components, where s is called a *working state* if the system works given s. Let $p(s)$ denote the probability of state s. Note that the operation L_{ij} would affect the system reliability only if the system works under exactly one of the two assignments, which implies that exactly one of the two components, i or j, works. Let S_{ij} denote the set of working states (so that the system works for all $s \in S_{ij}$) in which element i works element j fails and L_{ij} is a failed system.

Lemma 2. $R(L) - R(L_{ij}) = (p_i - p_j)\left(\sum_{s\in S_{ij}} \frac{p(s)}{p_i q_j} - \sum_{s\in S_{ji}} \frac{p(s)}{p_j q_i}\right)$.

Proof. For any state s, define a dual state s^* that is the same as s except that the status of components i and j have both altered. For example, if i works and j fails in s then i fails and j works in s^* while all other components behave the same way in s^* as they do in s. Now if s is in S_{ij} the system L works but L_{ij} fails while i works and j fails. But this implies that, for the state s^*, L will fail and L_{ij} will work. Thus

$$\begin{aligned} R(L) - R(L_{ij}) &= \sum_{s\in S_{ij}} (p(s) - p(s^*)) + \sum_{s\in S_{ji}} (p(s) - p(s^*)) \\ &= \sum_{s\in S_{ij}} \frac{p(s)}{p_i q_j}(p_i q_j - q_i p_j) + \sum_{s\in S_{ji}} \frac{p(s)}{p_j q_i}(p_j q_i - q_j p_i) \\ &= (p_i q_j - q_i p_j)\left(\sum_{s\in S_{ij}} \frac{p(s)}{p_i q_j} - \sum_{s\in S_{ji}} \frac{p(s)}{p_j q_i}\right) \\ &= (p_i - p_j)\left(\sum_{s\in S_{ij}} \frac{p(s)}{p_i q_j} - \sum_{s\in S_{ji}} \frac{p(s)}{p_j q_i}\right). \qquad \square \end{aligned}$$

Lemma 3. *$p_1 = p_{[1]}$, $p_k = p_{[n-1]}$, $p_{k+1} = p_{[n]}$ and $p_n = p_{[2]}$ in L^*.*

This means that the component with the lowest reliability should be placed first and the component with second lowest reliability last; the two most reliable components should occupy the two middle positions, with the more reliable of the two in position $k+1$.

Proof of Lemma 3. By Lemma 1 and the assumption $p_k < p_{k+1}$ we obtain $p_{k+1} = p_{[n]}$.

By Lemma 2, if we interchange the two middle components,

$$0 \leqslant R(L^*) - R(L^*_{k,k+1}) = (p_k - p_{k+1})\left(\prod_{i=1}^{k-1} p_i - \prod_{i=k+2}^{n} p_i\right),$$

because $S_{k,k+1}$ consists of all states for which the first k components work and the $(k+1)$th fails while $S_{k+1,k}$ consists of all states for which the last k components work but the kth fails.

Hence, because we assume $p_k < p_{k+1}$, we must have

$$\prod_{i=1}^{k-1} p_i \leqslant \prod_{i=k+2}^{n} p_i. \tag{2.2}$$

On the other hand, on interchanging the first and last components, we have

$$0 \leqslant R(L^*) - R(L^*_{1n}) = (p_1 - p_n)\left(q_{k+1}\prod_{i=2}^{k} p_i - q_k \prod_{i=k+1}^{n-1} p_i\right), \tag{2.3}$$

because S_{1n} consists of all states such that the first k components work but the $(k+1)$th and the nth components fail, while S_{n1} consists of all states such that the last k components work and the first and the kth components fail.

The rest of the proof contains some results obtained by contradiction. Suppose first that $p_1 > p_n$: then (2.3) implies that

$$q_{k+1}\prod_{i=2}^{k} p_i \geqslant q_k \prod_{i=k+1}^{n-1} p_i \quad \text{which implies,}$$

$$\prod_{i=2}^{k} p_i > \prod_{i=k+1}^{n-1} p_i \quad \text{which further implies,}$$

$$\prod_{i=1}^{k-1} p_i > \prod_{i=k+2}^{n} p_i,$$

since $p_k < p_{k+1}$. The last inequality contradicts (2.2) and so the assumption $p_1 > p_n$ is false. Thus $p_1 < p_n$ and so Lemma 1 implies that $p_1 = p_{[1]}$.

Now let L' be obtained from L^* by shifting the first component to the end of the line and shifting every other component one place to the left. Then, from (2.1),

$$\begin{aligned} 0 \leqslant R(L^*) - R(L') &= \left(\prod_{i=1}^{k} p_i - \prod_{i=1}^{k+1} p_i\right) - \left(\prod_{i=k+2}^{n} p_i p_1 - \prod_{i=k+1}^{n} p_i p_1\right) \\ &= q_{k+1}\prod_{i=1}^{k} p_i - q_{k+1}\prod_{i=k+2}^{n} p_i p_1, \end{aligned}$$

as the first term in parentheses is the only term from (2.1) that occurs in $R(L^*)$ but not in $R(L')$ while the second term in parentheses is the only term that occurs in $R(L')$ but not in $R(L^*)$. It follows that

$$\prod_{i=2}^{k} p_i \geqslant \prod_{i=k+2}^{n} p_i. \tag{2.4}$$

Also, by interchanging the kth and $(k+2)$nd components, Lemma 2 gives

$$
\begin{aligned}
0 &\leqslant R(L^*) - R(L^*_{k,k+2}) \\
&= (p_k - p_{k+2})\left[\left(\prod_{i=2}^{k-1} p_i\right) p_{k+1} + \left(\prod_{i=1}^{k-1} p_i\right) q_{k+1} - \left(\prod_{i=k+3}^{n} p_i\right) p_{k+1}\right] \\
&= (p_k - p_{k+2})\left[\left(\prod_{i=1}^{k-1} p_i\right) q_{k+1} + p_{k+1}\left(\frac{\prod_{i=2}^{k} p_i}{p_k} - \frac{\prod_{i=k+2}^{n} p_i}{p_{k+2}}\right)\right]. \qquad (2.5)
\end{aligned}
$$

This is because $S_{k,k+2}$ consists of states for which component $k+2$ fails and either all components from 2 to $k+1$ are good or the $(k+1)$th fails and the previous k are all good. $S_{k+2,k}$ consists of states for which the kth component fails and all k following components are good.

Suppose now that $p_k < p_{k+2}$: then (2.5) implies that $\prod_{i=2}^{k} p_i < \prod_{i=k+2}^{n} p_i$. This contradicts (2.4) and so $p_k > p_{k+2}$.

Finally, interchange components 2 and n and Lemma 2 gives

$$
\begin{aligned}
0 &\leqslant R(L^*) - R(L^*_{2,n}) \\
&= (p_2 - p_n)\left[p_1\left(\prod_{i=3}^{k} p_i\right) q_{k+1} + \left(\prod_{i=3}^{k+1} p_i\right) q_{k+2} - q_k\left(\prod_{i=k+1}^{n-1} p_i\right)\right] \\
&= (p_2 - p_n)\left[p_1\left(\prod_{i=3}^{k} p_i\right) q_{k+1}\right. \\
&\qquad \left. + p_{k+1}\left(\frac{q_{k+2}\prod_{i=2}^{k} p_i}{p_2} - \frac{q_k\prod_{i=k+2}^{n} p_i}{p_n}\right)\right], \qquad (2.6)
\end{aligned}
$$

because S_{2n} consists of states for which component n fails and either the first k components are all good and the $(k+1)$th fails or the $(k+2)$th fails and the previous k are all good. S_{n2} consists of states for which the kth component fails and all k following components are good, while the second component fails.

Finally, suppose that $p_2 < p_n$: then (2.6) implies that $\prod_{i=2}^{k} p_i < \prod_{i=k+2}^{n} p_i$. This contradicts (2.4) and so $p_2 > p_n$. Then it follows from Lemma 1 that $p_n = p_{[2]}$. □

Theorem. *The unique (up to line reversing) invariant consecutive-k-out-of-n: G line for* $n = 2k$ *is* $\alpha = ([1], [3], [5], \ldots, [6], [4], [2])$.

Proof. The theorem is trivially true for $k = 1$ and follows from Lemma 3 for $k = 2$. We prove the general case by induction on k. We now consider $k > 2$ and suppose the theorem is true for all $k' < k$.

Suppose to the contrary that $(p'_1, \ldots, p'_n) = \beta \neq \alpha$ is an optimal line. By Lemma 3, $p'_1 = p_{[1]}$, $p'_k = p_{[n-1]}$, $p'_{k+1} = p_{[n]}$ and $p'_n = p_{[2]}$. Let $F_\alpha, S_\alpha, F_\beta, S_\beta$ denote the product of reliabilities for the first or second half of α or β, respectively. From the proof of

Lemma 3, we have

$$F_x = \left(\prod_{i=1}^{k-1} p_i\right) p_k < p_{k+1}\left(\prod_{i=k+2}^{n} p_i\right) = S_x \text{ for } x \in \{\alpha, \beta\}, \text{ while}$$

$$F_\alpha S_\alpha = F_\beta S_\beta = \prod_{i=1}^{n} p_i.$$

Let σ be a sequence of components and let $t(\sigma) \equiv t_\sigma$ be the subsequence obtained by removing the component with the smallest reliability and the one with the greatest reliability. If β is optimal, then

$$\begin{aligned} 0 \geqslant R(\alpha) - R(\beta) &= \{q_{k+1}F_\alpha + p_{k+1}R(t_\alpha)\} - \{q_{k+1}F_\beta + p_{k+1}R(t_\beta)\} \\ &= q_{k+1}(F_\alpha - F_\beta) + p_{k+1}[R(t_\alpha) - R(t_\beta)], \end{aligned} \tag{2.7}$$

since the smallest and greatest reliabilities occur in positions 1 and $k+1$ for both sequences. Moreover, either system will work if the $(k+1)$th component fails and all of the first k work; the system will also work if the $(k+1)$th component works and there is a sequence of at least $k-1$ working components in the subsequence t_α (in the case of system α) or in subsequence t_β (in the case of system β). Note that in this circumstance the behaviour of the first component is irrelevant.

To clarify this by example, consider the case $k = 3$. If the fourth component is working then any sequence of at least 2 consecutive working components in the subsequence (2 3 5 6) will, together with the fourth component, form a sequence of at least 3 working components in the sequence (2 3 4 5 6), the behaviour of component 1 being irrelevant. To be specific, the consecutive working sequences (2 3), (3 5), (5 6), (2 3 5), (3 5 6) and (2 3 5 6) in the subsequence lead, respectively, to consecutive working sequences (2 3 4), (3 4 5), (4 5 6), (2 3 4 5), (3 4 5 6) and (2 3 4 5 6) in the full sequence.

By the induction hypothesis, t_α is the reverse of a uniquely optimal arrangement for $k-1$. It follows that the second term in (2.7) is strictly positive, so that the first term must be strictly negative, i.e. $F_\alpha < F_\beta$ and hence also $S_\alpha > S_\beta$. Thus we have

$$0 < F_\alpha < F_\beta < S_\beta < S_\alpha < 1.$$

Consequently,

$$S_\alpha - S_\beta = S_\alpha - F_\alpha S_\alpha / F_\beta = (F_\beta - F_\alpha) S_\alpha / F_\beta > F_\beta - F_\alpha. \tag{2.8}$$

We can expand $R(t_\alpha) - R(t_\beta)$ in (2.7) in a similar manner to the expansion of $R(\alpha) - R(\beta)$, bearing in mind that t_α is the reversal of the standard order. Thus

$$\begin{aligned} 0 \geqslant R(\alpha) - R(\beta) &= q_{k+1}(F_\alpha - F_\beta) + p_{k+1}\left\{ q_k \left(\prod_{i=k+2}^{n} p_i - \prod_{i=k+2}^{n} p_i' \right) \right. \\ &\quad \left. + p_k[R(t(t_\alpha)) - R(t(t_\beta))] \right\} \\ &\geqslant q_{k+1}(F_\alpha - F_\beta) + q_k(S_\alpha - S_\beta). \end{aligned}$$

since $t(t_\alpha)$ is optimal for $k-2$ by the inductive hypothesis.

But $q_{k+1} < q_k$, hence $F_\beta - F_\alpha > S_\alpha - S_\beta$. This contradicts (2.8) and so β is not optimal. This is true for all $\beta \neq \alpha$ so α must be optimal. The theorem is thus true by induction, as we already know it is true for $k = 1$ and 2. □

Corollary. *For $n \leqslant 2k$, an invariant line is* $([1],[3],[5],\ldots,B,\ldots,[6],[4],[2])$ *where B is a centre block of $2k - n$ largest reliabilities in any permutation.*

Proof. Any working state must have every component in B working. Therefore B should consist of the largest reliabilities. Furthermore, the system L works if and only if all components in B work and some $n-k$ components consecutive in $L \setminus B$ all work.

Therefore, the components not in B should be arranged according to the Theorem with $n' = 2k'$ and $k' = n - k$. □

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