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Revisiting dual-role factors in data envelopment analysis: derivation and implications

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Data Envelopment Analysis (DEA) is a mathematical programming method to evaluate relative performance. Typical DEA studies consider a production process transforming inputs to outputs. In some cases, however, some factors can be both inputs and outputs simultaneously and are termed dual-role factors. For example, research funding can be an input that strengthens a university's academic performance and the actual funds can be an output. This article investigates the problem of how to incorporate dual-role factors in DEA. Rather than proposing an *ad hoc* evaluation model directly, this article considers the concept of "joint technology," two individual production processes acting in common by summarizing the intuitive thinking. The efficiency evaluation models, based on variant assumptions, thus can be axiomatically derived, validated, and extended. How to determine the input/output tendency of a dual-role factor based on the evaluating results is shown and explained from different aspects. It is concluded that the tendency is a property on the projected boundary, not the data point itself.

Keywords: Data envelopment analysis, dual-role factor, efficiency

1. Introduction

Data Envelopment Analysis (DEA), coined and popularized by Charnes *et al*. (1978), is a mathematical programming method to evaluate relative performance by peer comparison. DEA considers multiple aspects of the performance simultaneously and aggregates different criteria values as a ratio of weighted output to weighted input without *a priori* weight assignments. At this analytical stage, DEA is an aggregating mechanism to aggregate multiple criteria into a single score. On the other hand, DEA is a stream of nonparametric production analysis, originated by Farrell (1957), to measure the technical efficiency of production units. The efficiency of a unit is measured relative to the production frontier, which is estimated by a set of data.

Typical DEA studies consider a production process of transforming multiple inputs to various outputs. In addition to those having a clear input/output role, however, some factors may play roles as both inputs and outputs simultaneously; these factors are referred to as *dual-role factors*. Beasley (1990, 1995) is the first to note dual-role factors in his study evaluating research productivity at a university. He finds that research funding is an important performance criterion (output) and a resource (input) that strengthens the institution's academic performance. Cook *et al*. (2006) looked at graduate students in higher education organizations and nurse trainees on staff in hospitals and found that the graduate student and nurse trainee are dual-role factors. In other words, they are the maximizing-oriented performance criteria (outputs) themselves and the resources (inputs) to provide publications and care for more patients.

A literature survey reveals different models for DEA evaluation studies with dual-role factors. Following the idea by Charnes *et al*. (1978), the models specify different output-to-input ratio forms while retaining the core spirit of DEA with minimum assumptions about weights determination. In Beasley (1990, 1995), a dual-role factor is in both the denominator (as a part of the weighted input) and the numerator (as a part of the weighted output). Cook *et al*. (2006) suggest moving the input role from the denominator to the numerator, the output side, but with the opposite sign in its weight. Specifically, they consider the dual-role factor as an exogenously fixed or non-discretionary variable (Banker and Morey, 1986), which is not controlled but can affect the DEA evaluation. In addition, Bi *et al*. (2009) attempt to address this issue from the angle of a production process, not multi-criteria performance aggregation. Rather than considering these factors as inputs and outputs simultaneously, Cook and Zhu (2007) suggest a method to classify them in DEA. To the best of our knowledge, the available literature does not discuss the validity of any proposed

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model or the axiomatic arguments. Although Cook *et al*. (2006) show that Beasley's treatment of dual-role factors is inappropriate, they do not provide any validation for their modified model.

Therefore, this article investigates how to handle dual-role factors in DEA. Rather than proposing an *ad hoc* evaluating model directly, we develop a concept of "joint technology," two individual production processes acting in common, by summarizing the intuitive thinking. Evaluating models based on different assumptions can be derived, validated, and extended axiomatically using the "joint technology." Thus, the proposed model is theoretically well defined and intuitively obvious.

The benefits of the proposed approach are at least threefold. First, the approach provides a clear framework for analyzing how to incorporate dual-role factors in DEA. In particular, we approach the problem from production process axioms but not the weighting viewpoints. Multiple dual-role factors can also be analyzed. Second, by focusing on the basic framework and not a solution to a specific problem, the approach provides a way to validate the proposed models in the extant literature, including Cook *et al*. (2006). It clearly shows the flawed logic in Beasley (1990, 1995) beyond taking a counterexample. Third, the proposed approach is easy to extend and integrate with variant DEA models with detailed specific requirements, such as Free Disposal Hull (FDH; Deprins *et al*. (1984)), non-discretionary variables, etc.

Based on the proposed "joint technology" framework, this article makes the following contributions to the literature. We develop a simple three-dimensional case for visualization, and we also generalize to multi-dimensional cases for considering the multiple factors. Furthermore, we determine the input/output behavior (tendency) of a dual-role factor based on the analysis results and explain it from multi-criteria performance aggregation, geometry, and economics perspectives. Importantly, we conclude that the tendency is a property on the projected boundary, not the data point itself. This conclusion sheds light for centralized/decentralized dual-role factor allocation problems.

The remainder of this article is organized as follows. Section 2 introduces DEA as a performance evaluation method. Section 3 proposes the axiomatic framework and derives the efficiency evaluation considering dual-role factors, including Cook *et al*. (2006). Section 4 provides a visualization example and motivates the geometric interpretations. Section 5 generalizes the findings to multiple dimensions, gives economic interpretations, and discusses some possible variant models, and Section 6 concludes.

2. Technology and radial efficiency

Consider a production process transforming various inputs into different outputs. Denote an input set I and an output set O, and let $\mathbf{x} \in \Re_+^{|I|}$ and $\mathbf{y} \in \Re_+^{|O|}$ be the value vectors for inputs and outputs, respectively. A production technology, represented by a set T, describes the possible input–output transformation as

$$T \equiv \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \text{ can produce } \mathbf{y}\}.$$

Given a set of records R with data $(\mathbf{x}_r, \mathbf{y}_r) \in \Re_+^{|I|+|O|}, r \in R$, we can approximate the underlying but unknown T based on the following assumptions:

1. Strong disposability: $(\mathbf{x}, \mathbf{y}) \in T, \mathbf{x}' \geq \mathbf{x}$ and $\mathbf{y}' \leq \mathbf{y}$ imply $(\mathbf{x}', \mathbf{y}') \in T$.
2. Convexity: $(\mathbf{x}, \mathbf{y}) \in T$ and $(\mathbf{x}', \mathbf{y}') \in T$ imply $\lambda(\mathbf{x}, \mathbf{y}) + (1-\lambda)(\mathbf{x}', \mathbf{y}') \in T$ for $\lambda \in [0, 1]$.
3. Constant Returns to Scale (CRS): $(\mathbf{x}, \mathbf{y}) \in T$ implies $\alpha(\mathbf{x}, \mathbf{y}) \in T$ for $\alpha \in [0, \infty)$.

One approximation is

$$\hat{T} \equiv \left\{(\mathbf{x}, \mathbf{y}) : \sum_{r\in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r\in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \lambda_r \geq 0, r \in R\right\}.$$

$\hat{T}$ imposes assumptions 1 to 3 and is referred to as the CRS technology. The CRS assumption can be relaxed by adding the convexity constraint $\sum_{r\in R} \lambda_r = 1$ and thus is termed the Variable Returns to Scale (VRS) technology. An important use of $\hat{T}$ is the Farrell input efficiency measure (Farrell, 1957). To measure input efficiency of $(\mathbf{x}_k, \mathbf{y}_k)$, $k \in R$, we solve the following problem:

$$\begin{aligned} \min \quad & \theta, \\ \text{s.t.} \quad & \sum_{r\in R} \mathbf{x}_r \lambda_r \leq \theta \mathbf{x}_k, \\ & \sum_{r\in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}_k, \\ & \lambda_r \geq 0, r \in R. \end{aligned} \tag{1}$$

The optimal value θ^* of Model (1) is the input efficiency of k associated with the CRS technology $\hat{T}$. Assuming CRS technology, it suggests that k can reduce its inputs x_k to $(100 \times \theta^*)\%$ while maintaining the same level of outputs y_k. There are variant models according to different technology assumptions and objectives, such as the orientations. Comprehensive explanations and variant models can be found in Färe *et al*. (1994a) and Cooper *et al*. (2007).

Model (1) can be transformed to the equivalent problem as follows (Charnes *et al*., 1978). For notational simplicity, the vector multiplication in this paper represents the dot product of two vectors.

$$\begin{aligned} \max \quad & \frac{\mathbf{y}_k \mathbf{v}}{\mathbf{x}_k \mathbf{u}}, \\ \text{s.t.} \quad & \frac{\mathbf{y}_r \mathbf{v}}{\mathbf{x}_r \mathbf{u}} \leq 1, r \in R, \\ & \mathbf{u} \geq 0, \mathbf{v} \geq 0. \end{aligned} \tag{2}$$

We can see Model (2) as a weight aggregation scheme for record $k \in R$. $\mathbf{u} \in \Re_+^{|I|}$ and $\mathbf{v} \in \Re_+^{|O|}$ are weights assigned for inputs and outputs, respectively, and all records' overall performance scores are calculated as ratios of weighted output to weighted input. Model (2) allows k to select weights favoring its performance score $\mathbf{y}_k\mathbf{v}/\mathbf{x}_k\mathbf{u}$, as long as all performance scores are normalized within one.

3. Modeling dual-role factors

Dual-role factors can play roles as both inputs and outputs simultaneously. When dual-role factors play roles as outputs, we term them dual-role outputs in contrast with the regular outputs $\mathbf{y}$, and when dual-role factors play roles as inputs, we term them dual-role inputs in contrast to the regular inputs $\mathbf{x}$. Consider the cases with dual-role factors in addition to inputs and outputs. Denote the dual-role factor set D and the value vector $\mathbf{w} \in \Re_+^{|D|}$. Collect the data $(\mathbf{x}_r, \mathbf{w}_r, \mathbf{y}_r) \in \Re_+^{|I|+|D|+|O|}$, $r \in R$.

3.1. *A joint technology*

The following two statements are apparently true and generally intuitive for production processes having dual-role factors in addition to regular inputs and outputs:

Regular inputs generate regular outputs and dual-role outputs, (3)

and

Regular inputs and dual-role inputs generate regular outputs. (4)

Dual-role factors can be considered as part of the outputs, which count for performance to be maximized. However, there is a cost, since dual-role outputs and regular outputs consume the same resources. On the other hand, when dual-role factors are also considered as part of inputs, together with regular inputs, they can contribute to provide more regular outputs. In the study by Cook *et al*. (2006, pp. 105) on the role of graduate students in evaluating researchers' performance, the authors make two important arguments and observations:

> while published research (articles in referred journals, etc.) is likely the predominant output for evaluating the researcher, the extent to which the research contributes to the training of highly qualified personnel is also an important component in the evaluation.

and

> at least two inputs contribute to the generation of research publications: (i) research dollars available to support publication; and (ii) the number of graduate assistants. . . .

Note that their first statement coincides with Observation (3) and that (4) generalizes the second argument.

Next, consider the example of nurse trainees in a hospital. Part of the responsibilities and performance of certified nursing staff is to guide and educate more trainees, which, of course, consumes more of the hospital's resources, such as the certified nurses' time and labor. However, the trainees help the nurses with several routine tasks requiring the least skill and experience.

Observations (3) and (4) describe two individual production processes. Observation (3) is formalized by T_1 as

$$T_1 \equiv \{(\mathbf{x}, \mathbf{w}, \mathbf{y}) : \mathbf{x} \text{ can produce } (\mathbf{w}, \mathbf{y})\}.$$

Observation (4) is represented by T_2 as

$$T_2 \equiv \{(\mathbf{x}, \mathbf{w}, \mathbf{y}) : (\mathbf{x}, \mathbf{w}) \text{ can produce } \mathbf{y}\}.$$

Observations (3) and (4) are observed simultaneously. In other words, we can formalize the process as a joint technology based on both Observations (3) and (4). The overall underlying technology with dual-role factors is the intersection of technologies T_1 and T_2 as

$$T_1 \cap T_2.$$

It is important to bear in mind that dual-role factors play two roles, inputs and outputs, simultaneously. They are neutral in terms of their input/output roles. We do not treat a dual-role factor as either an input or an output *ex ante*. We only *ex post* conclude that a particular record prefers the dual-role factor to be an input or output by comparing what it gains from both roles as discussed later.

Without loss of generality, denote $\psi(\mathbf{y}) = \{(\mathbf{x}, \mathbf{w}) : (\mathbf{x}, \mathbf{w}, \mathbf{y}) \in T_1 \cap T_2\}$ as the feasible set of $(\mathbf{x}, \mathbf{w})$ for a given $\mathbf{y}$. The following proposition shows that $\psi(\mathbf{y})$ is nested in $\mathbf{y}$ due to the strong disposability of $\mathbf{y}$.

Proposition 1. $\psi(\mathbf{y}) \subseteq \psi(\mathbf{y}')$ *if* $\mathbf{y}' \leq \mathbf{y}$.

Proof. $\mathbf{y}$ exhibits strong disposability for T_1 and T_2, and we have

$$\text{If } \mathbf{y}' \leq \mathbf{y}, \text{ then } (\mathbf{x}, \mathbf{w}, \mathbf{y}) \in T_1 \Rightarrow (\mathbf{x}, \mathbf{w}, \mathbf{y}') \in T_1.$$
$$\text{If } \mathbf{y}' \leq \mathbf{y}, \text{ then } (\mathbf{x}, \mathbf{w}, \mathbf{y}) \in T_2 \Rightarrow (\mathbf{x}, \mathbf{w}, \mathbf{y}') \in T_2.$$

Therefore, for $\mathbf{y}' \leq \mathbf{y}$, $(\mathbf{x}, \mathbf{w}, \mathbf{y}) \in T_1 \cap T_2$ implies $(\mathbf{x}, \mathbf{w}, \mathbf{y}') \in T_1 \cap T_2$. According to the definition of $\psi(\mathbf{y})$, we can also rewrite the statement as $(\mathbf{x}, \mathbf{w}) \in \psi(\mathbf{y}) \Rightarrow (\mathbf{x}, \mathbf{w}) \in \psi(\mathbf{y}')$; i.e., $\psi(\mathbf{y}) \subseteq \psi(\mathbf{y}')$, for $\mathbf{y}' \leq \mathbf{y}$. ■

Following the idea of approximating technology based on collected data set R, a possible approximation of $T_1 \cap T_2$ is

$$\widehat{T_1 \cap T_2} \equiv \Bigg\{ (\mathbf{x}, \mathbf{w}, \mathbf{y}) : \sum_{r\in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r\in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \sum_{r\in R} \mathbf{w}_r \lambda_r \geq \mathbf{w}; \sum_{r\in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r\in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \sum_{r\in R} \mathbf{w}_r \lambda_r \leq \mathbf{w}; \lambda_r \geq 0, r \in R \Bigg\}.$$

The approximation imposes assumptions 1 to 3 as does $\hat{T}$. The first three constraints associate with T_1, and the next three associate with T_2. Any $(\mathbf{x}, \mathbf{w}, \mathbf{y})$ (as seen on the right-hand side of the inequalities) that satisfies all constraints are in $\widehat{T_1 \cap T_2}$. Note that $\widehat{T_1 \cap T_2}$ approximates $T_1 \cap T_2$ directly and are joint by applying the same λ_r values corresponding to two technologies.

3.2. *Radial efficiency*

The classic Farrell input efficiency is a radial efficiency measure relative to the boundaries of the technology by proportionately reducing all "inputs," including any factor that could provide function as inputs. We thus reduce dual-role factors together with regular inputs since dual-role factors are part of inputs. As an analog to Model (1), solving the following problem evaluates $(\mathbf{x}_k, \mathbf{w}_k, \mathbf{y}_k)$, $k \in R$:

$$\begin{aligned}
\theta_k^{\text{radial}} = \min\quad & \theta, \\
\text{s.t.}\ & \sum_{r\in R} \mathbf{x}_r\lambda_r \le \theta\mathbf{x}_k, \\
& \sum_{r\in R} \mathbf{y}_r\lambda_r \ge \mathbf{y}_k, \\
& \sum_{r\in R} \mathbf{w}_r\lambda_r \ge \theta\mathbf{w}_k, \\
& \sum_{r\in R} \mathbf{x}_r\lambda_r \le \theta\mathbf{x}_k, \\
& \sum_{r\in R} \mathbf{y}_r\lambda_r \ge \mathbf{y}_k, \\
& \sum_{r\in R} \mathbf{w}_r\lambda_r \le \theta\mathbf{w}_k, \\
& \lambda_r \ge 0, r \in R.
\end{aligned} \tag{P1}$$

Constraints represent a joint technology of $\widehat{T_1 \cap T_2}$, where the top three constraints associate with T_1 and the next three constraints associate with T_2. The optimal value of (P1), θ_k^{radial}, is the efficiency of k in a process with dual-role factors; i.e., (P1) proportionately reduces all "inputs" to θ times of its current level, including $\mathbf{x}_k$ and $\mathbf{w}_k$. Reducing $\mathbf{w}_k$ together with $\mathbf{x}_k$ is clear with respect to T_2, in which dual-role factors play roles of inputs, and this reduction result is denoted as $(\theta\mathbf{x}_k, \theta\mathbf{w}_k, \mathbf{y}_k)$. A feasible θ means that $(\theta\mathbf{x}_k, \theta\mathbf{w}_k, \mathbf{y}_k) \in \widehat{T_1 \cap T_2}$, and $(\theta\mathbf{x}_k, \theta\mathbf{w}_k, \mathbf{y}_k)$ should satisfy the constraints associated with both T_1 and T_2. Thus, we have $\theta\mathbf{w}_k$, not $\mathbf{w}_k$, in the right-hand side of the third constraint. Rather than interpreting it as simply minimizing (dual-role) outputs, this setting truthfully and passively reflects the results of reducing both regular and dual-role inputs $\mathbf{x}_k$ and $\mathbf{w}_k$, which define the radial efficiency measure here. Note that reducing $\mathbf{w}_k$ implies that the dual-role factors are adjustable (other efficiency measures with additional assumptions are discussed in Sections 3.3 and 5.3).

(P1) is simplified by removing identical constraints as

$$\begin{aligned}
\theta_k^{\text{radial}} = \min\quad & \theta, \\
\text{s.t.}\ & \sum_{r\in R} \mathbf{x}_r\lambda_r \le \theta\mathbf{x}_k, \\
& \sum_{r\in R} \mathbf{y}_r\lambda_r \ge \mathbf{y}_k, \\
& \sum_{r\in R} \mathbf{w}_r\lambda_r \ge \theta\mathbf{w}_k, \\
& \sum_{r\in R} \mathbf{w}_r\lambda_r \le \theta\mathbf{w}_k, \\
& \lambda_r \ge 0, r \in R.
\end{aligned}$$

The dual problem follows as

$$\begin{aligned}
\max\quad & \mathbf{y}_k\mathbf{v}, \\
\text{s.t.}\quad & -\mathbf{x}_r\mathbf{u} + \mathbf{y}_r\mathbf{v} + \mathbf{w}_r\boldsymbol{\gamma}^o - \mathbf{w}_r\boldsymbol{\gamma}^i \le 0, r \in R, \\
& \mathbf{x}_k\mathbf{u} - \mathbf{w}_k\boldsymbol{\gamma}^o + \mathbf{w}_k\boldsymbol{\gamma}^i = 1, \\
& \mathbf{u} \ge 0, \mathbf{v} \ge 0, \boldsymbol{\gamma}^o \ge 0, \boldsymbol{\gamma}^i \ge 0,
\end{aligned} \tag{5}$$

where $\mathbf{u} \in \Re_+^{|I|}$, $\mathbf{v} \in \Re_+^{|O|}$, $\boldsymbol{\gamma}^o \in \Re_+^{|D|}$, and $\boldsymbol{\gamma}^i \in \Re_+^{|D|}$ are dual variables corresponding to the constraints from the top to the bottom. The equivalent linear fractional programming problem (Charnes and Cooper, 1962; Charnes *et al.*, 1978) is

$$\begin{aligned}
\max\quad & \frac{\mathbf{y}_k\mathbf{v}}{\mathbf{x}_k\mathbf{u} - \mathbf{w}_k\boldsymbol{\gamma}^o + \mathbf{w}_k\boldsymbol{\gamma}^i}, \\
\text{s.t.}\quad & -\mathbf{x}_r\mathbf{u} + \mathbf{y}_r\mathbf{v} + \mathbf{w}_r\boldsymbol{\gamma}^o - \mathbf{w}_r\boldsymbol{\gamma}^i \le 0, r \in R, \\
& \mathbf{u} \ge 0, \mathbf{v} \ge 0, \boldsymbol{\gamma}^o \ge 0, \boldsymbol{\gamma}^i \ge 0.
\end{aligned}$$

It can be rearranged as

$$\begin{aligned}
\max\quad & \frac{\mathbf{y}_k\mathbf{v}}{\mathbf{x}_k\mathbf{u} + \mathbf{w}_k(\boldsymbol{\gamma}^i - \boldsymbol{\gamma}^o)} \\
\text{s.t.}\quad & \frac{\mathbf{y}_r\mathbf{v}}{\mathbf{x}_r\mathbf{u} + \mathbf{w}_r(\boldsymbol{\gamma}^i - \boldsymbol{\gamma}^o)} \le 1, r \in R, \\
& \mathbf{u} \ge 0, \mathbf{v} \ge 0, \boldsymbol{\gamma}^o \ge 0, \boldsymbol{\gamma}^i \ge 0.
\end{aligned} \tag{P1R}$$

(P1R) is a standard ratio form of DEA (Charnes *et al.*, 1978) and has properties similar to Model (2). $\mathbf{u}$ and $\mathbf{v}$ are the weight vectors for regular inputs and outputs, respectively. $\boldsymbol{\gamma}^o$ is the weight vector for dual-role outputs, and $\boldsymbol{\gamma}^i$ is for the $\mathbf{w}_r$ serving as inputs. $\boldsymbol{\gamma}^i - \boldsymbol{\gamma}^o$ can be either positive or negative, although $\boldsymbol{\gamma}^o$ and $\boldsymbol{\gamma}^i$ are both non-negative.

We give the implication on the sign of $\gamma_d^{i*} - \gamma_d^{o*}$ $d \in D$ in a more intuitive and straightforward manner. Similar to Model (2), Problem (P1R) allows k to select weights to favor its performance (efficiency). In (P1R), $\gamma_d^{i*} < \gamma_d^{o*}$ $d \in D$ implies that k prefers factor d as an output rather than being an input and weights factor d more on its output role. As a result, $\gamma_d^{i*} - \gamma_d^{o*} < 0$ provides a smaller denominator value and results in a better performance. $\gamma_d^{i*} < \gamma_d^{o*}$ suggests that factor d benefits k from an output role and has an overall negative impact on d as an input. In contrast, $\gamma_d^{i*} > \gamma_d^{o*}$ means that k prefers factor d as an input. $\gamma_d^{i*} = \gamma_d^{o*}$ suggests indifference to d being an output or an input.

3.3. *Being exogenously fixed*

Instead of minimizing both $\mathbf{x}_k$ and $\mathbf{w}_k$ proportionately as shown in (P1), we may want to treat dual-role factors as exogenously fixed or non-discretionary (Banker and Morey, 1986). The concept is adopted by Cook *et al*. (2006). Here, we adopt this assumption and derive the formulation using joint technology. Solving the following problem evaluates k, but we assume that the dual-role factors are exogenously fixed:

$$\begin{aligned} \theta_k^{\text{fix}} = \min \quad & \theta, \qquad \text{(P2)} \\ \text{s.t.} \quad & \sum_{r\in R} \mathbf{x}_r\lambda_r \le \theta \mathbf{x}_k, \\ & \sum_{r\in R} \mathbf{y}_r\lambda_r \ge \mathbf{y}_k, \\ & \sum_{r\in R} \mathbf{w}_r\lambda_r \ge \mathbf{w}_k, \\ & \sum_{r\in R} \mathbf{x}_r\lambda_r \le \theta \mathbf{x}_k, \\ & \sum_{r\in R} \mathbf{y}_r\lambda_r \ge \mathbf{y}_k, \\ & \sum_{r\in R} \mathbf{w}_r\lambda_r \le \mathbf{w}_k, \\ & \lambda_r \ge 0, r \in R. \end{aligned}$$

The interpretation of (P2) is similar to (P1). However, unlike (P1), (P2) minimizes only the regular inputs. $\sum_{r\in R} \mathbf{w}_r\lambda_r \ge \mathbf{w}_k$ specifies exogenously fixed dual-role outputs associated with T_1, and $\sum_{r\in R} \mathbf{w}_r\lambda_r \le \mathbf{w}_k$ associates non-discretionary dual-role inputs associated with T_2.

Following the same procedure deriving (P1R) from (P1), the dual of the equivalence of (P2) is

$$\begin{aligned} \max \quad & \mathbf{y}_k\mathbf{v} + \mathbf{w}_k\boldsymbol{\gamma}^o - \mathbf{w}_k\boldsymbol{\gamma}^i, \qquad (6) \\ \text{s.t.} \quad & -\mathbf{x}_r\mathbf{u} + \mathbf{y}_r\mathbf{v} + \mathbf{w}_r\boldsymbol{\gamma}^o - \mathbf{w}_r\boldsymbol{\gamma}^i \le 0, r \in R, \\ & \mathbf{x}_k\mathbf{u} = 1, \\ & \mathbf{u} \ge 0, \mathbf{v} \ge 0, \boldsymbol{\gamma}^o \ge 0, \boldsymbol{\gamma}^i \ge 0. \end{aligned}$$

The equivalent DEA ratio form is

$$\begin{aligned} \max \quad & \frac{\mathbf{y}_k\mathbf{v} + \mathbf{w}_k(\boldsymbol{\gamma}^o - \boldsymbol{\gamma}^i)}{\mathbf{x}_k\mathbf{u}}, \qquad \text{(P2R)} \\ & \frac{\mathbf{y}_r\mathbf{v} + \mathbf{w}_r(\boldsymbol{\gamma}^o - \boldsymbol{\gamma}^i)}{\mathbf{x}_r\mathbf{u}} \le 1, r \in R, \\ & \mathbf{u} \ge 0, \mathbf{v} \ge 0, \boldsymbol{\gamma}^o \ge 0, \boldsymbol{\gamma}^i \ge 0. \end{aligned}$$

The interpretations of decision variables are identical to (P1R), but the dual-role factors are now in the numerator with weight vector $\boldsymbol{\gamma}^o - \boldsymbol{\gamma}^i$, not in the denominator with $\boldsymbol{\gamma}^i - \boldsymbol{\gamma}^o$ as in (P1R).

Indeed, (P2R) derived from (P2) is identical to the formulation in Cook *et al*. (2006); more precisely, (P2R) is a general model considering multiple dual-role factors. Arguing that $\gamma_d^{o*}\gamma_d^{i*} = 0$, Cook *et al*. (1986) propose the implication of sign by comparing (P2R) to the VRS ratio form (Banker *et al*., 1984) without dual-role factors. The difference between Model (2) and (P2R) is the second term of the numerator, as is the difference between Model (2) and the VRS ratio form. Their reasoning is based on the Returns To Scale (RTS) characteristics drawn from the VRS ratio form. $\gamma_d^{o*} - \gamma_d^{i*} < 0$ is similar to the decreasing RTS cases, and thus the dual-role factor d acts like an input. Similarly, $\gamma_d^{o*} - \gamma_d^{i*} > 0$ indicates that d behaves like an output, and $\gamma_d^{o*} - \gamma_d^{i*} = 0$ suggests that d is at an equilibrium or optimal level.

Cook *et al*. (2006) are correct in judging a dual-role factor's input/output behavior based on the sign of $\gamma_d^{o*} - \gamma_d^{i*}$, but it is not necessary to have $\gamma_d^{o*}\gamma_d^{i*} = 0$. Without loss of generality, if $\gamma_d^{o*} = 0$ and $\gamma_d^{i*} > 0$, $\gamma_d^{o*} + \delta$ and $\gamma_d^{i*} + \delta$ for $\delta \ge 0$ give the same optimal value, then $\gamma_d^{o*} + \delta$ and $\gamma_d^{i*} + \delta$ for $\delta \ge 0$ are also optimal solutions to (P2R). Although the mathematical formulation structure of Cook *et al*. (2006) gives a nice comparison to judge the importance, or tendency, of the role of dual factors, the managerial interpretations are not yet clear. Thus, we suggest using the same arguments for (P1R): k prefers d as an output if and only if $\gamma_d^{i*} < \gamma_d^{o*}$ gives a positive weighted value of d in the numerator and d contributes positively as an output.

Comparing (P1R) and (P2R) shows that (P1R) has dual-role factors in the denominators with weight vector $\boldsymbol{\gamma}^i - \boldsymbol{\gamma}^o$ while (P2R) has them in the numerators with $\boldsymbol{\gamma}^o - \boldsymbol{\gamma}^i$. Our proposed judging rules and interpretations yield a consistent result in both cases. In contrast, relying on RTS characteristics may limit to (P2R). Moreover, to derive the problem in a ratio form, note that both $\mathbf{w}_k\boldsymbol{\gamma}^i$ and $\mathbf{w}_k\boldsymbol{\gamma}^o$ should always be in either the unity constraint as in Model (5) or the objective function as in Model (6). Otherwise, since the objective function and unity constraint correspond to the decision whether to reduce dual-role factors in $\widehat{T_1 \cap T_2}$, $\mathbf{w}_k$ will be minimized by θ for its input role as $\sum_{r\in R} \mathbf{w}_r\lambda_r \le \theta\mathbf{w}_k$ and also being exogenously fixed for its output role as $\sum_{r\in R} \mathbf{w}_r\lambda_r \ge \mathbf{w}_k$ (or, similarly, $\sum_{r\in R} \mathbf{w}_r\lambda_r \ge \theta\mathbf{w}_k$ and $\sum_{r\in R} \mathbf{w}_r\lambda_r \le \mathbf{w}_k$); thus $\theta^* = 1$. Therefore, we can conclude that the model proposed by Beasley (1990, 1995) is problematic and does not align with the general intuitive understanding of the dual-role factors.

4. Visualization and geometric implications

Here, we visualize the joint technology using a simple, three-dimensional case with one regular input, one regular output, and one dual-role factor, based on a portion of the real-world data collected by Beasley (1990) in his study evaluating research productivity at a university. The purpose of the visualization is to understand the generalized DEA models with dual-role factors, the difference between (P1) and (P2), and the possible extensions.

4.1. *Data set*

First, we recall that (P1) and (P1R) are equivalent, as are (P2) and (P2R). For the purpose of visualization, from Beasley (1990), we arbitrarily use equipment expenditure as the only input (x), PG research as the only output (y), and research income as the only dual-role factor (w). We exclude Universities 7 and 49 due to zero or extremely little equipment expenditure; however, this exclusion does not affect our visualization. We approximate and visualize $\widehat{T_1 \cap T_2}$, in which w is dual-role, and also $\hat{T}_1$, w as an output, and $\hat{T}_2$, w as an input, for comparison. All technology approximations employ the CRS.

Table 1 shows the data and the analysis results for our simple visualization. Columns 5 and 6 are the input-oriented efficiency measures associated with $\hat{T}_1$ and $\hat{T}_2$. For θ_k^2, w is minimized together with x regarding $\hat{T}_2$, but w is not minimized together with x regarding $\hat{T}_1$ to obtain θ_k^1. Universities 24 and 36 have $\theta_k^1 = 1$ and are efficient with respect to $\hat{T}_1$. Universities 9, 19, and 36 have $\theta_k^2 = 1$ and are efficient with respect to $\hat{T}_2$. Column 7 is the radial efficiency measures θ_k^{radial} for $\widehat{T_1 \cap T_2}$, which is obtained by (P1), and followed by the associated weight for w, $\gamma_d^{i*} - \gamma_d^{o*}$ computed by (P1R). θ_k^{fix} is the efficiency measure for k considering w as non-discretionary computed by (P2). Column 10 is the weight of w under this assumption. As expected, the efficiency measures, weights of the dual-role factors, and signs of the weights are different for the two models.

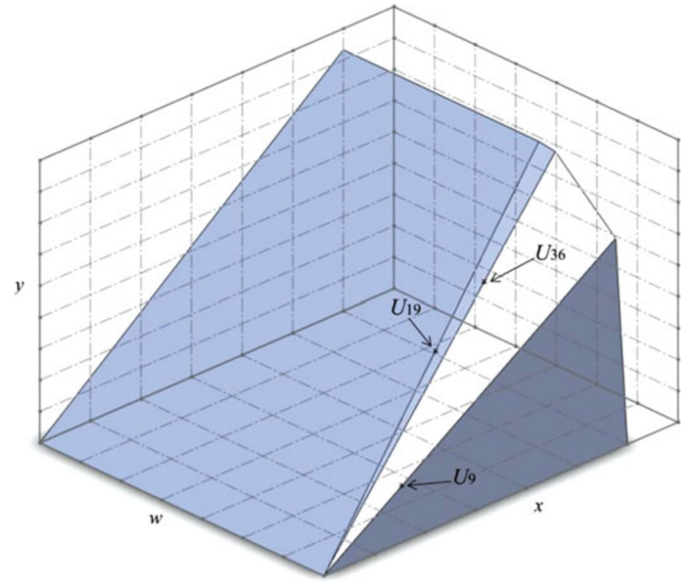

Fig. 2. A partial three-dimensional visualization of $\hat{T}_2$.

Figures 1 to 3 visualize $\hat{T}_1$, $\hat{T}_2$, and $\widehat{T_1 \cap T_2}$. One block represents 20 in x, 500 in w, and 10 in y. All figures are bounded at $x = 120$ and Fig. 3 is also bounded at $w = 3000$. Doing so produces a more focused illustration of the technology boundaries. Figure 1 represents $\hat{T}_1$, in which research income w is an output. Universities 24 and 36 form the boundaries of $\hat{T}_1$ (also see Table 1). Figure 2 represents $\hat{T}_2$, where research income is an input. Universities 9, 19, and 36 form the boundaries of $\hat{T}_2$ (also see Table 1). Figure 3 shows the joint technology $\widehat{T_1 \cap T_2}$. Comparing three figures show the differences of treating a dual-role factor in three different settings.

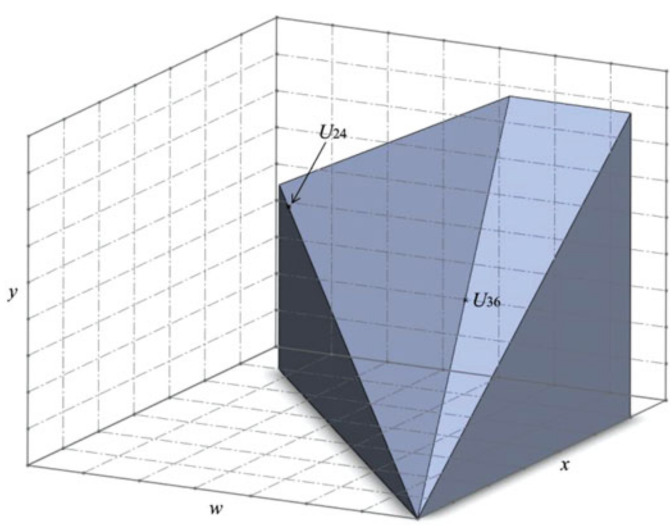

Fig. 1. A partial three-dimensional visualization of $\hat{T}_1$.

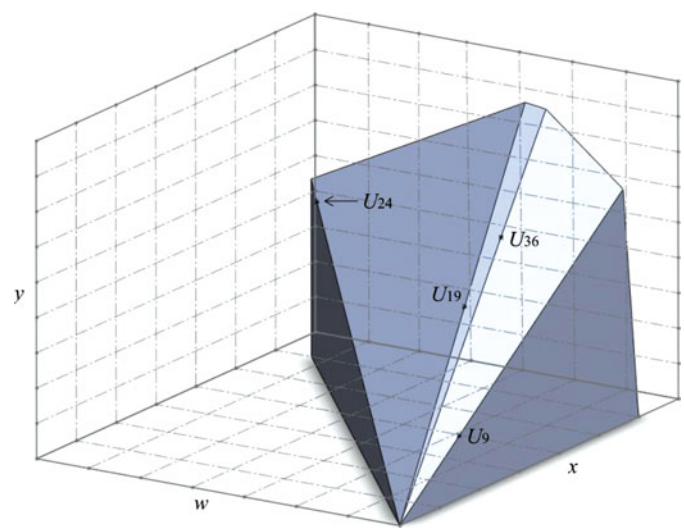

Fig. 3. A partial three-dimensional visualization of $\widehat{T_1 \cap T_2}$.

Table 1. Data set and analysis results

Univ.	x	w	y	θ_k^1	θ_k^2	(P1) & (P1R) θ_k^{radial}	(P1) & (P1R) $\gamma^{i*}-\gamma^{o*}$	(P2) & (P2R) θ_k^{fix}	(P2) & (P2R) $\gamma^{i*}-\gamma^{o*}$
1	64	254	26	0.59	0.68	0.68	0.000 654	0.61	0.000 784
2	301	1485	54	0.31	0.29	0.29	0.000 134	0.31	−0.000 063
3	23	45	3	0.19	0.24	0.24	0.001 986	0.19	−0.000 826
4	485	940	48	0.15	0.18	0.18	0.000 094	0.15	−0.000 039
5	90	106	22	0.35	0.48	0.48	0.009 434	0.43	0.000 557
6	767	2967	166	0.33	0.36	0.36	0.000 055	0.33	−0.000 025
8	126	776	32	0.42	0.39	0.39	0.000 304	0.42	−0.000 151
9	32	39	17	0.77	1.00	1.00	0.001 477	1.00	0.001 568
10	87	353	27	0.45	0.52	0.52	0.000 479	0.45	0.000 071
11	161	293	20	0.18	0.23	0.23	0.000 286	0.18	−0.000 118
12	91	781	37	0.65	0.59	0.59	0.000 065	0.65	−0.000 209
13	109	215	19	0.25	0.32	0.32	0.000 419	0.25	0.000 057
14	77	269	24	0.45	0.53	0.53	0.000 554	0.45	0.000 081
15	121	392	31	0.37	0.44	0.44	0.000 357	0.37	0.000 051
16	128	546	31	0.37	0.40	0.40	0.000 323	0.37	−0.000 148
17	116	925	24	0.40	0.30	0.30	0.000 051	0.40	−0.000 164
18	571	764	27	0.08	0.09	0.09	0.000 082	0.08	−0.000 033
19	83	615	57	0.99	1.00	1.00	0.000 441	1.00	0.000 075
20	267	3182	153	0.91	0.83	0.88	−0.000 092	0.91	−0.000 071
21	226	791	53	0.35	0.40	0.40	0.000 189	0.35	−0.000 084
22	81	741	29	0.60	0.52	0.52	−0.000 284	0.60	−0.000 234
23	450	347	32	0.10	0.21	0.21	0.002 882	0.10	0.000 014
24	112	2945	47	1.00	0.61	1.00	−0.000 339	1.00	−0.000 170
25	74	453	9	0.26	0.19	0.19	0.000 519	0.26	−0.000 257
26	841	2331	65	0.14	0.14	0.14	0.000 052	0.14	−0.000 023
27	81	695	37	0.71	0.66	0.66	0.000 073	0.71	−0.000 234
28	50	98	23	0.66	0.84	0.84	0.000 914	0.82	0.001 003
29	170	879	38	0.36	0.35	0.35	0.000 234	0.36	−0.000 112
30	628	4838	217	0.56	0.50	0.50	0.000 009	0.56	−0.000 030
31	77	490	26	0.52	0.51	0.51	0.000 494	0.52	−0.000 247
32	61	291	25	0.59	0.66	0.66	0.000 664	0.59	0.000 102
33	39	327	18	0.71	0.67	0.67	0.000 151	0.71	−0.000 487
34	131	956	50	0.59	0.56	0.56	0.000 280	0.59	−0.000 145
35	119	512	48	0.58	0.66	0.66	0.000 347	0.59	0.000 422
36	62	563	43	1.00	1.00	1.00	0.000 095	1.00	0.000 100
37	235	714	36	0.24	0.27	0.27	0.000 185	0.24	−0.000 081
38	94	297	23	0.35	0.42	0.42	0.000 461	0.35	0.000 066
39	46	277	19	0.61	0.63	0.63	0.000 838	0.61	−0.000 413
40	28	154	7	0.40	0.39	0.39	0.001 404	0.40	−0.000 678
41	40	531	23	0.94	0.83	0.92	−0.000 635	0.94	−0.000 475
42	68	305	23	0.49	0.55	0.55	0.000 602	0.49	−0.000 279
43	82	85	9	0.16	0.24	0.24	0.011 765	0.17	0.000 612
44	26	130	11	0.61	0.68	0.68	0.001 543	0.61	0.000 238
45	123	1043	39	0.54	0.46	0.46	0.000 048	0.54	−0.000 154
46	149	1523	51	0.60	0.49	0.51	−0.000 158	0.60	−0.000 127
47	89	743	30	0.56	0.49	0.49	0.000 066	0.56	−0.000 213
48	82	513	47	0.83	0.87	0.87	0.000 466	0.83	0.000 076
50	95	485	32	0.50	0.54	0.54	0.000 420	0.50	−0.000 200

4.2. $\psi(23)$

Figure 4, the x-w cutting plane of $\widehat{T_1 \cap T_2}$ at $y = 23$, can represent the feasible collection of (x, w) associated with $y = 23$—i.e., $\psi(23)$—because $\psi(y)$ is nested as shown in Proposition 1. The x-axis is the value of x and the y-axis is the value of w. The east region, bounded by $aABCDd$, represents part of the feasible region of (x, w) when $y = 23$. In particular, $(x_A, w_A) = (43.29, 52.8)$, $(x_B, w_B) = (33.49, 248.1)$, $(x_C, w_C) = (33.16, 301.2)$, and $(x_D, w_D) =$

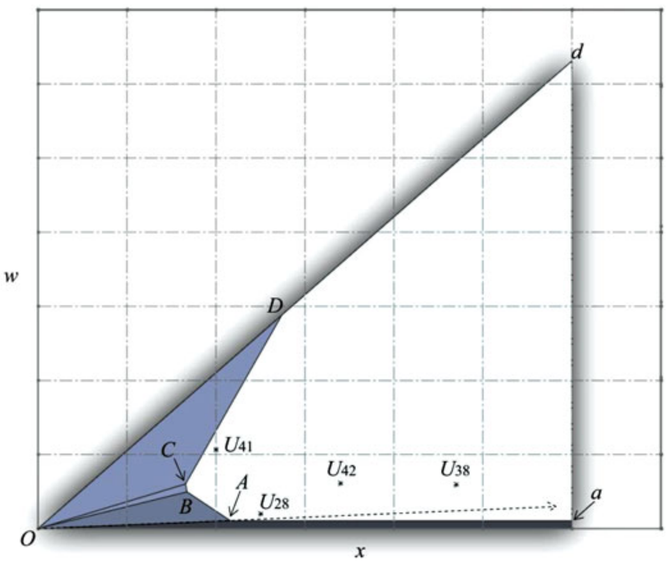

Fig. 4. A partial x–w cutting plane at $y = 23$.

(54.81, 1441.2). The lower boundary Aa is parallel to the x-axis, not the extension of OA (the dashed line in Fig. 4); the lower boundary indicates the strong disposability of regular input x. Segments AB and BC are downward sloping; a reduction of one dimension requires the other dimension to increase for any points on AB and BC. Segments CD and Dd are upward sloping, meaning that for points on these boundaries, increasing w while keeping x the same will result in a smaller y, since the point is outside $\psi(23)$ and $\psi(y)$ is nested. To increase w for points on the upward sloping boundaries also requires an increase in x, but the opposite is not true.

Furthermore, in Fig. 4, the upward sloping boundary segments occur only when w is larger than 301.2 at C. We also note that the normal vector values for AB and BC have the same sign for x and w. In contrast, segments CD and Dd have opposite signs on the normal vector values for x and w.

4.3. *Projection*

In Fig. 4, U_{28}, U_{38}, U_{41}, and U_{42} are x-w values for Universities 28, 38, 41, and 42, for which $y = 23$; Table 1 shows the detailed values. We can compute the efficiency measure based on (P2), where w is non-discretionary and is fixed at its current level, by reducing only x horizontally. Therefore, (P2) suggests U_{28} and U_{38} project onto segments AB and BC, respectively, and both U_{41} and U_{42} project onto CD, which is upward sloping.

On the other hand, we apply (P1) if adjustment on w is permitted. (P1) considers and measures the proportionate reduction on both x and w. As a result, all records evaluated try to move toward the origin, and we can observe different projected points on the boundaries by comparing with (P2). For example, U_{42} projects onto AB, which is downward sloping, instead of CD.

Interestingly, regardless of the model, a projection on upward sloping boundaries has a negative weight $\gamma^{i*} - \gamma^{o*}$, such as U_{41} and U_{42} in (P2) and U_{41} in (P1) (Table 1). Projecting onto downward-sloping boundaries coincides with positive $\gamma^{i*} - \gamma^{o*}$, such as U_{28}, U_{38}, and U_{42} in (P1) and U_{28} and U_{38} in (P2). U_{42} has opposite signs in the two models; one projects onto downward-sloping AB, leading to $\gamma^{i*} - \gamma^{o*} > 0$, and the other projects onto upward-sloping CD, leading to $\gamma^{i*} - \gamma^{o*} < 0$. In summary, the sign of $\gamma^{i*} - \gamma^{o*}$ only depends on the projected point, not the data point itself. A model determines the improvement (projection) path. Different improvement paths, due to different objectives and assumptions, lead to different projected points on the boundary. Two different points and two different models may project onto the same point on the boundary, leading to the same sign of $\gamma^{i*} - \gamma^{o*}$. This explains why, in Table 1, (P2R) gives more universities with negative $\gamma^{i*} - \gamma^{o*}$ than does (P1R). It is simply because (P1R) proportionately reduces x and w, for which the projection moves toward the southwest and, thus, the projecting points will more likely be on the downward-sloping boundaries.

$\gamma^{i*} - \gamma^{o*}$ is a boundary property, not the property of the point before a projection, and it associates the ideal benchmark without inefficiency. We thus suggest interpreting the sign of $\gamma^{i*} - \gamma^{o*}$ as preferring dual-role factor w for its input (output) role, without further interpretation for any improvement direction. As addressed in Section 3, $\gamma^{i*} - \gamma^{o*} > 0$ indicates that dual-role factor w benefits k from being an input; however, this does not lead to the conclusion of reducing w for improvement. Since the projected point is on the downward-sloping boundary, any reduction on w results in infeasible or less y. In contrast, when $\gamma^{i*} - \gamma^{o*} < 0$, the projecting boundary is upward sloping, and any increase on w results in infeasible or less y. Now, factor w does not contribute as do the regular inputs.

The reallocation of w, such that the efficiency of (P2) is maximized, or other objectives can give multiple solutions. Taking U_{41} as an example, both increasing and reducing w will yield identity efficiency in (P2), although the corresponding weight $\gamma^{i*} - \gamma^{o*}$ indicates that U_{41} prefers w as an output.

5. Generalization and extensions

Now, we generalize the observations in Section 4 and investigate $\boldsymbol{\gamma}^{i*} - \boldsymbol{\gamma}^{o*}$ from the viewpoints of geometry and economics in addition to the weights in the performance aggregation scheme (see Section 3). We also discuss some possible extensions.

5.1. *Supporting hyperplanes*

Note that the first type of constraints in Models (5) and (6) are identical. This formulation takes the joint technology composed by the two technologies as one technology. In DEA, these constraints associate with a hyperplane, $-\mathbf{x}\mathbf{u}^* + \mathbf{y}\mathbf{v}^* - \mathbf{w}(\boldsymbol{\gamma}^{i*} - \boldsymbol{\gamma}^{o*}) = 0$, of the joint technology $\widehat{T_1 \cap T_2}$ (also termed the supporting hyperplane; see e.g., Banker *et al*. (1984)). All $(\mathbf{x}_r, \mathbf{w}_r, \mathbf{y}_r)\ r \in R$ are either on this hyperplane (if = holds) or beneath it (<). $(\mathbf{x}_k, \mathbf{w}_k, \mathbf{y}_k)$ satisfying the equation is on the supporting hyperplane (boundary, frontier), and k is efficient. "<" indicates $(\mathbf{x}_k, \mathbf{w}_k, \mathbf{y}_k)$ is beneath the hyperplane and within $\widehat{T_1 \cap T_2}$, and there must exist $(\mathbf{x}_r, \mathbf{w}_r, \mathbf{y}_r)\ r \in R\backslash\{k\}$ on the plane. In this case, this hyperplane is the one on which k should project.

$(-\mathbf{u}^*, \mathbf{v}^*, -(\boldsymbol{\gamma}^{i*} - \boldsymbol{\gamma}^{o*}))$ is the normal vector for the hyperplane $-\mathbf{x}\mathbf{u}^* + \mathbf{y}\mathbf{v}^* - \mathbf{w}(\boldsymbol{\gamma}^{i*} - \boldsymbol{\gamma}^{o*}) = 0$, and $\mathbf{u}^* \geq 0$ and $\mathbf{v}^* \geq 0$ in standard DEA. Comparing the normal vector values, $-(\gamma_d^{i*} - \gamma_d^{o*}) < 0\ d \in D$ indicates that the hyperplane's orientation on d is like the orientation of the inputs, since it has the same sign with $-\mathbf{u}^*$. In contrast, $-(\gamma_d^{i*} - \gamma_d^{o*}) > 0\ d \in D$ suggests that the hyperplane's orientation on d is like an output. This observation gives geometric implication on the sign of $\gamma_d^{i*} - \gamma_d^{o*}$, and the input/output behavior judging result is consistent with the results discussed in Sections 3.2 and 3.3.

5.2. *Economic implications*

$\mathbf{u}$ and $\mathbf{v}$ are dual variables corresponding to the input and output constraints, and $\mathbf{u}^*$ and $\mathbf{v}^*$ specify the shadow prices for the inputs and outputs. $\boldsymbol{\gamma}^o$ and $\boldsymbol{\gamma}^i$ are dual variables associated with constraints of dual-role factors in their output and input roles, respectively, and $\boldsymbol{\gamma}^{o*}$ and $\boldsymbol{\gamma}^{i*}$ are the shadow prices for their output and input roles. $\boldsymbol{\gamma}^{o*} - \boldsymbol{\gamma}^{i*}$ (or $\boldsymbol{\gamma}^{i*} - \boldsymbol{\gamma}^{o*}$) is thus the (additive) composited shadow prices for dual-role factors. $\gamma_d^{o*} - \gamma_d^{i*}$ can be deemed as the "net value" per unit of d. $\gamma_d^{o*} - \gamma_d^{i*} > 0$ suggests an additional unit of d gives a net benefit. $\gamma_d^{o*} - \gamma_d^{i*} < 0$ indicates the input tendency of d since it shows a net cost. Regardless of the setting of evaluation models—e.g., (P1) and (P2)—the shadow price gives a simple and consistent interpretation.

Without loss of generality, consider the boundary (hyperplanes, isoquant) of $\psi(\mathbf{y})$ as defined in Section 3.1. Every point on the boundary has a corresponding normal vector $(-\mathbf{u}^*, -(\boldsymbol{\gamma}^{i*} - \boldsymbol{\gamma}^{o*}))$. A positive vector entry value, say entry i, indicates that the boundary at this point is backward bending in the dimension i and a negative marginal productivity of the corresponding factor i. Namely, increasing factor i at this point causes a decline in level of $\mathbf{y}$.

If $-(\gamma_d^{i*} - \gamma_d^{o*}) > 0\ d \in D$, d has negative marginal productivity on $\mathbf{y}$, and an increase in d will not be in $\psi(\mathbf{y})$ but in a $\psi(\mathbf{y}')$ such that $\mathbf{y}' \leq \mathbf{y}$, since $\psi(\mathbf{y})$ is nested in $\mathbf{y}$. It also suggests that the composted shadow price $\gamma_d^{i*} - \gamma_d^{o*}$ is negative. Bear in mind that the discussion here concerns the projected boundary corresponding to $(\mathbf{x}_k, \mathbf{w}_k, \mathbf{y}_k)$, not $(\mathbf{x}_k, \mathbf{w}_k, \mathbf{y}_k)$ itself; the projecting hyperplane depends on the projection paths, such as radial or non-discretionary.

The change along the boundaries of a dual-role factor d's orientation and composted shadow price $\gamma_d^{i*} - \gamma_d^{o*}$ reflect the tendency and importance of its input/output role. The input-output tendency is for factor d but not for any particular point under evaluation. That is, we consider cases without inefficiency and show the interactions among d and other factors. Starting from a small value for d, the major role of d is to support regular inputs to produce regular outputs at this stage. It should be noted that a reduction in d leads to less regular outputs. The composted shadow prices are relative prices among d and other inputs and relate to the marginal rates of substitution in consumption. The law of diminishing marginal rates of substitution, which results in $\psi(\mathbf{y})$ being convex, can be observed in this region (e.g., Fig. 4).

Unlike regular inputs, increasing d comes with the cost of consuming regular inputs. As mentioned, nurse trainees who need guidance from certified staff consume the hospital's resources of staff time and labor. This phenomenon is significant when the value of dual-role factor d is large; a further increasing d has a negative impact on regular outputs. Thus, we will observe a backward bending hyperplane and the negative composted shadow price $\gamma_d^{i*} - \gamma_d^{o*}$.

Increasing d with negative impact on $\mathbf{y}$ does not necessarily imply that an increasing d is "bad." In contrast, d contributes to overall output directly, because d itself is an important component of outputs, yet it does not contribute indirectly as an input to generate regular outputs. Therefore, the net benefit derives more from the output side than the input side. The corresponding hyperplane orientation and shadow price behave like an output. Increasing d at this stage yields a reduction of $\mathbf{y}$, which is the tradeoff between d and $\mathbf{y}$ or, in the real world, the possible transformation in production between d and regular output.

5.3. *Extensions*

It is easy and straightforward to extend our proposed models and to interpret the results. Assumptions of variant DEA models can be classified (Cherchye and Post, 2003) as (i) production technology, the underlying characteristics of the input/output transformation processes, such as assumptions 1 to 3 listed in Section 2; (ii) the data generating process, the sample data used to estimate T; and (iii) the objective function, the uses of T or its estimation; e.g., (P1) and (P2). Any DEA model is a combination of these three components. Discussions of the variant technologies can be found in Kuosmanen (2003) and Briec *et al*. (2004).

Based on the three classifications and given a sample data set, it is simple to model different combinations of production technologies and objective functions. For example, both (P1) and (P2) are based on the technologies assuming CRS and convexity, which may be strong in some cases.

When we adopt VRS together with assumptions 1 and 2, the joint technology is

$$\Bigg\{ (\mathbf{x}, \mathbf{w}, \mathbf{y}) : \sum_{r \in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r \in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \sum_{r \in R} \mathbf{w}_r \lambda_r \geq \mathbf{w}; \sum_{r \in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r \in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \sum_{r \in R} \mathbf{w}_r \lambda_r \leq \mathbf{w}; \sum_{r \in R} \lambda_r = 1; \lambda_r \geq 0, r \in R \Bigg\}. \quad (7)$$

Furthermore, if we relax the convexity, the FDH technology is applied as

$$\Bigg\{ (\mathbf{x}, \mathbf{w}, \mathbf{y}) : \sum_{r \in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r \in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \sum_{r \in R} \mathbf{w}_r \lambda_r \geq \mathbf{w}; \sum_{r \in R} \mathbf{x}_r \lambda_r \leq \mathbf{x}; \sum_{r \in R} \mathbf{y}_r \lambda_r \geq \mathbf{y}; \sum_{r \in R} \mathbf{w}_r \lambda_r \leq \mathbf{w}; \sum_{r \in R} \lambda_r = 1; \lambda_r = \{0, 1\}, r \in R \Bigg\}. \quad (8)$$

Given any technology approximations such as Models (7) and (8), we can apply different objectives with different interpretations. For example, if the dual-role factors are fixed, we can apply the objective similar to (P2) to Model (7) or (8). Note that deriving the performance evaluation model is more flexible and easier to interpret than using the ratio form proposed by (P1R) and (P2R). For example, there is no ratio form if FDH is applied, and the ratio will not be similar to standard VRS or CRS DEA models if we use other objective functions, such as additive models, cost minimization, or profit maximization. Moreover, it is simple to measure the distance function (Shephard, 1970) for any given $(\mathbf{x}, \mathbf{w}, \mathbf{y})$ using the joint technology proposed. Thus, we can measure the Malmquist productivity index and its decomposition (Färe *et al.*, 1994b) for cases with dual-role factors.

6. Conclusions

Typical DEA studies consider processes transforming inputs to outputs. In some cases, however, some factors can be both inputs and outputs simultaneously. Such factors are termed dual-role factors and their ambiguous role definitions make performance evaluation challenging. Rather than proposing an *ad hoc* model directly, this article proposed an axiomatic framework using joint technology, which is developed based on intuitive thinking. Under the proposed joint technology, evaluation models can be mathematically derived and/or axiomatically validated. We noted that variant models based on different assumptions and needs can be easily but rigorously extended. We showed that the input/output behavior of dual factors can be explained from different perspectives such as multi-criteria performance aggregation, geometry, and economics. We developed a simple three-dimensional case that we also generalized to multi-dimensional cases for considering the multiple dual-role factors. We found that the input/output tendency of a dual-role factor is a property on the projected boundary, not the data point itself. Different projecting paths associated with different objectives produced different weights and different implications. We concluded that the weights relate to the ideal performance improvement target. In other words, the benchmark on the boundary is the status after the improvement and does not imply the future improvement.

The finding of negative weights for dual-role factors in $\psi(\mathbf{y})$ suggests that increasing dual-role factors comes with a cost. We can simplify two types of constraints associated with dual-role factors as equality constraints when computing efficiency scores, but the corresponding dual variables should be interpreted with care. The literature imposes weak disposability to model the cases where increasing a factor comes with a cost (e.g., Färe et al. (1994a)). Noting that the real connection to weak disposability is beyond the scope of this article, we propose it as a topic for future research. In addition, more research is needed on RTS and the rate of technical substitutability for the production process with dual-role factors.

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Biography

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