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WKB analysis of the Schrödinger–KdV system

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Dedicated to Professor I-Liang Chern on his 60th birthday

Abstract

We consider the behavior of solutions to the water wave interaction equations in the limit $\varepsilon \to 0^+$. To justify the semiclassical approximation, we reduce the water wave interaction equation into some hyperbolic-dispersive system by using a modified Madelung transform. The reduced system causes loss of derivatives which prevents us to apply the classical energy method to prove the existence of solution. To overcome this difficulty we introduce a modified energy method and construct the solution to the reduced system.

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1. Introduction

The purpose of this paper is to study the zero dispersion limit or WKB approximation of solutions to the water wave interaction equations

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$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\partial_x^2 u^\varepsilon = \alpha u^\varepsilon v^\varepsilon + \beta\left|u^\varepsilon\right|^2 u^\varepsilon, & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ \partial_t v^\varepsilon + \partial_x^3 v^\varepsilon + \partial_x (v^\varepsilon)^2 = \gamma\partial_x\left(\left|u^\varepsilon\right|^2\right), & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ u^\varepsilon(0,x) = A_0^\varepsilon(x)\exp\left(\dfrac{i}{\varepsilon}S_0^\varepsilon(x)\right), \qquad v^\varepsilon(0,x) = v_0^\varepsilon(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where $u^\varepsilon : \mathbb{R}^+\times\mathbb{R}\to\mathbb{C}$ and $v^\varepsilon : \mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ are unknown functions, $A_0^\varepsilon : \mathbb{R}\to\mathbb{C}$ and $S_0, v_0^\varepsilon : \mathbb{R}\to\mathbb{R}$ are given functions, and α, β, γ are real constants. Throughout this paper we always assume that $\beta > 0$, which corresponds to the defocusing nonlinear Schrödinger equation when the coupling coefficient α vanishes, i.e., $\alpha = 0$. The parameter ε is analogous to Planck's constant in the quantum mechanics.

It is well known that a nonlinear interaction between long and short waves can occur strongly if the phase velocity of the long wave coincides with the group velocity of the short wave. Under the assumption of long wave short wave resonance, Benny [3] proposed several systems of dispersive equations. One of the systems is given by (1) which describes an interaction phenomenon between the long and short waves arising in various physical situations such as an electron-plasma, ion field interaction and the water wave theory. In (1), the short wave is described by the Schrödinger type equation and the long wave is described by KdV type equation. The reader is referred to Kawahara–Sugimoto–Kakutani [13] for the physical background of (1) in the theory of capillary-gravity waves.

Concerning the mathematical issues for (1), the time local well-posedness for (1) has been studied by many authors where the time interval of solution depends on the parameter ε, see [2,8,17]. Recently, Wang–Cui [18] proved the local well-posedness for (1) in $L^2\times H^{-1}$. Their proofs heavily depend on the dispersive properties of the Schrödinger equation. Therefore the time interval of solution to (1) depends on ε. In this paper we consider the semiclassical limit as $\varepsilon\to 0$ to solution to (1). To this end, we have to prove the existence of solution to (1) in some time interval independent of $\varepsilon\in(0,1]$. Therefore our first task is to derive this existence result.

There are two approaches to justify the semiclassical approximation. Concerning the more detail for the semiclassical or WKB approximation, the reader is referred to the books [5,19] and references therein. The first approach is to use Madelung's transform defined by

$$u^\varepsilon(t,x) = \sqrt{\rho^\varepsilon(t,x)}\exp\left(\frac{i}{\varepsilon}S^\varepsilon(t,x)\right),$$

where $\rho^\varepsilon = |u^\varepsilon|^2$ and S^ε are real-valued functions. According to this design, the first equation in (1) is rewritten as

$$\begin{aligned} &\left(-2\rho^\varepsilon\partial_t S^\varepsilon - \rho^\varepsilon\left(\partial_x S^\varepsilon\right)^2 - 2\alpha\rho^\varepsilon v^\varepsilon - 2\beta\left(\rho^\varepsilon\right)^2\right) \\ &\quad + \varepsilon\left(i\partial_t\rho^\varepsilon + i\partial_x\rho^\varepsilon\partial_x S^\varepsilon + i\rho^\varepsilon\partial_x^2 S^\varepsilon\right) + \varepsilon^2\left(-\frac{1}{4}\frac{(\partial_x\rho^\varepsilon)^2}{\rho^\varepsilon} + \frac{1}{2}\partial_x^2\rho^\varepsilon\right) = 0. \end{aligned}$$

We split the above equation into the following two equations:

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x \left(\rho^\varepsilon \partial_x S^\varepsilon\right) = 0, \\ \partial_t S^\varepsilon + \dfrac{1}{2}\left(\partial_x S^\varepsilon\right)^2 + \alpha v^\varepsilon + \beta \rho^\varepsilon = \dfrac{\varepsilon^2}{2} \dfrac{\partial_x^2 \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}. \end{cases}$$

Let $w^\varepsilon = \partial_x S^\varepsilon$ and apply ∂_x to the second equation, then we have

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x \left(\rho^\varepsilon w^\varepsilon\right) = 0, \\ \partial_t w^\varepsilon + w^\varepsilon \partial_x w^\varepsilon + \partial_x \left(\alpha v^\varepsilon + \beta \rho^\varepsilon\right) = \dfrac{\varepsilon^2}{2} \partial_x \left(\dfrac{\partial_x^2 \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right), \end{cases}$$

which have the form of a perturbation of the compressible Euler equation with v^ε satisfying

$$\partial_t v^\varepsilon + \partial_x^3 v^\varepsilon + \partial_x \left(v^\varepsilon\right)^2 = \gamma \partial_x \rho^\varepsilon.$$

Although this representation suggests the limiting equations as $\varepsilon \to 0$, but the above equations do not have a meaning in the point where $\rho^\varepsilon = 0$ and the phase function S^ε may be undefined. As suggested by Grenier [11], the modified Madelung's transform can be utilized in the study of the semiclassical or zero dispersion limit. In fact, we have

$$u^\varepsilon(t,x) = A^\varepsilon(t,x) \exp\left(\frac{i}{\varepsilon} S^\varepsilon(t,x)\right),$$

where A^ε and S^ε are complex and real valued functions. It will reduce the first equation in (1) into

$$\left(-A^\varepsilon \partial_t S^\varepsilon - \frac{1}{2} A^\varepsilon \left(\partial_x S^\varepsilon\right)^2 - \alpha A^\varepsilon v^\varepsilon - \beta \left|A^\varepsilon\right|^2 A^\varepsilon\right) \\ + \varepsilon \left(i \partial_t A^\varepsilon + i \partial_x A^\varepsilon \partial_x S^\varepsilon + \frac{i}{2} A^\varepsilon \partial_x^2 S^\varepsilon\right) + \frac{\varepsilon^2}{2} \partial_x^2 A^\varepsilon = 0.$$

We decompose the above equation as follows:

$$\begin{cases} i \partial_t A^\varepsilon + \dfrac{\varepsilon}{2} \partial_x^2 A^\varepsilon + i \partial_x A^\varepsilon \partial_x S^\varepsilon + \dfrac{i}{2} A^\varepsilon \partial_x^2 S^\varepsilon = 0, \\ \partial_t S^\varepsilon + \dfrac{1}{2}\left(\partial_x S^\varepsilon\right)^2 + \beta \left|A^\varepsilon\right|^2 = -\alpha v^\varepsilon. \end{cases}$$

Furthermore, putting $A^\varepsilon = a^\varepsilon + i b^\varepsilon$, $w^\varepsilon = \partial_x S^\varepsilon$, we have

$$\begin{cases} \partial_t a^\varepsilon + \dfrac{\varepsilon}{2} \partial_x^2 b^\varepsilon + w^\varepsilon \partial_x a^\varepsilon + \dfrac{1}{2} a^\varepsilon \partial_x w^\varepsilon = 0, \\ \partial_t b^\varepsilon - \dfrac{\varepsilon}{2} \partial_x^2 a^\varepsilon + w^\varepsilon \partial_x b^\varepsilon + \dfrac{1}{2} b^\varepsilon \partial_x w^\varepsilon = 0, \\ \partial_t w^\varepsilon + w^\varepsilon \partial_x w^\varepsilon + 2\beta a^\varepsilon \partial_x a^\varepsilon + \alpha \partial_x v^\varepsilon + 2\beta b^\varepsilon \partial_x b^\varepsilon = 0. \end{cases}$$

This approach is used by Grenier [11] to justify the semiclassical approximation in Sobolev space for the Schrödinger equation with power type nonlinearity, see also [10] and [12] for analytic initial data. Some extensions of [11] were given by several authors, see for instance Alazard–Carles [1] and Chiron–Rousset [7] for nonlinear Schrödinger equation and Carles–Masaki [6] for Hartree equation.

For the convenience, let us put

$$\mathcal{U}^\varepsilon = \begin{bmatrix} a^\varepsilon \\ b^\varepsilon \\ w^\varepsilon \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} 0 & \partial_x^2 & 0 \\ -\partial_x^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\mathcal{N}(\mathcal{U}^\varepsilon) = \begin{bmatrix} w^\varepsilon & 0 & a^\varepsilon/2 \\ 0 & w^\varepsilon & b^\varepsilon/2 \\ 2\beta a^\varepsilon & 2\beta b^\varepsilon & w^\varepsilon \end{bmatrix}, \qquad \mathcal{V}^\varepsilon = \begin{bmatrix} 0 \\ 0 \\ \partial_x v^\varepsilon \end{bmatrix}.$$

Then $(\mathcal{U}^\varepsilon, v^\varepsilon)$ satisfies

$$\begin{cases} \partial_t \mathcal{U}^\varepsilon + \dfrac{\varepsilon}{2}\mathcal{L}\mathcal{U}^\varepsilon + \mathcal{N}(\mathcal{U}^\varepsilon)\partial_x \mathcal{U}^\varepsilon = -\alpha \mathcal{V}^\varepsilon, \\ \partial_t v^\varepsilon + \partial_x^3 v^\varepsilon + \partial_x (v^\varepsilon)^2 = \gamma \partial_x (|a^\varepsilon|^2 + |b^\varepsilon|^2), \end{cases} \tag{2}$$

where $\varepsilon \in [0, 1]$. We first prove the existence of local smooth solution to (2) with the initial condition

$$\mathcal{U}^\varepsilon(0, x) = \mathcal{U}_0^\varepsilon(x), \qquad v_\varepsilon(0, x) = v_0^\varepsilon(x). \tag{3}$$

Theorem 1.1. *Let $m \geqslant 3$ be an integer. Assume that there is a constant $C_1 > 0$ such that $\|\mathcal{U}_0^\varepsilon\|_{H^m} + \|v_0^\varepsilon\|_{H^{m-1}} \leqslant C_1$ for all $\varepsilon \in [0, 1]$. Then there exists a time $T > 0$ independent of the parameter ε and a unique solution $(\mathcal{U}^\varepsilon, v^\varepsilon) \in C([0, T), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ to the initial value problem (2)–(3). Furthermore, there exits a constant $C_2 > 0$ such that*

$$\sup_{t\in[0,T)} \left(\|\mathcal{U}^\varepsilon(t)\|_{H^m} + \|v^\varepsilon(t)\|_{H^{m-1}}\right) \leqslant C_2$$

for any $\varepsilon \in [0, 1]$.

Remark. The order pair $(\mathcal{U}^\varepsilon, v^\varepsilon)$ is served as a column vector and the same for the rest of this paper. We point out that we may well be able to extend Theorem 1.1 to the case when m is not an integer combining our proof with the estimates for the fractional derivatives. In this paper we do not touch on this issue.

Remark. It is natural to raise the following question: Can we extend the local smooth solution of (1) to the global one? Although some conservation laws for (1) are derived in [15], so far, we do not know whether the local smooth solution of (1) can be extended to infinite time interval or not.

In (2), taking $\varepsilon \to 0$, we obtain the following system

$$\begin{cases} \partial_t \mathcal{U}_0 + \mathcal{N}(\mathcal{U}_0)\partial_x \mathcal{U}_0 = -\alpha \mathcal{V}_0 \\ \partial_t v_0 + \partial_x^3 v_0 + \partial_x (v_0^2) = \gamma \partial_x (|a_0|^2 + |b_0|^2), \end{cases} \tag{4}$$

where

$$\mathcal{U}_0 = \begin{bmatrix} a_0 \\ b_0 \\ w_0 \end{bmatrix}, \qquad \mathcal{N}(\mathcal{U}_0) = \begin{bmatrix} w_0 & 0 & a_0/2 \\ 0 & w_0 & b_0/2 \\ 2\beta a_0 & 2\beta b_0 & w_0 \end{bmatrix}, \qquad \mathcal{V}_0 = \begin{bmatrix} 0 \\ 0 \\ \partial_x v_0 \end{bmatrix}.$$

The main purpose of this paper is to show that the solution to (2) converges to the solution to (4) as $\varepsilon \to 0$. The following theorem justifies the WKB approximation for (1).

Theorem 1.2. *Let $m \geqslant 3$ be an integer. Assume that there exists a constant $C > 0$ such that $\|\mathcal{U}_0^\varepsilon\|_{H^m} + \|v_0^\varepsilon\|_{H^{m-1}} \leqslant C$ for any $\varepsilon \in [0,1]$. Let $(\mathcal{U}^\varepsilon, v^\varepsilon) \in C([0,T), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ be the solution to (2) with the initial data $(\mathcal{U}_0^\varepsilon, v_0^\varepsilon) \in H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R})$. Let $(\mathcal{U}_0, v_0) \in C([0,T), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ be the solution to (4) with the initial data $(\mathcal{U}_{0,0}, v_{0,0}) \in H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R})$. Then we have*

$$\sup_{t\in[0,T)} \left(\left\| \mathcal{U}^\varepsilon(t) - \mathcal{U}_0(t) \right\|_{H^{m-2}} + \left\| v^\varepsilon(t) - v_0(t) \right\|_{H^{m-3}} \right) \leqslant C\left(\left\| \mathcal{U}_0^\varepsilon - \mathcal{U}_{0,0} \right\|_{H^{m-2}} + \left\| v_0^\varepsilon - v_{0,0} \right\|_{H^{m-3}} + \varepsilon \right),$$

where the constant $C > 0$ depends on T but independent of $\varepsilon \in [0,1]$.

As a corollary of Theorem 1.2, we obtain the semiclassical expansion of (1).

Corollary 1.1. *Let $m \geqslant 3$ be an integer. Assume that $(\mathcal{U}_{0,0}, v_{0,0}), (\mathcal{U}_0^\varepsilon, v_0^\varepsilon) \in H^m \times H^{m-1}$ and $\|\mathcal{U}_0^\varepsilon\|_{H^m} + \|v_0^\varepsilon\|_{H^{m-1}} \leqslant C$ for some positive constant C independent of ε. Assume further that*

$$\lim_{\varepsilon\to 0} \left(\left\| \mathcal{U}_0^\varepsilon - \mathcal{U}_{0,0} \right\|_{H^m} + \left\| v_0^\varepsilon - v_{0,0} \right\|_{H^{m-1}} \right) = 0.$$

Then the unique solution $(\mathcal{U}^\varepsilon, v^\varepsilon) \in C([0,T), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ to (2)–(3) satisfies

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T)} \left(\left\| \mathcal{U}^\varepsilon(t) - \mathcal{U}_0(t) \right\|_{H^{m-2}} + \left\| v^\varepsilon(t) - v_0(t) \right\|_{H^{m-3}} \right) = 0,$$

where $(\mathcal{U}_0, v_0) \in C([0,T), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ is the unique solution to (4) with the initial data $(\mathcal{U}_{0,0}, v_{0,0})$.

Corollary 1.1 gives the zeroth order approximation of $(\mathcal{U}^\varepsilon, v^\varepsilon)$ as $\varepsilon \to 0$. In Section 5, we shall consider the first order approximation of $(\mathcal{U}^\varepsilon, v^\varepsilon)$.

We give an outline of proofs for Theorems 1.1 and 1.2. They follow from the combination of parabolic regularization and a priori estimates for the approximate solutions. In proving a priori estimates for approximate solutions, we meet so called "loss of derivatives". More precisely, if we apply the classical energy method (see [16] for instance) to the first equation of (2), then we have

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\partial_x^m a^\varepsilon(t)\right\|_{L^2}^2+\left\|\partial_x^m b^\varepsilon(t)\right\|_{L^2}^2+\frac{1}{4\beta}\left\|\partial_x^m w^\varepsilon(t)\right\|_{L^2}^2\right)$$
$$=-\alpha\int_{\mathbb{R}}\partial_x^m w^\varepsilon\partial_x^{m+1}v^\varepsilon\,dx+\text{lower order terms}$$

Therefore, to obtain the H^m estimate for $\mathcal{U}^\varepsilon$, we have to evaluate the H^{m+1} norm of v^ε. For this purpose, we also apply the classical energy method to the second equation of (2). Indeed, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_x^{m+1}v^\varepsilon(t)\right\|_{L^2}^2=\gamma\int_{\mathbb{R}}\partial_x^{m+2}\left(\left|a^\varepsilon\right|^2\right)\partial_x^{m+1}v^\varepsilon dx+\gamma\int_{\mathbb{R}}\partial_x^{m+2}\left(\left|b^\varepsilon\right|^2\right)\partial_x^{m+1}v^\varepsilon\,dx$$
$$+\text{lower order terms}$$

Thus to close the H^{m+1}-estimate for v^ε, we need the H^{m+2}-bound of $(a^\varepsilon, b^\varepsilon)$. However those terms can not be controlled in terms of the H^m-norm of $(a^\varepsilon, b^\varepsilon)$. To overcome this difficulty we take the idea from Kwon [14] and introduce a modified energy given by

$$E_m\left(\mathcal{U}^\varepsilon, v^\varepsilon\right)(t)=\left\|a^\varepsilon(t)\right\|_{H^m}^2+\left\|b^\varepsilon(t)\right\|_{H^m}^2+\frac{1}{4\beta}\left\|w^\varepsilon(t)\right\|_{H^m}^2+C_{\alpha,\beta}\left\|v^\varepsilon(t)\right\|_{H^{m-1}}^2$$
$$-2\alpha\int_{\mathbb{R}}\partial_x^m w^\varepsilon\partial_x^{m-2}v^\varepsilon\,dx,$$

where the positive constant $C_{\alpha,\beta}$ is chosen sufficiently large so that E_m is equivalent to the $H^m\times H^{m-1}$-norm. Thanks to the modification of energy, we can obtain a priori estimates for solution to (2). Since it does not depend on the dispersive effect of $\mathcal{L}$ in (2), we can show that the time interval of existence of solution does not depend on $\varepsilon\in[0,1]$ which is the crucial part of the proof.

To obtain Theorem 1.2, we need to evaluate the difference $\mathcal{U}^\varepsilon-\mathcal{U}_0$ and $v^\varepsilon-v_0$. In this step, loss of derivatives also prevents us to apply the classical energy method. So we again employ the modified energy $E_m(\mathcal{U}^\varepsilon, v^\varepsilon)(t)$ and justify the WKB approximation.

We introduce several notation and function spaces used in this paper. We denote the Fourier and its inverse transforms by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively:

$$\mathcal{F}[f](\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-ix\cdot\xi}f(x)\,dx,\qquad \mathcal{F}^{-1}[f](x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{+ix\cdot\xi}f(\xi)\,d\xi.$$

Let $(X_j,\|\cdot\|_{X_j})$ $(j=1,\dots,N)$ be Banach spaces. For the vector valued function $(f_1,\dots,f_N)\in X_1\times\dots\times X_N$, we define its norm by $\|(f_1,\dots,f_N)\|_{X_1\times\dots\times X_N}=\|f_1\|_{X_1}+\dots+\|f_N\|_{X_N}$. If $X_1=\dots=X_N=X$, we write $\|(f_1,\dots,f_N)\|_{X_1\times\dots\times X_N}=\|(f_1,\dots,f_N)\|_X$. For the notational convenience, we introduce

$$\|\mathcal{U}\|_{H^m}^2 = \|a\|_{H^m}^2 + \|b\|_{H^m}^2 + \frac{1}{4\beta}\|w\|_{H^m}^2$$

for $\mathcal{U} = (a, b, w)$.

The plan of this paper is as follows. Section 2 is devoted to the parabolic regularization associated to (2). In Section 3, we introduce the modified energy and give an a-priori estimate for the solution to (2). Then we prove the existence and uniqueness of solution to (2). Section 4 is devoted to proving that the solution to (2) converges to the solution to (4) as $\epsilon \to 0$. Finally, in Section 5, we consider the first order semiclassical approximation of (1).

2. Parabolic regularization

To prove Theorem 1.1, we consider the regularized problem. To this end, we introduce the regularizing sequence used in Bona–Smith [4]. Let $\varphi \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \varphi(\xi) \leqslant 1$ for $\xi \in \mathbb{R}$, $\varphi(0) = 1$, $\varphi^{(k)}(0) = 0$ for $k = 1, 2, \dots$, and $\varphi(\xi)$ tends to 0 exponentially as $|\xi| \to \infty$. We define for $\delta \in (0, 1]$,

$$\phi^\delta(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \varphi(\delta\xi)\hat{\phi}(\xi)\, d\xi.$$

Then, $\{\phi^\delta\}_{\delta>0} \subset H^\infty(\mathbb{R})$ and $\|\phi - \phi^\delta\|_{H^m} \to 0$ as $\delta \to 0$. Furthermore, for any $l \geqslant 0$, we have the following inequalities

$$\begin{aligned} \|\phi^\delta\|_{H^{m+l}} &\leqslant C\delta^{-l}\|\phi\|_{H^m}, \\ \|\phi - \phi^\delta\|_{H^m} &\leqslant C\|\phi\|_{H^m}, \\ \|\phi - \phi^\delta\|_{H^{m-l}} &\leqslant C\delta^l\|\phi\|_{H^m}. \end{aligned}$$

Let us consider the regularized problem associated to (2):

$$\begin{cases} \partial_t \mathcal{U}^{\varepsilon,\delta} + \delta\tilde{\mathcal{L}}\mathcal{U}^{\varepsilon,\delta} + \frac{\varepsilon}{2}\mathcal{L}\mathcal{U}^{\varepsilon,\delta} + \mathcal{N}(\mathcal{U}^{\varepsilon,\delta})\partial_x\mathcal{U}^{\varepsilon,\delta} = -\alpha\mathcal{V}^{\varepsilon,\delta} \\ \partial_t v^{\varepsilon,\delta} + \delta\partial_x^4 v^{\varepsilon,\delta} + \partial_x^3 v^{\varepsilon,\delta} + \partial_x(v^{\varepsilon,\delta})^2 = \gamma\partial_x(|a^{\varepsilon,\delta}|^2 + |b^{\varepsilon,\delta}|^2), \\ \mathcal{U}^{\varepsilon,\delta}(0,x) = \mathcal{U}_0^{\varepsilon,\delta}(x), \qquad v^{\varepsilon,\delta}(0,x) = v_0^{\varepsilon,\delta}(x), \end{cases} \tag{5}$$

where $\varepsilon \in [0, 1]$, $\delta \in (0, 1]$, and

$$\mathcal{U}^{\varepsilon,\delta} = \begin{bmatrix} a^{\varepsilon,\delta} \\ b^{\varepsilon,\delta} \\ w^{\varepsilon,\delta} \end{bmatrix}, \qquad \tilde{\mathcal{L}} = \begin{bmatrix} \partial_x^4 & 0 & 0 \\ 0 & \partial_x^4 & 0 \\ 0 & 0 & \partial_x^4 \end{bmatrix}, \qquad \mathcal{V}^{\varepsilon,\delta} = \begin{bmatrix} 0 \\ 0 \\ \partial_x v^{\varepsilon,\delta} \end{bmatrix}.$$

Lemma 2.1. *Let $m \geqslant 2$ be an integer and let $\varepsilon \in [0, 1]$ and $\delta \in (0, 1]$. Assume that there is a constant $C > 0$ such that $\|\mathcal{U}_0^\varepsilon\|_{H^m} + \|v_0^\varepsilon\|_{H^{m-1}} \leqslant C$ for any $\varepsilon \in [0, 1]$. Then, there exists a time $T^\delta > 0$ independent of $\varepsilon > 0$ and a unique solution $(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ of (5) satisfying*

$$(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta}) \in C([0, T^\delta), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R})).$$

Proof. Since the system (5) is a semi-linear parabolic system of the standard type, the proof of Lemma 2.1 follows from the Banach fixed point theorem via the integral equation, see [9] for instance. Hence we omit the detail. □

3. Modified energy – existence and uniqueness

In this section, we give a priori estimate for the solutions to the regularized system (5). Let us evaluate the $H^m \times H^{m-1}$-norm of $(\mathcal{U}_\varepsilon^\delta, v_\varepsilon^\delta)$. As explained in the introduction, Eq. (5) causes the loss of derivatives. To overcome this difficulty, we introduce the modified energy given by

$$E_m(\mathcal{U}, v)(t) = \|a\|_{H^m}^2 + \|b\|_{H^m}^2 + \frac{1}{4\beta}\|w\|_{H^m}^2 + C_{\alpha,\beta}\|v\|_{H^{m-1}}^2 - 2\alpha \int_{\mathbb{R}} \partial_x^m w \partial_x^{m-2} v \, dx,$$

for $\mathcal{U} = (a, b, w) \in H^m$ and $v \in H^{m-1}$, where the positive constant $C_{\alpha,\beta}$ is chosen so that $E_m(\mathcal{U}, v)(t) > 0$ for all $0 < t < T$. This is possible since by the Gagliardo–Nirenberg inequality we have

$$-2\alpha \int_{\mathbb{R}} \partial_x^m w \partial_x^{m-2} v \, dx \geqslant -\frac{1}{8\beta}\|w\|_{H^m}^2 - D_{\alpha,\beta}\|v\|_{H^{m-1}}^2$$

for some positive constant $D_{\alpha,\beta}$ which depends only on α and β. So it suffices to choose $C_{\alpha,\beta} = D_{\alpha,\beta} + 1$.

We note that there exists a constant $\tilde{C}_{\alpha,\beta} > 0$ such that for any $(\mathcal{U}, v) \in H^m \times H^{m-1}$ the following inequality holds,

$$\tilde{C}_{\alpha,\beta}^{-1}\big\|(\mathcal{U}, v)\big\|_{H^m \times H^{m-1}}^2 \leqslant E_m(\mathcal{U}, v)(t) \leqslant \tilde{C}_{\alpha,\beta}\big\|(\mathcal{U}, v)\big\|_{H^m \times H^{m-1}}^2.$$

Lemma 3.1. *Let $m \geqslant 3$ be an integer and let $\varepsilon \in [0, 1]$ and $\delta \in (0, 1]$. Let $(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta}) \in C([0, T^\delta), H^m(\mathbb{R})) \times C([0, T^\delta), H^{m-1}(\mathbb{R}))$ be a solution to (5) obtained by Lemma 2.1. Then there exist positive constants C and T independent of $\varepsilon \in [0, 1]$ and $\delta \in (0, 1]$ such that*

$$\big\|\big(\mathcal{U}^{\varepsilon,\delta}(t), v^{\varepsilon,\delta}(t)\big)\big\|_{H^m \times H^{m-1}} \leqslant C(T)\big\|\big(\mathcal{U}_0^\varepsilon, v_0^\varepsilon\big)\big\|_{H^m \times H^{m-1}}$$

for any $t \in [0, T)$.

Proof. Applying the operator

$$\begin{bmatrix} \partial_x^2 & 0 & 0 \\ 0 & \partial_x^2 & 0 \\ 0 & 0 & (4\beta)^{-1}\partial_x^2 \end{bmatrix}$$

to the both sides of the first equation in (5) and taking the inner product of the resultant equations with $\partial_x^m \mathcal{U}^{\varepsilon,\delta}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\partial_x^m a^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\left\|\partial_x^m b^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\frac{1}{4\beta}\left\|\partial_x^m w^{\varepsilon,\delta}(t)\right\|_{L^2}^2\right)$$
$$+\delta\left(\left\|\partial_x^{m+2} a^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\left\|\partial_x^{m+2} b^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\frac{1}{4\beta}\left\|\partial_x^{m+2} w^{\varepsilon,\delta}(t)\right\|_{L^2}^2\right)$$
$$=-\alpha\int_{\mathbb{R}}\partial_x^m w^{\varepsilon,\delta}\partial_x^{m+1}v^{\varepsilon,\delta}\,dx+\mathcal{R}_1(t), \tag{6}$$

where $\mathcal{R}_1(t)$ satisfies

$$\begin{aligned}\left|\mathcal{R}_1(t)\right| &\leqslant C\left(\left\|a^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|b^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|w^{\varepsilon,\delta}(t)\right\|_{H^m}^2\right)\left\|w^{\varepsilon,\delta}(t)\right\|_{H^m}\\ &\leqslant CE_m^{3/2}\left(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\right)(t)\\ &\leqslant CE_m\left(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\right)(t)+CE_m^2\left(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\right)(t),\end{aligned} \tag{7}$$

for any $t\in[0,T^\delta)$ with constant C independent of ε and δ.

Applying $(m-1)$-derivatives with respect to x on both sides of the second equation in (5) and taking the inner product of the resultant equations with $\partial_x^{m-1}v^{\varepsilon,\delta}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\partial_x^{m-1}v^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\delta\left\|\partial_x^{m+1}v^{\varepsilon,\delta}(t)\right\|_{L^2}^2\\ &\quad\leqslant C\left(\left\|a^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|b^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|v^{\varepsilon,\delta}(t)\right\|_{H^{m-1}}^2\right)\left\|v^{\varepsilon,\delta}(t)\right\|_{H^{m-1}},\end{aligned} \tag{8}$$

where constant C is again independent of ε and δ.

On the other hand

$$\begin{aligned}&-\alpha\frac{d}{dt}\int_{\mathbb{R}}\partial_x^m w^{\varepsilon,\delta}\partial_x^{m-2}v^{\varepsilon,\delta}\,dx\\ &=\alpha\int_{\mathbb{R}}\partial_x^m\left(\delta\partial_x^4 w^{\varepsilon,\delta}+\alpha\partial_x v^{\varepsilon,\delta}+w^{\varepsilon,\delta}\partial_x w^{\varepsilon,\delta}+2\beta a^{\varepsilon,\delta}\partial_x a^{\varepsilon,\delta}+2\beta b^{\varepsilon,\delta}\partial_x b^{\varepsilon,\delta}\right)\partial_x^{m-2}v^{\varepsilon,\delta}\,dx\\ &\quad+\alpha\int_{\mathbb{R}}\partial_x^m w^{\varepsilon,\delta}\partial_x^{m-2}\left(\delta\partial_x^4 v^{\varepsilon,\delta}+\partial_x^3 v^{\varepsilon,\delta}+2v^{\varepsilon,\delta}\partial_x v^{\varepsilon,\delta}-\gamma\partial_x\left(\left|a^{\varepsilon,\delta}\right|^2+\left|b^{\varepsilon,\delta}\right|^2\right)\right)dx\\ &=\alpha\int_{\mathbb{R}}\partial_x^m w^{\varepsilon,\delta}\partial_x^{m+1}v^{\varepsilon,\delta}\,dx+2\alpha\delta\int_{\mathbb{R}}\partial_x^{m+2}w^{\varepsilon,\delta}\partial_x^m v^{\varepsilon,\delta}\,dx+\mathcal{R}_2(t),\end{aligned}$$

where $\mathcal{R}_2(t)$ satisfies

$$\begin{aligned}\left|\mathcal{R}_2(t)\right| &\leqslant C\left(\left\|a^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|b^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|w^{\varepsilon,\delta}(t)\right\|_{H^m}^2+\left\|v^{\varepsilon,\delta}(t)\right\|_{H^{m-1}}^2\right)\\ &\quad\times\left(\left\|w^{\varepsilon,\delta}(t)\right\|_{H^m}+\left\|v^{\varepsilon,\delta}(t)\right\|_{H^{m-1}}\right)\\ &\leqslant CE_m^{3/2}\left(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\right)(t)\leqslant CE_m\left(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\right)(t)+CE_m^2\left(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\right)(t)\end{aligned} \tag{9}$$

with constant C independent of ε and δ. Hence

$$\left|-\alpha\frac{d}{dt}\int_{\mathbb{R}}\partial_x^m w^{\varepsilon,\delta}\partial_x^{m-2}v^{\varepsilon,\delta}\,dx-\alpha\int_{\mathbb{R}}\partial_x^m w^{\varepsilon,\delta}\partial_x^{m+1}v^{\varepsilon,\delta}\,dx\right|$$
$$\leqslant\delta\left(\frac{1}{4\beta}\left\|\partial_x^{m+2}w^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\left\|\partial_x^{m+1}v^{\varepsilon,\delta}(t)\right\|_{L^2}^2\right)+C_{\alpha,m}\left\|v^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\mathcal{R}_2(t). \tag{10}$$

The L^2 estimates for $(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta})$ can be obtained by using the standard energy method. Therefore, from (6), (7), (8), (9) and (10), we obtain

$$\frac{d}{dt}E_m\big(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\big)(t)$$
$$+\delta\left(2\left\|\partial_x^{m+2}a^{\varepsilon,\delta}(t)\right\|_{L^2}^2+2\left\|\partial_x^{m+2}b^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\frac{1}{4\beta}\left\|\partial_x^{m+2}w^{\varepsilon,\delta}(t)\right\|_{L^2}^2+\left\|\partial_x^{m+1}v^{\varepsilon,\delta}(t)\right\|_{L^2}^2\right)$$
$$\leqslant C_{\alpha,m}E_m\big(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\big)(t)+E_m^2\big(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\big)(t).$$

We note that the constant $C_{\alpha,m}$ is independent of $\varepsilon\in[0,1]$ and $\delta\in(0,1]$. Therefore the Gronwall inequality yields

$$E_m\big(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\big)(t)\leqslant\frac{E_m(\mathcal{U}_0^{\varepsilon,\delta},v_0^{\varepsilon,\delta})e^{C_{\alpha,m}t}}{1-C_{\alpha,m}^{-1}E_m(\mathcal{U}_0^{\varepsilon,\delta},v_0^{\varepsilon,\delta})(e^{C_{\alpha,m}t}-1)}\leqslant 2E_m\big(\mathcal{U}_0^{\varepsilon,\delta},v_0^{\varepsilon,\delta}\big)e^{C_{\alpha,m}t}$$

for

$$0\leqslant t<\min\big\{T^\delta,C^{-1}\log\big(1+1/\big(2E_m\big(\mathcal{U}_0^{\varepsilon,\delta},v_0^{\varepsilon,\delta}\big)\big)\big)\big\}.$$

Let $T=C^{-1}\log(\frac{1}{2}E_m(\mathcal{U}_0^{\varepsilon,\delta},v_0^{\varepsilon,\delta})+1)$. Combining above inequality with

$$E_m\big(\mathcal{U}_0^{\varepsilon,\delta},v_0^{\varepsilon,\delta}\big)\leqslant C\big(\left\|\mathcal{U}_0^\varepsilon\right\|_{H^m}^2+\left\|v_0^\varepsilon\right\|_{H^{m-1}}^2\big),$$

we obtain

$$E_m\big(\mathcal{U}^{\varepsilon,\delta},v^{\varepsilon,\delta}\big)(t)\leqslant C\big(\left\|\mathcal{U}_0^\varepsilon\right\|_{H^m}^2+\left\|v_0^\varepsilon\right\|_{H^{m-1}}^2\big)e^{C_{\alpha,m}t},$$

for any $0\leqslant t<\min\{T^\delta,T\}$. If $T^\delta<T$, we can apply Lemma 2.1 to extend the solution in the same class to the interval $[0,T)$. Therefore we obtain the desired result. □

Using Lemma 3.1 we obtain the existence of the solution to (2):

Lemma 3.2. *Let $m\geqslant 3$ be an integer. Assume that there exists a constant $C>0$ such that $\|\mathcal{U}_0^\varepsilon\|_{H^m}+\|v_0^\varepsilon\|_{H^{m-1}}\leqslant C$ for any $\varepsilon\in[0,1]$. Then there exists a time $T>0$ independent of $\varepsilon\in[0,1]$ and a solutions $(\mathcal{U}^\varepsilon,v^\varepsilon)$ of (2) satisfying*

$$\mathcal{U}^{\varepsilon} \in C\big([0,T), L^2_x(\mathbb{R})\big) \cap L^{\infty}\big([0,T), H^m(\mathbb{R})\big),$$
$$v^{\varepsilon} \in C\big([0,T), L^2_x(\mathbb{R})\big) \cap L^{\infty}\big([0,T), H^{m-1}(\mathbb{R})\big).$$

Furthermore, there exits a constant $C > 0$ such that

$$\sup_{t\in[0,T)} \big(\big\|\mathcal{U}^{\varepsilon}(t)\big\|_{H^m} + \big\|v^{\varepsilon}(t)\big\|_{H^{m-1}}\big) \leqslant C$$

for any $\varepsilon \in [0,1]$.

Proof. Since the proof of Lemma 3.2 follows from the standard compactness argument, we give the outline of the proof. Let $(\mathcal{U}^{\varepsilon}_0, v^{\varepsilon}_0) \in H^m \times H^{m-1}$ and let $\{(\mathcal{U}^{\varepsilon,\delta}_0, v^{\varepsilon,\delta}_0)\}_{\delta} \subset H^{\infty}(\mathbb{R}) \times H^{\infty}(\mathbb{R})$ be a Bona–Smith approximation of $(\mathcal{U}^{\varepsilon}_0, v^{\varepsilon}_0)$. Then Lemma 2.1 leads that there exists a unique solution $(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta}) \in C([0,T_{\epsilon}), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ to (5). Lemma 3.1 yields that there exists $T > 0$ which is independent of $\epsilon \in [0,1]$ and $\delta \in (0,1]$ such that $\{(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta})\}_{\delta}$ is uniformly bounded in $L^{\infty}(0,T,H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ with respect to $\epsilon \in [0,1]$ and $\delta \in (0,1]$. By the classical compactness argument and diagonalization process, there exists a subsequence, still denoted by $\{(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta})\}_{\delta}$ and $(\mathcal{U}^{\varepsilon}, v^{\varepsilon}) \in L^{\infty}(0,T,H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$, a solution of (2), such that $(\mathcal{U}^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ converges in $L^{\infty}(0,T,H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ weak* to $(\mathcal{U}^{\varepsilon}, v^{\varepsilon})$. We can also show that $(\mathcal{U}^{\varepsilon}, v^{\varepsilon}) \in C([0,T) \times \mathbb{R})^2$ by Sobolev imbedding theorem. This completes the proof of Lemma 3.2. □

Next, we give the proof of Theorem 1.1 by showing uniqueness of the solution. Let $(\mathcal{U}^{\varepsilon}_1, v^{\varepsilon}_1)$ and $(\mathcal{U}^{\varepsilon}_2, v^{\varepsilon}_2)$ be two solutions to (2) with the same initial data in $H^m \times H^{m-1}$ satisfying

$$\mathcal{U}^{\varepsilon}_j \in C\big([0,T), L^2(\mathbb{R})\big) \cap L^{\infty}\big([0,T), H^m(\mathbb{R})\big),$$
$$v^{\varepsilon}_j \in C\big([0,T), L^2(\mathbb{R})\big) \cap L^{\infty}\big([0,T), H^{m-1}(\mathbb{R})\big),$$

$j = 1,2$. We shall prove that $(\mathcal{U}^{\varepsilon}_1, v^{\varepsilon}_1) \equiv (\mathcal{U}^{\varepsilon}_2, v^{\varepsilon}_2)$ for $t \in [0,T)$. To prove this, it suffices to show that $\mathcal{U}^{\varepsilon} = \mathcal{U}^{\varepsilon}_2 - \mathcal{U}^{\varepsilon}_1$ and $v^{\varepsilon} = v^{\varepsilon}_2 - v^{\varepsilon}_1$ satisfy $\|\mathcal{U}^{\varepsilon}\|_{H^1} \equiv \|v^{\varepsilon}\|_{L^2} \equiv 0$ for any $t \in [0,T)$.

Define $a^{\varepsilon} = a^{\varepsilon}_1 - a^{\varepsilon}_2$, $b^{\varepsilon} = b^{\varepsilon}_1 - b^{\varepsilon}_2$, $w^{\varepsilon} = w^{\varepsilon}_1 - w^{\varepsilon}_2$ and $v^{\varepsilon} = v^{\varepsilon}_1 - v^{\varepsilon}_2$. Then we see that $(\mathcal{U}^{\varepsilon}, v^{\varepsilon})$ satisfies

$$\begin{cases} \partial_t \mathcal{U}^{\varepsilon} + \dfrac{\varepsilon}{2}\mathcal{L}\mathcal{U}^{\varepsilon} + \mathcal{N}\big(\mathcal{U}^{\varepsilon}_1\big)\partial_x \mathcal{U}^{\varepsilon} + \mathcal{N}\big(\mathcal{U}^{\varepsilon}\big)\partial_x \mathcal{U}^{\varepsilon}_2 = -\alpha \mathcal{V}^{\varepsilon} \\ \partial_t v^{\varepsilon} + \partial_x^3 v^{\varepsilon} + 2v^{\varepsilon}_1 \partial v^{\varepsilon} + 2v^{\varepsilon}\partial v^{\varepsilon}_2 = \gamma \partial_x\big(a^{\varepsilon}_1 \bar{a}^{\varepsilon} + a\bar{a}^{\varepsilon}_2 + b^{\varepsilon}_1 \bar{b}^{\varepsilon} + b\bar{b}^{\varepsilon}_2\big). \end{cases} \tag{11}$$

The classical energy estimate leads to

$$\begin{aligned} &\left|\frac{1}{2}\frac{d}{dt}\big\|\mathcal{U}^{\varepsilon}(t)\big\|^2_{H^1} + \alpha \int_{\mathbb{R}} \partial_x^2 v^{\varepsilon} \partial_x w^{\varepsilon}\, dx\right| \\ &\quad \leqslant C\Big\{\sup_{t\in[0,T)} \big(\big\|\mathcal{U}^{\varepsilon}_1(t)\big\|_{H^2} + \big\|\mathcal{U}^{\varepsilon}_2(t)\big\|_{H^2}\big)\Big\}\big\|\mathcal{U}^{\varepsilon}(t)\big\|^2_{H^1} + C\big\|\mathcal{U}^{\varepsilon}(t)\big\|_{H^1}\big\|v^{\varepsilon}(t)\big\|_{L^2} \\ &\quad \leqslant C\big(\big\|\mathcal{U}^{\varepsilon}(t)\big\|^2_{H^1} + \big\|v^{\varepsilon}(t)\big\|^2_{L^2}\big), \end{aligned} \tag{12}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v^\varepsilon(t)\|_{L_x^2}^2 \\
&\quad\leqslant C\Big\{\sup_{t\in[0,T)}\big(\|\mathcal{U}_1^\varepsilon(t)\|_{H^1}+\|\mathcal{U}_2^\varepsilon(t)\|_{H^1}+\|v_1^\varepsilon(t)\|_{H^1}+\|v_2^\varepsilon(t)\|_{H^1}\big)\Big\} \\
&\qquad\times\big(\|\mathcal{U}^\varepsilon(t)\|_{H^1}^2+\|v^\varepsilon(t)\|_{L^2}^2\big) \\
&\quad\leqslant C\big(\|\mathcal{U}^\varepsilon(t)\|_{H^1}^2+\|v^\varepsilon(t)\|_{L^2}^2\big)
\end{aligned} \tag{13}$$

after employing the fact that there exists $C>0$ such that

$$\sup_{t\in[0,T)}\big(\|\mathcal{U}_j^\varepsilon(t)\|_{H^2}+\|v_j^\varepsilon(t)\|_{H^1}\big)\leqslant C,\quad \varepsilon\in[0,1],\ j=1,2$$

for some $C>0$. On the other hand

$$\begin{aligned}
&\left|\alpha\frac{d}{dt}\int_{\mathbb{R}} v^\varepsilon w^\varepsilon\,dx-\alpha\int_{\mathbb{R}}\partial_x^2 v^\varepsilon\partial_x w^\varepsilon\,dx\right| \\
&\quad\leqslant C\Big\{\sup_{t\in[0,t)}\big(\|\mathcal{U}_1^\varepsilon(t)\|_{H^1}+\|\mathcal{U}_2^\varepsilon(t)\|_{H^1}+\|v_1^\varepsilon(t)\|_{H^1}+\|v_2^\varepsilon(t)\|_{H^1}\big)\Big\} \\
&\qquad\times\big(\|\mathcal{U}^\varepsilon(t)\|_{H^1}^2+\|v^\varepsilon(t)\|_{L^2}^2\big) \\
&\quad\leqslant C\big(\|\mathcal{U}^\varepsilon(t)\|_{H^1}^2+\|v^\varepsilon(t)\|_{L^2}^2\big).
\end{aligned} \tag{14}$$

Here we introduce the modified energy:

$$E_1^\natural(\mathcal{U},v)(t)=\|a\|_{H^1}^2+\|b\|_{H^1}^2+\frac{1}{4\beta}\|w\|_{H^1}^2+10|\alpha|\beta^2\|v\|_{L^2}^2+2\alpha\int_{\mathbb{R}} wv\,dx.$$

We note that there exists a constant $C_{\alpha,\beta}>0$ such that for any $(\mathcal{U},v)\in H^1\times L^2$,

$$C_{\alpha,\beta}^{-1}\big\|\big(\mathcal{U}(t),v(t)\big)\big\|_{H^1\times L^2}^2\leqslant E_1^\natural(\mathcal{U},v)(t)\leqslant C_{\alpha,\beta}\big\|\big(\mathcal{U}(t),v(t)\big)\big\|_{H^1\times L^2}^2,$$

for all $t\in[0,T)$. Then, from (12), (13) and (14), we obtain

$$\frac{d}{dt}E_1^\natural\big(\mathcal{U}^\varepsilon,v^\varepsilon\big)(t)\leqslant CE_1^\natural\big(\mathcal{U}^\varepsilon,v^\varepsilon\big)(t),$$

where C is independent of $\varepsilon\in[0,1]$. Hence Gronwall's lemma and $(\mathcal{U}_0^\varepsilon,v_0^\varepsilon)\equiv(0,0)$ yields

$$0\leqslant E_1^\natural\big(\mathcal{U}^\varepsilon,v^\varepsilon\big)(t)\leqslant CE_1^\natural\big(\mathcal{U}_0^\varepsilon,v_0^\varepsilon\big)e^{Ct}=0,$$

for any $t\in[0,T)$. Therefore $E_1^\natural(\mathcal{U}^\varepsilon,v^\varepsilon)\equiv 0$ and hence $(\mathcal{U}^\varepsilon,v^\varepsilon)\equiv 0$. This completes the proof of uniqueness.

Remark. We can construct S^ε from w^ε defined in Theorem 1.1. Indeed, let us define S^ε as

$$S^\varepsilon(t,x) = S_0^\varepsilon(x) - \int_0^t \left(\frac{1}{2} w_2^\varepsilon + \beta \left|A^\varepsilon\right|^2 + \alpha v^\varepsilon\right)(\tau,x)\,d\tau.$$

Since $\partial_t(\partial_x S^\varepsilon - w^\varepsilon) = \partial_x\partial_t S^\varepsilon - \partial_t w^\varepsilon = 0$ we have $w^\varepsilon = \partial_x S^\varepsilon$.

4. Semiclassical limit

In this section we will devoted to the proof of Theorem 1.2. To his end, we have to evaluate the difference $\mathcal{U}^\varepsilon - \mathcal{U}_0$ and $v^\varepsilon - v_0$, where $(\mathcal{U}^\varepsilon, v^\varepsilon)$ and $(\mathcal{U}_0, v_0)$ are solutions to (2) and (4), respectively. We only give the proof of Theorem 1.2 with $m = 3$ since the case $m \geqslant 4$ being similar.

Putting

$$\mathcal{U} = \begin{bmatrix} a \\ b \\ w \end{bmatrix} = \begin{bmatrix} a^\varepsilon - a_0 \\ b^\varepsilon - b_0 \\ w^\varepsilon - w_0 \end{bmatrix}, \qquad \mathcal{V} = \begin{bmatrix} 0 \\ 0 \\ \partial_x v \end{bmatrix}, \quad v = v^\varepsilon - v_0$$

we see that $(\mathcal{U}, v)$ satisfies

$$\begin{cases} \partial_t \mathcal{U} + \dfrac{\varepsilon}{2}\mathcal{L}\mathcal{U}^\varepsilon + \mathcal{N}(\mathcal{U}^\varepsilon)\partial_x\mathcal{U} + \mathcal{N}(\mathcal{U})\partial_x\mathcal{U}_0 = -\alpha\mathcal{V}, \\ \partial_t v + \partial_x^3 v + v^\varepsilon\partial_x v + v\partial_x v_0 = \gamma\partial_x\left(a\bar{a}^\varepsilon + \bar{a}a_0 + b\bar{b}^\varepsilon + \bar{b}b_0\right). \end{cases} \tag{15}$$

As in the previous section, the classical energy method yields

$$\begin{aligned} &\left|\frac{1}{2}\frac{d}{dt}\left\|\mathcal{U}(t)\right\|_{H^1}^2 + \alpha\int_{\mathbb{R}} \partial_x^2 v\partial_x w\,dx\right| \\ &\quad\leqslant C\Big\{\sup_{t\in[0,T)}\left(\left\|\mathcal{U}^\varepsilon(t)\right\|_{H^2} + \left\|v^\varepsilon(t)\right\|_{H^2}\right)\Big\}\left(\left\|\mathcal{U}(t)\right\|_{H^1}^2 + \left\|v(t)\right\|_{L^2}^2\right) \\ &\qquad + C\varepsilon\left\|\mathcal{U}^\varepsilon(t)\right\|_{H^3}\left\|\mathcal{U}(t)\right\|_{H^1}, \end{aligned} \tag{16}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|v(t)\right\|_{L^2}^2 \\ &\quad\leqslant C\Big\{\sup_{t\in[0,T)}\left(\left\|\mathcal{U}^\varepsilon(t)\right\|_{H^1} + \left\|\mathcal{U}_0(t)\right\|_{H^1} + \left\|v^\varepsilon(t)\right\|_{H^1} + \left\|v_0(t)\right\|_{H^1}\right)\Big\} \\ &\qquad \times\left(\left\|\mathcal{U}(t)\right\|_{H^1}^2 + C\left\|v(t)\right\|_{L^2}^2\right), \end{aligned} \tag{17}$$

$$\begin{aligned} &\left|\alpha\frac{d}{dt}\int_{\mathbb{R}} wv\,dx - \alpha\int_{\mathbb{R}} \partial_x^2 v\partial_x w\,dx\right| \\ &\quad\leqslant C\Big\{\sup_{t\in[0,T)}\left(\left\|\mathcal{U}^\varepsilon(t)\right\|_{H^1} + \left\|\mathcal{U}_0(t)\right\|_{H^1} + \left\|v^\varepsilon(t)\right\|_{H^1} + \left\|v_0(t)\right\|_{H^1}\right)\Big\} \\ &\qquad \times\left(\left\|\mathcal{U}(t)\right\|_{H^1}^2 + C\left\|v(t)\right\|_{L^2}^2\right). \end{aligned} \tag{18}$$

We introduce the modified energy

$$E_1^{\flat}(\mathcal{U}, v)(t) = \|\mathcal{U}(t)\|_{H^1}^2 + C_\alpha \|v(t)\|_{L^2}^2 + 2\alpha \int_{\mathbb{R}} wv\,dx,$$

where we chose the positive constant C_α so that $E_1^{\flat}$ is equivalent to $H^1 \times L^2$ norm. From (16), (17) and (18), we obtain

$$\frac{d}{dt} E_1^{\flat}(\mathcal{U}, v)(t) \leqslant C\big(E_1^{\flat}(\mathcal{U}, v)(t) + \varepsilon^2\big),$$

where $C > 0$ is independent of $\varepsilon \in [0, 1]$. By Gronwall's lemma, we have

$$E_1^{\flat}(\mathcal{U}, v)(t) \leqslant C E_1^{\flat}(\mathcal{U}, v)(0)e^{Ct} + C\varepsilon^2\big(e^{Ct} - 1\big),$$

for any $t \in [0, T)$. Therefore we have

$$\begin{aligned} &\sup_{t\in[0,T)} \big(\|\mathcal{U}^\varepsilon(t) - \mathcal{U}_0(t)\|_{H^1} + \|v^\varepsilon(t) - v_0(t)\|_{L^2}\big) \\ &\quad \leqslant C\big(\|\mathcal{U}_0^\varepsilon - \mathcal{U}_{0,0}\|_{H^1} + \|v_0^\varepsilon - v_{0,0}\|_{L^2}\big) + C\varepsilon, \end{aligned}$$

where the constant $C > 0$ is independent of $\varepsilon \in [0, 1]$. This completes the proof of Theorem 1.2.

5. The first order approximation

In this section, we consider the first order approximation of a solution to (1). We give some formal calculation. Let

$$u^\varepsilon = \big(A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\big)e^{\frac{i}{\varepsilon}(S_0+\varepsilon S_1+\varepsilon^2 S_2+\cdots)}, \tag{19}$$

$$v^\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots. \tag{20}$$

Substituting above two equations into the first equation in (1), we have

$$F_0 + \varepsilon F_2 + \varepsilon^2 F_2 + \cdots = 0,$$

where

$$\begin{aligned} F_0 &= -A_0\left[\partial_t S_0 + \frac{1}{2}(\partial_x S_0)^2 + \alpha v_0 + \beta|A_0|^2\right], \\ F_1 &= -A_1\left[\partial_t S_0 + \frac{1}{2}(\partial_x S_0)^2 + \alpha v_0 + \beta|A_0|^2\right] \\ &\quad - A_0\big[\partial_t S_1 + \partial_x S_0 \partial_x S_1 + \alpha v_1 + \beta(\overline{A}_0 A_1 + A_0\overline{A}_1)\big] \\ &\quad + i\left(\partial_t A_0 + \partial_x A_0 \partial_x S_0 + \frac{1}{2} A_0 \partial_x^2 S_0\right) \\ &\equiv F_{1,1} + F_{1,2} + F_{1,3}, \end{aligned}$$

$$
\begin{aligned}
F_2 = &-A_2\left[\partial_t S_0 + \frac{1}{2}(\partial_x S_0)^2 + \alpha v_0 + \beta|A_0|^2\right] \\
&- 2A_1\left[\partial_t S_1 + \partial_x S_0 \partial_x S_1 + \alpha V_1 + \beta(\overline{A}_0 A_1 + A_0\overline{A}_1)\right] \\
&- A_0\left[\partial_t S_2 + (\partial_x S_1)^2 + \partial_x S_0 \partial_x S_2 + \alpha v_2 + 2\beta|A_1|^2 + \beta(\overline{A}_0 A_2 + A_0\overline{A}_2)\right] \\
&+ i\left(\partial_t A_1 - i\frac{1}{2}\partial_x^2 A_0 + \partial_x A_1 \partial_x S_0 + \partial_x A_0 \partial_x S_1 + \frac{1}{2}A_1\partial_x^2 S_0 + \frac{1}{2}A_0\partial_x^2 S_1\right) \\
\equiv &\, F_{2,1} + F_{2,2} + F_{2,3} + F_{2,3}, \\
&\vdots
\end{aligned}
$$

Substituting (19) and (20) into the second equation in (1), we obtain

$$G_0 + \varepsilon G_2 + \varepsilon^2 G_2 + \cdots = 0,$$

where

$$
\begin{aligned}
G_0 &= \partial_t v_0 + \partial_x^3 v_0 + 2v_0\partial_x v_0 - \gamma\partial_x\left(|A_0|^2\right), \\
G_1 &= \partial_t v_1 + \partial_x^3 v_1 + 2v_0\partial_x v_1 + 2v_1\partial_x v_0 - \gamma\partial_x(\overline{A}_0 A_1 + A_0\overline{A}_1), \\
G_2 &= \partial_t v_2 + \partial_x^3 v_2 + 2v_0\partial_x v_2 + 4v_1\partial_x v_1 + 2v_2\partial_x v_0 \\
&\quad - \gamma\partial_x\left(\overline{A}_0 A_2 + 2|A_1|^2 + A_0\overline{A}_2\right), \\
&\;\;\vdots
\end{aligned}
$$

We split above equations so that $F_{j,k} \equiv G_l \equiv 0$. Then we see that (A_0, S_0, v_0) satisfies

$$
\begin{cases}
\partial_t A_0 + \partial_x A_0 \partial_x S_0 + \frac{1}{2}A_0\partial_x^2 S_0 = 0, \\
\partial_t S_0 + \frac{1}{2}(\partial_x S_0)^2 + \alpha v_0 + \beta|A_0|^2 = 0, \\
\partial_t v_0 + \partial_x^3 v_0 + 2v_0\partial_x v_0 = \gamma\partial_x\left(|A_0|^2\right),
\end{cases}
\tag{21}
$$

and (A_1, S_1, v_1) satisfies

$$
\begin{cases}
\partial_t A_1 + \partial_x A_1 \partial_x S_0 + \partial_x A_0 \partial_x S_1 + \frac{1}{2}A_1\partial_x^2 S_0 + \frac{1}{2}A_0\partial_x^2 S_1 = i\frac{1}{2}\partial_x^2 A_0, \\
\partial_t S_1 + \partial_x S_0 \partial_x S_1 + \alpha v_1 + \beta(\overline{A}_0 A_1 + A_0\overline{A}_1) = 0, \\
\partial_t v_1 + \partial_x^3 v_1 + 2v_0\partial_x v_1 + 2v_1\partial_x v_0 = \gamma\partial_x(\overline{A}_0 A_1 + A_0\overline{A}_1).
\end{cases}
\tag{22}
$$

Next, we justify the semiclassical approximation of solution to (1) up to the first order. To this end, putting $\mathcal{U}_0 = (\operatorname{Re} A_0, \operatorname{Im} A_0, \partial_x S_0)^{\mathrm{t}}$ and $\mathcal{U}_1 = (\operatorname{Re} A_1, \operatorname{Im} A_1, \partial_x S_1)^{\mathrm{t}}$, we rewrite (21) and (22) as follow

$$\begin{cases} \partial_t \mathcal{U}_0 + \mathcal{N}(\mathcal{U}_0)\partial_x \mathcal{U}_0 = -\alpha \mathcal{V}_0 \\ \partial_t v_0 + \partial_x^3 v_0 + \partial_x (v_0^2) = \gamma \partial_x \left(|a_0|^2 + |b_0|^2\right), \end{cases} \tag{23}$$

and

$$\begin{cases} \partial_t \mathcal{U}_1 + \mathcal{N}(\mathcal{U}_0)\partial_x \mathcal{U}_1 + \mathcal{N}(\mathcal{U}_1)\partial_x \mathcal{U}_0 = \frac{1}{2}\tilde{\mathcal{L}}\mathcal{U} - \alpha \mathcal{V}_1 \\ \partial_t v_1 + \partial_x^3 v_1 + \partial_x (v_0 v_1) = \gamma \partial_x (a_0 a_1 + b_0 b_1), \end{cases} \tag{24}$$

where

$$\tilde{\mathcal{L}} = \begin{bmatrix} i\partial_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{V}_1 = \begin{bmatrix} 0 \\ 0 \\ \partial_x v_1 \end{bmatrix}.$$

From Theorem 1.1, if $m \geqslant 3$ is integer and $\|\mathcal{U}_{0,0}\|_{H^m} + \|v_{0,0}\|_{H^{m-1}} \leqslant C$, then there exists a time $T > 0$ and a unique solution $(\mathcal{U}_0, v_0) \in C([0,T), H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}))$ to (23) with $(\mathcal{U}_0(0), v_0(0)) = (\mathcal{U}_{0,0}, v_{0,0})$. Furthermore, if $m \geqslant 5$ is integer and $\|\mathcal{U}_{1,0}\|_{H^{m-2}} + \|v_{1,0}\|_{H^{m-3}} \leqslant C$, then there exists a time $T > 0$ and a unique solution $(\mathcal{U}_1, v_1) \in C([0,T), H^{m-2}(\mathbb{R})) \cap C([0,T), H^{m-3}(\mathbb{R}))$ to (24) with $(\mathcal{U}_1(0), v_1(0)) = (\mathcal{U}_{1,0}, v_{1,0})$. Then next theorem gives the first order approximation of solution to (1).

Theorem 5.1. *Let $m \geqslant 5$ be an integer and let $(\mathcal{U}_{0,0}, v_{0,0}), (\mathcal{U}_{1,0}, v_{1,0}) \in H^m \times H^{m-1}$. If $(\mathcal{U}_0^\varepsilon, v_0^\varepsilon) \in H^m \times H^{m-1}$ satisfies*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\left\|\mathcal{U}_0^\varepsilon - \mathcal{U}_{0,0} - \varepsilon \mathcal{U}_{1,0}\right\|_{H^m} + \left\|v_0^\varepsilon - v_{0,0} - \varepsilon v_{1,0}\right\|_{H^{m-1}}\right) = 0.$$

Then the unique solution $(\mathcal{U}^\varepsilon, v^\varepsilon)$ to (2) satisfies

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{t \in [0,T)} \left(\left\|\mathcal{U}^\varepsilon(t) - \mathcal{U}_0(t) - \varepsilon \mathcal{U}_1(t)\right\|_{H^{m-4}} + \left\|v^\varepsilon(t) - v_0(t) - \varepsilon v_1(t)\right\|_{H^{m-5}}\right) = 0,$$

where $(\mathcal{U}_0, v_0)$ and $(\mathcal{U}_1, v_1)$ are solutions to (23) and (24), respectively.

By a similar way, we formally obtain some hyperbolic-dispersive system for $(\mathcal{U}_j, v_j)$, $j \geqslant 2$. Since the higher order approximation of the $(\mathcal{U}^\varepsilon, v^\varepsilon)$ is obtained by the similar argument as the proof of Theorem 5.1, we do not give the proof of the higher order approximation of the $(\mathcal{U}^\varepsilon, v^\varepsilon)$.

Proof of Theorem 5.1. We only give the proof for Theorem 5.1 with $m = 5$. We put $\mathcal{U} = \mathcal{U}^\varepsilon - \mathcal{U}_0 - \varepsilon \mathcal{U}_1$ and $v = v^\varepsilon - v_0 - \varepsilon v_1$ with $\mathcal{U} = (a, b, w)$. Then $(\mathcal{U}, v)$ satisfies

$$\begin{aligned} &\partial_t \mathcal{U} + \frac{\varepsilon}{2}\mathcal{L}\mathcal{U}^\varepsilon - \frac{\varepsilon}{2}\mathcal{L}\mathcal{U}_0 + \mathcal{N}(\mathcal{U})\partial_x \mathcal{U}^\varepsilon + \mathcal{N}(\mathcal{U}_0)\partial_x \mathcal{U} \\ &\quad + \varepsilon \mathcal{N}(\mathcal{U}_1)\partial_x \mathcal{U} + \varepsilon^2 \mathcal{N}(\mathcal{U}_1)\partial_x \mathcal{U}_1 = \frac{1}{2}\tilde{\mathcal{L}}\mathcal{U} - \alpha \mathcal{V}, \end{aligned}$$

$$\begin{aligned}&\partial_t v+\partial_x^3 v+2\big(v\partial_x v^\varepsilon+v_0\partial_x v+\varepsilon v_1\partial_x v+\varepsilon^2 v_1\partial_x v_1\big)\\&\quad-2\gamma\big(a\partial_x a^\varepsilon+a_0\partial_x a+\varepsilon a_1\partial_x a+\varepsilon^2 a_1\partial_x a_1\big)\\&\quad-2\gamma\big(b\partial_x b^\varepsilon+b_0\partial_x b+\varepsilon b_1\partial_x b+\varepsilon^2 b_1\partial_x b_1\big)=0,\end{aligned}$$

where $\mathcal{V}=(0,0,\partial_x v)$. By using the energy method, we obtain

$$\begin{aligned}&\left|\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|_{H^1}^2+\alpha\int_{\mathbb{R}}\partial_x^2 v\partial_x w\,dx\right|\\&\quad\leqslant C\big(\big(\|\mathcal{U}^\varepsilon\|_{H^2}+\|\mathcal{U}_0\|_{H^2}\big)\|\mathcal{U}\|_{H^1}^2+\|\mathcal{U}\|_{H^1}\|v\|_{L^2}\big)\\&\qquad+C\varepsilon\big(\|\mathcal{U}^\varepsilon-\mathcal{U}_0\|_{H^3}\|\mathcal{U}\|_{H^1}+\|\mathcal{U}_1\|_{H^2}\|\mathcal{U}\|_{H^1}^2\big)+C\varepsilon^2\|\mathcal{U}_1\|_{H^2}^2\|\mathcal{U}\|_{H^1}\\&\quad\leqslant C\big(\|\mathcal{U}\|_{H^1}^2+\|v\|_{L^2}^2\big)+C\varepsilon\|\mathcal{U}\|_{H^1}^2+C\varepsilon^2\|\mathcal{U}\|_{H^1}\\&\quad\leqslant CE^\flat(\mathcal{U},v)+C\varepsilon^4.\end{aligned}$$

By a similar way

$$\frac{d}{dt}\|v\|_{L^2}^2\leqslant CE^\flat(\mathcal{U},v)+C\varepsilon^4,$$

$$\left|\alpha\frac{d}{dt}\int_{\mathbb{R}}wv\,dx-\alpha\int_{\mathbb{R}}\partial_x^2 v\partial_x w\,dx\right|\leqslant CE^\flat(\mathcal{U},v)+C\varepsilon^4,$$

where C is independent of $\varepsilon\in[0,1]$. Collecting above three inequalities, we have

$$\frac{d}{dt}E^\flat(\mathcal{U},v)\leqslant CE^\flat(\mathcal{U},v)+C\varepsilon^4.$$

Hence the Gronwall lemma leads the inequality

$$\begin{aligned}&\sup_{t\in[0,T)}\big(\|\mathcal{U}^\varepsilon-\mathcal{U}_0-\varepsilon\mathcal{U}_1\|_{H^1}+\|v^\varepsilon-v_0-\varepsilon v_1\|_{L^2}\big)\\&\quad\leqslant C\big(\|\mathcal{U}_0^\varepsilon-\mathcal{U}_{0,0}-\varepsilon\mathcal{U}_{1,0}\|_{H^1}+\|v_0^\varepsilon-v_{0,0}-\varepsilon v_{1,0}\|_{L^2}\big)+\varepsilon^2=o(\varepsilon).\end{aligned}$$

This completes the proof of Theorem 5.1. □

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References

[1] T. Alazard, Carles Rémi, Supercritical geometric optics for nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 194 (2009) 315–347.
[2] D. Bekiranov, T. Ogawa, G. Ponce, Weak solvability and well-posedness of a coupled Schrödinger–Korteweg de Vries equation for capillary-gravity wave interactions, Proc. Amer. Math. Soc. 125 (1997) 2907–2919.
[3] D.J. Benny, A general theory for interactions between short and long waves, Stud. Appl. Math. 56 (1977) 81–94.
[4] J.L. Bona, R. Smith, The initial-value problem for the Korteweg–de Vries equation, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 278 (1975) 555–601.
[5] Carles Rémi, Semi-Classical Analysis for Nonlinear Schrödinger Equations, World Scientific Publishing, 2008.
[6] Carles Rémi, S. Masaki, Semiclassical analysis for Hartree equation, Asymptot. Anal. 58 (2008) 211–227.
[7] D. Chiron, F. Rousset, Geometric optics and boundary layers for nonlinear-Schrödinger equations, Comm. Math. Phys. 288 (2009) 503–546.
[8] A.J. Corcho, F. Linares, Well-posedness for the Schrödinger–Korteweg–de Vries system, Trans. Amer. Math. Soc. 359 (2007) 4089–4106.
[9] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin–New York, 1981.
[10] P. Gérard, Remarques sur l'analyse semi-classique de l'equation de Schrödinger non linéaire, Séminaire EDP de l'Ecole Polytechnique, Palaiseau, France, 1992–1993, lecture no. XIII.
[11] E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, Proc. Amer. Math. Soc. 126 (1998) 523–530.
[12] S. Jin, C.D. Levermore, D.W. McLaughlin, The semiclassical limit of the defocussing NLS hierarchy, Comm. Pure Appl. Math. 52 (1999) 613–654.
[13] T. Kawahara, N. Sugimoto, T. Kakutani, Nonlinear interaction between short and long capillary-gravity waves, J. Phys. Soc. Jpn. 39 (1975) 1379–1386.
[14] S. Kwon, On the fifth-order KdV equation: local well-posedness and lack of uniform continuity of the solution map, J. Differential Equations 245 (2008) 2627–2659.
[15] C.-K. Lin, Y.-S. Wong, Zero-dispersion limit of the short-wave-long-wave interaction equations, J. Differential Equations 228 (2006) 87–110.
[16] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, Appl. Math. Sci., vol. 53, Springer-Verlag, New York, 1984.
[17] M. Tsutsumi, Well-posedness of the Cauchy problem for a coupled Schrödinger–KdV equation, Math. Sci. Appl. 2 (1993) 513–528.
[18] H. Wang, S. Cui, The Cauchy problem for the Schrödinger–KdV system, J. Differential Equations 250 (2011) 3559–3583.
[19] P. Zhang, Wigner Measure and Semiclassical Limits of Nonlinear Schrödinger Equations, Courant Lect. Notes Math., vol. 17, AMS, 2008.