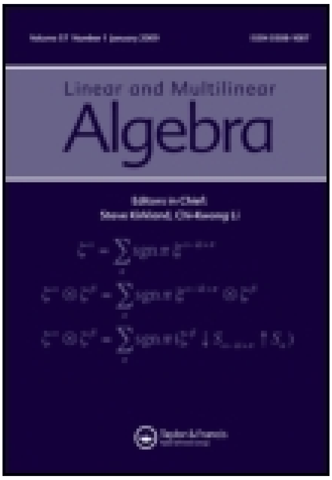

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Equality of higher-rank numerical ranges of matrices

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Let $\Lambda_k(A)$ denote the rank-k numerical range of an n-by-n complex matrix A. We give a characterization for $\Lambda_{k_1}(A) = \Lambda_{k_2}(A)$, where $1 \leq k_1 \leq k_2 \leq n$, via the compressions and the principal submatrices of A. As an application, the matrix A satisfying $W(A) = \Lambda_k(A)$, where $W(A)$ is the classical numerical range of A and $1 \leq k \leq n$, is under consideration. We show that if $W(A) = \Lambda_k(A)$ for some $k > n/3$, then A is unitarily similar to $\underbrace{B \oplus B \oplus \cdots \oplus B}_{3k-n \text{ copies}} \oplus C$, where B is a 2-by-2 matrix, C is a $(3n-6k)$-by-$(3n-6k)$ matrix and $W(A) = W(B) = W(C) = \Lambda_{n-2k}(C)$.

Keywords: numerical range; higher-rank numerical range; compression; principal submatrix

AMS Subject Classification: 15A60

1. Introduction

The *rank-k numerical range* $(1 \leq k \leq n)$ of an n-by-n complex matrix A is the subset of the complex plane:

$$\Lambda_k(A) \equiv \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogoanl projection } P\}.$$

Therefore, $\lambda \in \Lambda_k(A)$ if and only if there is an n-by-n unitary matrix U such that λI_k is the leading principal submatrix of U^*AU. Here, I_k denotes the k-by-k identity matrix. The investigation of the higher-rank numerical range was started in [1]. Specifically, it is introduced when constructing the quantum error correction code in quantum computing (cf. [2]). It is already known that $\Lambda_k(A)$ is always a convex compact set, invariant under unitary similarity and $\Lambda_1(A) \supseteq \Lambda_2(A) \supseteq \cdots \supseteq \Lambda_n(A)$. For other properties, we refer the readers to [1,3–7]. In particular, the rank-one numerical range $\Lambda_1(A)$ is exactly the classical numerical range $W(A) \equiv \{\langle Ax, x\rangle : x \in \mathbb{C}^n \text{ and } \|x\| = 1\}$ of A, where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{C}^n$ and $\|\cdot\|$ is the corresponding norm. In this paper, the characterization of matrix A which satisfies $\Lambda_{k_1}(A) = \Lambda_{k_2}(A)$, where $1 \leq k_1 \leq k_2 \leq n$, is obtained.

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We study this property by analysing the compressions and the principal submatrices of the matrix A.

Recall that an ℓ-by-ℓ matrix B, $1 \le \ell \le n$, is a *compression* of an n-by-n matrix A if there is an n-by-n unitary matrix V such that $V^*AV = \begin{bmatrix} B & * \\ * & * \end{bmatrix}$. In this case, A is called a dilation of B. Notice that $\Lambda_k(B) \subseteq \Lambda_k(A)$ for all $1 \le k \le \ell$. On the other hand, for any index set $K = \{j_1, j_2, \cdots, j_p\} \subseteq \{1, 2, \cdots, n\}$, let $A[K]$ or $A[j_1, j_2, \cdots, j_p]$ be the principal submatrix of A obtained by deleting its rows and columns indexed by $j_1, \ldots, j_p$. We also define $A[K] \equiv A$ if $K = \emptyset$. It is obvious that $A[K]$ is a compression of A. However, for a compression B of A, $B = A[K]$ for some K is not always true. Our main result is that, for $1 \le k_1 \le k_2 \le n$, $\Lambda_{k_1}(A) = \Lambda_{k_2}(A)$ if and only if $\Lambda_{k_1}(A) = \Lambda_{\ell+k_2-n}(B)$ for all its ℓ-by-ℓ compression B, $n + k_1 - k_2 \le \ell \le n$. This is also equivalent to that $\Lambda_{k_1}(A) = \Lambda_{k_1}(A[K'])$ for all index set $K' \subseteq \{1, 2, \cdots, n\}$ with $\#K' = k_2 - k_1$ when $\Lambda_{k_1}(A)$ has no corner (Theorem 2.2). Here $\#S$ is the cardinal number of the set S. As an application, we investigate those matrix A satisfying $W(A) = \Lambda_k(A)$ for some $k > n/3$, and obtain a decomposition of A (Theorem 2.10). Consequently, such matrix A must be unitarily reducible.

We conclude this section with some notations frequently used in the discussions below. Let M_n be the algebra of all n-by-n complex matrices. For $A \in M_n$, we use A^T, Re A, Im A, tr A, det A and rank A to denote its transpose, real part $(A + A^*)/2$, imaginary part $(A - A^*)/(2i)$, trace, determinant and rank, respectively. Denote by $\sigma(A)$ the spectrum of A. Also, let I_n and diag $(a_1, \ldots, a_n)$ be the n-by-n identity matrix and diagonal matrix with diagonal entries $a_1, \ldots, a_n$, respectively. Denote by $\bigvee \mathcal{S}$ the subspace generated by the vectors in $\mathcal{S} \subseteq \mathbb{C}^n$ (or the span of $\mathcal{S}$). For a subset $\triangle$ of $\mathbb{C}$, let $\triangle^\wedge$, $\#\triangle$ and $\partial\triangle$ denote the convex hull, the cardinal number and the boundary of $\triangle$, respectively. In addition, for an n-by-n Hermitian matrix H and $j = 1, 2, \cdots, n$, let $\lambda_j(H)$ be the jth largest eigenvalue of H.

2. Main results

In [7], Li and Sze gave a nice characterization of higher-rank numerical ranges of matrices. More specifically, they showed that, for $A \in M_n$ and $1 \le k \le n$,

$$\Lambda_k(A) = \bigcap_{\theta \in [0, 2\pi)} \left\{ z \in \mathbb{C} : \operatorname{Re}(ze^{i\theta}) \le \lambda_k(\operatorname{Re}(e^{i\theta}A)) \right\} \tag{2.1}$$

(cf. [7, Theorem 2.2]). On the other hand, the kth numerical range of A is defined by

$$W_k(A) = \left\{ \frac{1}{k} \sum_{j=1}^{k} \langle Ax_j, x_j \rangle : \{x_1, \ldots, x_k\} \text{ is an orthonormal set in } \mathbb{C}^n \right\}.$$

When $k = 1$, $W_k(A)$ reduces to the classical numerical range of A, which has been studied extensively (e.g. see [8]). Moreover, for θ in $[0, 2\pi)$, the line

$$L(k, \theta) = \left\{ z \in \mathbb{C} : \operatorname{Re} z = \frac{1}{k} \sum_{j=1}^{k} \lambda_j(\operatorname{Re}(e^{i\theta}A)) \right\}$$

is the right supporting line of the convex set $W_k(e^{i\theta}A) = e^{i\theta}W_k(A)$ (e.g. see [9]). Since $\lambda_k(\mathrm{Re}\,(e^{i\theta}A)) \leq (1/k)\sum_{j=1}^{k}\lambda_j(\mathrm{Re}\,(e^{i\theta}A))$ for all real θ, by (2.1), we infer that

$$\Lambda_k(A) \subseteq W_k(A)$$

for all k, $1 \leq k \leq n$. Using this inclusion and [8, Theorem 2.1], we have the following property.

PROPOSITION 2.1 *Suppose $A \in M_n$ and $1 \leq k \leq n$. The following conditions are equivalent:*

(a) $\Lambda_1(A) = \Lambda_k(A)$.
(b) *There exists m with $1 \leq m < k$ such that $W_m(A) = W_k(A)$.*
(c) $W_r(A) = W_s(A)$ *for all* $1 \leq r < s \leq k$.
(d) $\lambda_1(\mathrm{Re}\,(e^{i\theta}A)) = \lambda_k(\mathrm{Re}\,(e^{i\theta}A))$ *for all* $\theta \in [0, 2\pi)$.

Proof By [8, Theorem 2.1], we obtain the equivalence of (b), (c) and (d). The implication (d) $\Rightarrow$ (a) follows directly from (2.1). Now, suppose (a) holds. Then, $\Lambda_k(A) \subseteq W_k(A) \subseteq W_1(A) = \Lambda_1(A) = \Lambda_k(A)$ implies $W_1(A) = W_k(A)$. Thus, condition (b) holds. □

We remark that for $1 < r < k \leq n$, if $W_r(A) = W_k(A)$, by Proposition 2.1, we have $W_1(A) = W_k(A)$. But, if $\Lambda_r(A) = \Lambda_k(A)$, the equality $\Lambda_1(A) = \Lambda_k(A)$ does not hold in general. For example, let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $\Lambda_2(A) = \Lambda_3(A) = \{z \in \mathbb{C} : |z| \leq 1/2\}$, but $\Lambda_1(A) = \{z \in \mathbb{C} : |z| \leq 1\} \neq \Lambda_3(A)$.

The next theorem characterizes the equality of higher-rank numerical ranges of a matrix via its compressions and principal submatrices.

THEOREM 2.2 *Let $A \in M_n$ and $1 \leq k_1 < k_2 \leq n$. The following statements are equivalent:*

(a) $\Lambda_{k_1}(A) = \Lambda_{k_2}(A)$.
(b) $\Lambda_{k_1}(A) = \Lambda_{\ell+k_2-n}(B)$ *for all ℓ-by-ℓ compressions B of A with* $n+k_1-k_2 \leq \ell \leq n$.
(c) *For some $\ell \in \{n + k_1 - k_2, \cdots, n\}$, $\Lambda_{k_1}(A) = \Lambda_{\ell+k_2-n}(B)$ for all ℓ-by-ℓ compressions B of A.*
If $\Lambda_{k_1}(A)$ has no corner, then the statements (a)–(c) are also equivalent to:
(d) $\Lambda_{k_1}(A) = \Lambda_{k_2-p}(B)$ *for all $(n-p)$-by-$(n-p)$ principal submatrices B of A with* $p \leq k_2 - k_1$.
(e) $\Lambda_{k_1}(A) = \Lambda_{k_1}(B)$ *for all $(n + k_1 - k_2)$-by-$(n + k_1 - k_2)$ principal submatrices B of A.*
(f) $\lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A)) = \lambda_{k_2}(\mathrm{Re}\,(e^{i\theta}A))$ *for all* $\theta \in [0, 2\pi)$.

We emphasize that in Theorem 2.2 (b)(c), the condition $\Lambda_{k_1}(A) = \Lambda_{\ell+k_2-n}(B)$ and the observation $\Lambda_{\ell+k_2-n}(B) \subseteq \Lambda_{k_1}(B) \subseteq \Lambda_{k_1}(A)$ together imply that $\Lambda_{k_1}(A) = \Lambda_{\ell+k_2-n}(B) = \Lambda_{k_1}(B)$.

Among other things, we remark that if $\Lambda_{k_1}(A)$ has a corner, then the implication (d) $\Rightarrow$ (a) does not hold in general. Here we give an example as following.

Example 2.3 Let

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

and $A = A_1 \oplus A_2 \in M_8$. Then A is a unitary matrix with eigenvalues $1, i, -1, -1, -1, -i, -i, -i$. Thus

$$\Lambda_2(A) = \{-1, -i, 0\}^\wedge \quad \text{and} \quad \Lambda_3(A) = \{-1, -i\}^\wedge.$$

It is clear that $\Lambda_2(A) \neq \Lambda_3(A)$.

On the other hand, for $j = 1, 2, 3, 4$, the principal submatrix $A[j]$ of A is unitarily similar to $J_3 \oplus A_2$, where J_3 is the 3-by-3 Jordan block. Moreover, we have

$$\lambda_2(\mathrm{Re}\,(e^{-i\theta} A[j])) = \begin{cases} 0 & \text{if } \theta \in [0, \pi/2], \\ \mathrm{Re}\,(-e^{-i\theta}) & \text{if } \theta \in [\pi/2, 5\pi/4], \\ \mathrm{Re}\,(-ie^{-i\theta}) & \text{if } \theta \in [5\pi/4, 2\pi]. \end{cases}$$

Thus, $\Lambda_2(A[j]) = \{-1, -i, 0\}^\wedge$ for $j = 1, 2, 3, 4$.

Next, $A[5] = A[6] = A_1 \oplus \mathrm{diag}\,(-1, -i, -i)$ is unitarily similar to $\mathrm{diag}\,(1, i, -1, -1, -i, -i, -i)$. It is easy to check that $\Lambda_2(A[j]) = \{-1, -i, 0\}^\wedge$ for $j = 5, 6$. Similarly, $A[7] = A[8] = A_1 \oplus \mathrm{diag}\,(-1, -1, -i)$ is unitarily similar to $\mathrm{diag}\,(1, i, -1, -1, -1, -i, -i)$. We also have $\Lambda_2(A[j]) = \{-1, -i, 0\}^\wedge$ for $j = 7, 8$. From above, we obtain $\Lambda_2(A[j]) = \{-1, -i, 0\}^\wedge = \Lambda_2(A)$ for all j. Hence, the matrix A satisfies the condition (d) of Theorem 2.2, but A does not satisfy the condition (a) of Theorem 2.2. □

For the proof of Theorem 2.2, we need to estimate the eigenvalues of the principal submatrices of a Hermitian matrix and analyse the corresponding eigenvectors. Next two lemmas provide useful approximation and can be found in [10, Theorem 4.3.15] and [11, Theorem 1], respectively.

LEMMA 2.4 *Let H_1 be an n-by-n Hermitian matrix and H_2 be any ℓ-by-ℓ principal submatrix of H_1, where $1 \leq \ell \leq n$. For each integer k with $1 \leq k \leq \ell$, we have*

$$\lambda_k(H_1) \geq \lambda_k(H_2) \geq \lambda_{k+n-\ell}(H_1).$$

LEMMA 2.5 *Suppose H is an n-by-n Hermitian matrix partitioned as*

$$H = \begin{pmatrix} H_1 & B^* \\ B & H_2 \end{pmatrix},$$

where H_1 is an m-by-m matrix. If there is an index set $J \subseteq \{1, 2, \ldots, m\}$ such that for any $j \in J$, either $\lambda_j(H) = \lambda_j(H_1)$ or $\lambda_{n-m+j}(H) = \lambda_j(H_1)$, then there is an orthonormal set $\{u_j\}_{j\in J}$ in $\mathbb{C}^m$ such that $Bu_j = 0$ and $H_1 u_j = \lambda_j(H_1) u_j$ for all $j \in J$.

Let K be a nonempty subset of $\{1, 2, \ldots, n\}$ with $\#K = p < n$. Suppose that $\{s_1, s_2, \ldots, s_{n-p}\} = \{1, 2, \ldots, n\} \setminus K$ with $s_1 < s_2 < \ldots < s_{n-p}$. For each $y = (y_1, y_2, \ldots, y_{n-p})^T \in \mathbb{C}^{n-p}$, we define $y^{[K]} = (y'_1, y'_2, \ldots, y'_n)^T \in \mathbb{C}^n$ by

$$y_i' = \begin{cases} y_j & \text{if } \ i = s_j \ \text{ for some } \ 1 \le j \le n-p, \\ 0 & \text{if } \ i \in K, \end{cases}$$

for $i = 1, 2, \ldots, n$. That is, $y^{[K]}$ is obtained from y by inserting zero in the ith entry for all $i \in K$. The following lemma plays an important role for establishing Theorem 2.2.

LEMMA 2.6 *Let H be an n-by-n Hermitian matrix,* $1 \le m < n$ *and* $1 \le r \le n-m$. *Then* $\lambda_r(H) = \lambda_r(H[K])$ *for all* $K \subseteq \{1, 2, \ldots, n\}$ *with* $\#K = m$ *if and only if* $\lambda_r(H) = \lambda_{r+1}(H) = \cdots = \lambda_{r+m}(H)$.

Proof The sufficiency follows directly from Lemma 2.4. We need only prove the necessity. Suppose that $\lambda_r(H) = \lambda_r(H[K])$ for all $K \subseteq \{1, 2, \ldots, n\}$ with $\#K = m$. Write

$$\begin{aligned} \lambda_1(H) &\ge \lambda_2(H) \ge \cdots \ge \lambda_{p-1}(H) \\ &> \lambda \equiv \lambda_p(H) = \lambda_{p+1}(H) = \cdots = \lambda_r(H) \\ &\ge \lambda_{r+1}(H) \ge \cdots \ge \lambda_n(H). \end{aligned} \tag{2.2}$$

We claim that $\lambda_p(H[K]) = \lambda_{p+1}(H[K]) = \cdots = \lambda_r(H[K]) = \lambda$ for all $K \subseteq \{1, 2, \cdots, n\}$ with $\#K = m$. Indeed, by Lemma 2.4, we have

$$\lambda_r(H) = \lambda_p(H) \ge \lambda_p(H[K]) \ge \lambda_{p+1}(H[K]) \ge \cdots \ge \lambda_r(H[K]) = \lambda_r(H),$$

hence the inequalities are indeed equalities. Let $K_1 \equiv \{n-m+1, n-m+2, \ldots, n\}$. We have $\#K_1 = m$ and above argument ensures that $\lambda_j(H) = \lambda_j(H[K_1]) = \lambda$ for all $p \le j \le r$. Lemma 2.5 yields that there is an orthonormal set $\{u_{1,1}, u_{1,2}, \ldots, u_{1,r-p+1}\}$ in $\mathbb{C}^{n-m}$ such that $H[K_1]u_{1,j} = \lambda u_{1,j}$ and $Hu_{1,j}^{[K_1]} = \lambda u_{1,j}^{[K_1]}$ for all j, $1 \le j \le r-p+1$. Let $y_j \equiv u_{1,j}^{[K_1]}$ for $1 \le j \le r-p+1$, then $\{y_1, y_2, \ldots, y_{r-p+1}\}$ is an orthonormal set in $\ker(\lambda I_n - H)$. Notice that for each index $i \in K_1$, the ith entry of y_j is zero for all $1 \le j \le r-p+1$. Let q_1 be the index such that the q_1th entry of y_{r-p+1} is nonzero and set $K_2 \equiv (K_1 \setminus \{n-m+1\}) \cup \{q_1\} = \{q_1, n-m+2, n-m+3, \ldots, n\}$. It is obvious that $q_1 \notin K_1$ and $\#K_2 = m$. As claimed before, $\lambda_j(H) = \lambda_j(H[K_2]) = \lambda$ for all $p \le j \le r$. Applying Lemma 2.5 again to obtain an orthonormal set $\{u_{2,1}, u_{2,2}, \ldots, u_{2,r-p+1}\}$ in $\mathbb{C}^{n-m}$ such that $H[K_2]u_{2,j} = \lambda u_{2,j}$ and $Hu_{2,j}^{[K_2]} = \lambda u_{2,j}^{[K_2]}$ for all j, $1 \le j \le r-p+1$. Therefore, $\mathcal{S}_2 \equiv \{u_{2,1}^{[K_2]}, u_{2,2}^{[K_2]}, \ldots, u_{2,r-p+1}^{[K_2]}\}$ forms an orthonormal set in $\ker(\lambda I_n - H)$. Since $q_1 \in K_2$, the q_1th entry of $u_{2,j}^{[K_2]}$ is zero for all j, $1 \le j \le r-p+1$. Thus, the q_1th entry of all vectors in $\bigvee \mathcal{S}_2$ is zero. We now check that $\bigvee \mathcal{S}_2 \not\subseteq \bigvee\{y_1, y_2, \ldots, y_{r-p+1}\}$. Indeed, since the q_1th entry of y_{r-p+1} is nonzero, then $\dim \bigvee(\mathcal{S}_2 \cup \{y_{r-p+1}\}) = r-p+2$. If $\bigvee \mathcal{S}_2 \subseteq \bigvee\{y_1, y_2, \ldots, y_{r-p+1}\}$, then $\bigvee(\mathcal{S}_2 \cup \{y_{r-p+1}\}) \subseteq \bigvee\{y_1, y_2, \ldots, y_{r-p+1}\}$ and $r-p+2 \le \dim \bigvee\{y_1, y_2, \ldots, y_{r-p+1}\} = r-p+1$, a contradiction. Hence, we can choose an unit vector $y_{r-p+2} \in \bigvee \mathcal{S}_2$ so that $Hy_{r-p+2} = \lambda y_{r-p+2}$ and $\{y_1, y_2, \ldots, y_{r-p+1}, y_{r-p+2}\}$ is an orthonormal set in $\ker(\lambda I_n - H)$. Let q_2 be the index so that the q_2th entry of y_{r-p+2} is nonzero and let $K_3 \equiv (K_2 \setminus \{n-m+2\}) \cup \{q_2\} = \{q_1, q_2, n-m+3, n-m+4, \ldots, n\}$. Then, $q_2 \notin K_2$ and $\#K_3 = m$. Similarly, we have $\lambda_j(H) = \lambda_j(H[K_3]) = \lambda$ for all j, $p \le j \le r$. Lemma 2.5 yields that there is an orthonormal set $\{u_{3,1}, u_{3,2}, \ldots, u_{3,r-p+1}\}$ in $\mathbb{C}^{n-m}$ such that $H[K_3]u_{3,j} = \lambda u_{3,j}$ and $Hu_{3,j}^{[K_3]} = \lambda u_{3,j}^{[K_3]}$ for all j, $1 \le j \le r-p+1$. Hence $\mathcal{S}_3 \equiv \{u_{3,1}^{[K_3]}, u_{3,2}^{[K_3]}, \ldots, u_{3,r-p+1}^{[K_3]}\}$ is an orthonormal set in $\ker(\lambda I_n - H)$. Since q_1 and q_2 are in K_3, we indicate that the q_1th and the q_2th entries of all vectors in $\bigvee \mathcal{S}_3$

are zero. On the other hand, we also have $\bigvee \mathcal{S}_3 \nsubseteq \bigvee\{y_1, y_2, \ldots, y_{r-p+2}\}$. Indeed, since the q_1th entry of y_{r-p+1} is nonzero, the q_1th entry of y_{r-p+2} is zero and the q_2th entry of y_{r-p+2} is nonzero, we deduce that $\dim \bigvee(\mathcal{S}_3 \cup \{y_{r-p+1}, y_{r-p+2}\}) = r-p+3$. If $\bigvee \mathcal{S}_3 \subseteq \bigvee\{y_1, y_2, \ldots, y_{r-p+2}\}$, then $\bigvee(\mathcal{S}_3 \cup \{y_{r-p+1}, y_{r-p+2}\}) \subseteq \bigvee\{y_1, y_2, \ldots, y_{r-p+2}\}$ and $r-p+3 \leq \dim \bigvee\{y_1, y_2, \ldots, y_{r-p+2}\} = r-p+2$, a contradiction. Therefore, there exists an unit vector $y_{r-p+3} \in \bigvee \mathcal{S}_3$ such that $Hy_{r-p+3} = \lambda y_{r-p+3}$ and $\{y_1, y_2, \ldots, y_{r-p+3}\}$ is an orthonormal set in $\ker(\lambda I_n - H)$. Repeating these arguments can obtain an orthonormal set $\{y_1, y_2, \ldots, y_{r-p+m+1}\}$ in $\ker(\lambda I_n - H)$. Combining this with (2.2) together, we conclude that $\lambda_r(H) = \lambda_{r+1}(H) = \cdots = \lambda_{r+m}(H)$ as asserted. □

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2 We will prove this result by establishing the equivalence of (a), (b) and (c), and the implications (f) ⇒ (a) ⇒ (b) ⇒ (d) ⇒ (e) ⇒ (f) for the case $\Lambda_{k_1}(A)$ has no corner.

Fix $\ell = n - k_2 + k_1, \cdots, n$. For any ℓ-by-ℓ compression B of A, we have that

$$\begin{aligned}\Lambda_{k_2}(A) &= \bigcap\{\Lambda_{\ell+k_2-n}(B') : B' \text{ is an } \ell\text{-by-}\ell \text{ compression of } A\} \\ &\subseteq \Lambda_{\ell+k_2-n}(B) \subseteq \Lambda_{k_1}(B) \subseteq \Lambda_{k_1}(A),\end{aligned} \tag{2.3}$$

where the equality is given in [5, Corollary 4.9]. Hence, the implications (a) ⇒ (b) ⇒ (c) are trivial. Suppose (c) holds. As indicated in the paragraph after Theorem 2.2, we obtain that $\Lambda_{\ell+k_2-n}(B') = \Lambda_{k_1}(A)$ for all ℓ-by-ℓ compressions B' of A. Then, (a) follows directly from the equality in (2.3).

We now suppose that $\Lambda_{k_1}(A)$ has no corner. The implication (f) ⇒ (a) follows from the Li-Sze characterization (2.1). The implications (b) ⇒ (d) ⇒ (e) are trivial. We now prove the implication (e) ⇒ (f). Suppose (e) holds, we want to show that $\lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A)) = \lambda_{k_2}(\mathrm{Re}\,(e^{i\theta}A))$ for all $\theta \in [0, 2\pi)$. Fix a $\theta \in [0, 2\pi)$. For any $K' \subseteq \{1, 2, \cdots, n\}$ with $\#K' = k_2 - k_1$, let L_θ be the right supporting line of the convex set $\Lambda_{k_1}(e^{i\theta}A[K'])$ and write

$$L_\theta = \{z \in \mathbb{C} : \mathrm{Re}\, z = d(\theta)\},$$

where $d(\theta) \in \mathbb{R}$. Then, $d(\theta) \leq \lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A[K']))$ by the Li-Sze characterization (2.1). On the other hand, the assumption $\Lambda_{k_1}(A) = \Lambda_{k_1}(A[K'])$ implies that L_θ is also the right supporting line of the convex set $\Lambda_{k_1}(e^{i\theta}A)$. Let α be a point in $L_\theta \cap \Lambda_{k_1}(e^{i\theta}A)$. Since $\Lambda_{k_1}(A)$ has no corner, L_θ is the unique supporting line of $\Lambda_{k_1}(e^{i\theta}A)$ which passing the point α. It forces that

$$L_\theta = \{z \in \mathbb{C} : \mathrm{Re}\, z = \lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A))\}$$

by the Li-Sze characterization (2.1). Thus, we have

$$\lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A)) = d(\theta) \leq \lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A[K'])) \leq \lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A)),$$

where the last inequality follows from Lemma 2.4. Hence, the inequalities are indeed equalities. We infer from above that $\lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A)) = \lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A[K']))$ for all $K' \subseteq \{1, 2, \ldots, n\}$ with $\#K' = k_2 - k_1$. Then, Lemma 2.6 yields that

$$\lambda_{k_1}(\mathrm{Re}\,(e^{i\theta}A)) = \lambda_{k_1+(k_2-k_1)}(\mathrm{Re}\,(e^{i\theta}A)) = \lambda_{k_2}(\mathrm{Re}\,(e^{i\theta}A)).$$

Since θ is arbitrary, hence condition (f) holds. □

We now restrict our attention to matrices A with $W(A) = \Lambda_k(A)$ for some k. We known that the boundary $\partial W(A)$ of the numerical range of a matrix A consists of arcs, flat portions and/or corners. We first consider the case that $W(A)$ has a corner. For this purpose, we need the Kippenhahn polynomial of a matrix. Recall that the *Kippenhahn polynomial* of an n-by-n matrix A is the degree-n real-coefficient homogeneous polynomial $p_A(x, y, z)$ given by $\det(x\mathrm{Re}\, A + y\mathrm{Im}\, A + zI_n)$. It relates to the numerical range of A by the fact that $W(A)$ equals the convex hull of the real part of the dual curve of $p_A(x, y, z) = 0$ (cf. [12, Theorem 10]).

PROPOSITION 2.7 *Let $A \in M_n$, $1 \le k \le n$ and $\alpha \in \mathbb{C}$ be a corner of $W(A)$. Then, the following statements are equivalent:*

(a) *α is a corner of $W_k(A)$.*
(b) *α is a corner of $\Lambda_k(A)$.*
(c) *$(z + xRe\,\alpha + yIm\,\alpha)^k$ divides $p_A(x, y, z)$.*
(d) *A is unitarily similar to $\alpha I_m \oplus C$ with $m \ge k$ and $\alpha \notin W(C)$.*

Proof The implication (a) $\Rightarrow$ (d) follows from [8, Lemma 4.1]. The implication (d) $\Rightarrow$ (c) is trivial.

Suppose (c) holds. Write $p_A(x, y, z) = (z + x\mathrm{Re}\,\alpha + y\mathrm{Im}\,\alpha)^k \cdot q(x, y, z)$. Since α is a corner of $W(A)$, there exists a $\theta_0 \in \mathbb{R}$ such that the line $L \equiv \{z \in \mathbb{C} : \mathrm{Re}\, z = \mathrm{Re}\,(e^{-i\theta_0}\alpha)\}$ intersects $W(A)$ with a singleton $\{e^{-i\theta_0}\alpha\}$. Note that

$$\begin{aligned}\det(zI_n - \mathrm{Re}\,(e^{-i\theta_0}A)) &= p_A(-\cos\theta_0, -\sin\theta_0, z)\\ &= (z - \cos\theta_0\,\mathrm{Re}\,\alpha - \sin\theta_0\,\mathrm{Im}\,\alpha)^k \cdot q(-\cos\theta_0, -\sin\theta_0, z)\\ &= (z - \mathrm{Re}\,(e^{-i\theta_0}\alpha))^k \cdot q(-\cos\theta_0, -\sin\theta_0, z).\end{aligned}$$

Thus, $\mathrm{Re}\,(e^{-i\theta_0}\alpha)$ is an eigenvalue of $\mathrm{Re}\,(e^{-i\theta_0}A)$ with multiplicity at least k. Moreover, let $M \equiv \ker(\mathrm{Re}\,(e^{-i\theta_0}\alpha)I_n - \mathrm{Re}\,(e^{-i\theta_0}A))$ and $m \equiv \dim M$, then $m \ge k$. On the other hand, for any unit vector $x \in M$, we have $\mathrm{Re}\,\langle(e^{-i\theta_0}A)x, x\rangle = \langle\mathrm{Re}\,(e^{-i\theta_0}A)x, x\rangle = \mathrm{Re}\,(e^{-i\theta_0}\alpha)$. Since $W(A) \cap L = \{e^{-i\theta_0}\alpha\}$, it forces that $\langle(e^{-i\theta_0}A)x, x\rangle = e^{-i\theta_0}\alpha$ or $\langle Ax, x\rangle = \alpha$ for all unit vector $x \in M$. That is, the numerical range of the compression B of A on M is the singleton $\{\alpha\}$. It follows that B is unitarily similarly to αI_m. Consequently, αI_m dilates to A, hence $\alpha \in \Lambda_m(A) \subseteq \Lambda_k(A)$. Since $\Lambda_k(A) \subseteq W(A)$ and α is a corner of $W(A)$, hence α is also a corner of $\Lambda_k(A)$.

Suppose (b) holds. Since $\Lambda_k(A) \subseteq W_k(A) \subseteq W(A)$, it follows that $\alpha \in W_k(A)$. Moreover, since $W_k(A) \subseteq W(A)$ and α is a corner of $W(A)$, hence α is also a corner of $W_k(A)$. □

Next, if $A \in M_n$ and $\partial W(A) \cap \partial\Lambda_k(A)$ contains an arc, Gau and Wu had gave a characterization as following [4, Lemma 5].

PROPOSITION 2.8 *Let A be an n-by-n matrix, q be an irreducible real homogeneous polynomial in x, y and z with degree at least two, and C be the real part of the dual curve of $q(x, y, z) = 0$. Then q^m divides p_A $(m \ge 1)$ if and only if $\partial\Lambda_{k_0}(A) \cap \partial\Lambda_{k_0-1}(A) \cap \cdots \cap \partial\Lambda_{k_0-m+1}(A)$ contains an arc of C for some k_0, $1 \le k_0 \le \lfloor n/2 \rfloor$.*

The next theorem gives a detailed characterization of matrices A with $W(A) = \Lambda_k(A)$ for some $k > n/3$.

THEOREM 2.9 *Let $A \in M_n$.*

(a) *If $k > n/2$, then $W(A) = \Lambda_k(A)$ if and only if A is a scalar matrix.*
(b) *If n is even, then $W(A) = \Lambda_{n/2}(A)$ if and only if A is unitarily similar to*

$$\underbrace{B \oplus B \oplus \cdots \oplus B}_{n/2 \text{ copies}},$$

where $B \in M_2$. Therefore, $W(A) = W(B)$.
(c) *If $n/3 < k < n/2$, then $W(A) = \Lambda_k(A)$ if and only if A is unitarily similar to*

$$\underbrace{B \oplus B \oplus \cdots \oplus B}_{3k-n \text{ copies}} \oplus C,$$

where $B \in M_2$ and $C \in M_{3n-6k}$, and $W(A) = W(B) = W(C) = \Lambda_{n-2k}(C)$.

For the proof of Theorem 2.9, we need a series of lemmas. Suppose $A \in M_n$ and $W(A) = \Lambda_k(A)$. Using Propositions 2.7 and 2.8, we can determine the shape of $W(A)$ when $k > n/3$.

LEMMA 2.10 *Let $A \in M_n$. If $k > n/3$ and $W(A) = \Lambda_k(A)$, then $W(A)$ is either a singleton set, a line segment or an elliptic disc.*

Proof Suppose $k > n/3$ and $W(A) = \Lambda_k(A)$. We first consider the case that $W(A)$ has a corner $a + ib$, where $a, b \in \mathbb{R}$. We claim that $W(A)$ is either a singleton set or a line segment. Indeed, Proposition 2.7 yields that $(ax + by + z)^k$ divides $p_A(x, y, z)$. If $\partial W(A)$ contains an arc, by Kippenhahn's result [12, Theorem 10], there exists an irreducible factor $p(x, y, z)$ of $p_A(x, y, z)$ with degree at least two such that $C_p \cap \partial W(A)$ contains an arc, where C_p is the real part of the dual curve of $p(x, y, z) = 0$. Since $W(A) = \Lambda_k(A)$, by Propositon 2.8, we obtain that p^k divides p_A. Then

$$n = \deg p_A \geq \deg \left(p^k \cdot (ax + by + z)^k\right) \geq 3k > n,$$

where $\deg f$ denotes the degree of the polynomial f, and this is a contradiction. Therefore, we infer that $\partial W(A)$ is a convex polygon. On the other hand, if $W(A)$ has at least three vertices a_1+ib_1, a_2+ib_2 and a_3+ib_3, then Proposition 2.7 yields that $\prod_{j=1}^{3}(a_jx+b_jy+z)^k$ divides $p_A(x, y, z)$. This implies that $n = \deg p_A \geq 3k > n$, a contradiction. Hence we conclude that $W(A)$ is either a singleton set or a line segment.

Next, we now suppose that $W(A)$ has no corner. By Kippenhahn's result, there exists an irreducible factor $q(x, y, z)$ of $p_A(x, y, z)$ with degree at least two such that $C_q \cap \partial W(A)$ contains an arc, where C_q is the real part of the dual curve of $q(x, y, z) = 0$. We indicate that $n = \deg p_A \geq k \cdot \deg q > (n/3) \cdot \deg q$ by Proposition 2.8. Therefore, the degree of $q(x, y, z)$ is exactly two and C_q is an ellipse. We want to show that $\partial W(A) = C_q$. Indeed, if it is not the case, there is another irreducible factor $p(x, y, z)$ of $p_A(x, y, z)$ with degree at least two such that $C_p \cap \partial W(A)$ contains an arc, where C_p is the real part of the dual curve of $p(x, y, z) = 0$. By Proposition 2.8 again, p^k divides p_A. As a result, we get that

$n = \deg p_A \ge \deg(q^k \cdot p^k) \ge 4k > n$, a contradiction. Hence $\partial W(A) = C_q$ and $W(A)$ is an elliptical disc. □

LEMMA 2.11 *Let* $A = \begin{bmatrix} B & x \\ y^* & \alpha \end{bmatrix} \in M_3$, *where* $B \in M_2$, $x, y \in \mathbb{C}^2$ *and* $\alpha \in \mathbb{C}$. *If* $W(A) = W(B)$ *then* $x = y = 0$. *In this case,* $A = B \oplus [\alpha]$.

Proof Since $W(A) = W(B)$, then $\lambda_1(\operatorname{Re} B) = \lambda_1(\operatorname{Re} A)$ and $\lambda_2(\operatorname{Re} B) = \lambda_3(\operatorname{Re} A)$. Note that

$$\operatorname{Re} A = \begin{bmatrix} \operatorname{Re} B & (x+y)/2 \\ (x+y)^*/2 & \operatorname{Re} \alpha \end{bmatrix}.$$

By Lemma 2.5, we have $(x+y)^* u_j = 0$ for $j = 1, 2$, where u_j is an eigenvector of $\operatorname{Re} B$ with respect to the eigenvalue $\lambda_j(\operatorname{Re} B)$ for $j = 1, 2$. Since $\operatorname{Re} B$ is a 2-by-2 Hermitian matrix and $\bigvee\{u_1, u_2\} = \mathbb{C}^2$, it forces that $x + y = 0$ or $y = -x$. On the other hand, $W(A) = W(B)$ implies that $\lambda_1(\operatorname{Im} B) = \lambda_1(\operatorname{Im} A)$ and $\lambda_2(\operatorname{Im} B) = \lambda_3(\operatorname{Im} A)$. Note that

$$\operatorname{Im} A = \begin{bmatrix} \operatorname{Im} B & -ix \\ ix^* & \operatorname{Im} \alpha \end{bmatrix}.$$

Similarly, Lemma 2.5 yields that $x^* v_j = 0$ for $j = 1, 2$, where v_j is an eigenvector of $\operatorname{Im} B$ with respect to the eigenvalue $\lambda_j(\operatorname{Im} B)$ for $j = 1, 2$. Since $\operatorname{Im} B$ is a 2-by-2 Hermitian matrix and $\bigvee\{v_1, v_2\} = \mathbb{C}^2$, hence we conclude that $x = 0$ as asserted. □

Using Lemma 2.11, we have the following corollary.

COROLLARY 2.12 *Let* $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \in M_n$ *on* $\mathbb{C}^n = \mathbb{C}^2 \oplus \mathbb{C}^{n-2}$. *If* $W(B) = W(A)$, *then* $A = B \oplus E$.

Proof Write $C = [x_1 \ \dots \ x_{n-2}]$, $D = [y_1 \ \dots \ y_{n-2}]^*$ and $E = [t_{ij}]_{i,j=1}^{n-2}$, where $x_1, \dots, x_{n-2}, y_1, \dots, y_{n-2} \in \mathbb{C}^2$. Let $T_j = \begin{bmatrix} B & x_j \\ y_j^* & t_{jj} \end{bmatrix} \in M_3$ for $j = 1, \dots, n-2$. Then, $W(B) \subseteq W(T_j) \subseteq W(A) = W(B)$ implies that $W(T_j) = W(B)$ for all j. Thus, Lemma 2.11 yields that $x_j = y_j = 0$ for all j. Hence, $C = 0$ and $D = 0$ as desired. □

The next example shows that the condition $B \in M_2$ in Corollary 2.12 is essential.

Example 2.13 Let

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array}\right] \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

By a direct computation, we obtain that

$$p_A(x, y, z) = \left(z^2 - \frac{x^2}{2} - \frac{y^2}{2}\right)\left(z^2 - \frac{x^2}{4} - \frac{y^2}{4}\right)$$

and

$$p_B(x, y, z) = z\left(z^2 - \frac{x^2}{2} - \frac{y^2}{2}\right).$$

Thus, $W(A) = W(B) = \{z \in \mathbb{C} : |z| \leq 1/\sqrt{2}\}$. Note that

$$A^2 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = 0 \quad \text{and} \quad A^4 = 0.$$

Thus, A is nilpotent. We now show that A is unitarily irreducible. Otherwise, A is unitarily similar to $[a] \oplus A_1$ for some $a \in \mathbb{C}$ and $A_1 \in M_3$ or to $A_2 \oplus A_3$ for some $A_2, A_3 \in M_2$. If A is unitarily similar to $[a] \oplus A_1$, where $a \in \mathbb{C}$ and $A_1 \in M_3$, then a is a reducing eigenvalue of A and $a = 0$. Therefore, we get $\ker A \cap \ker A^* \neq \{0\}$. But,

$$\ker A = \bigvee \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \end{bmatrix} \right\}, \quad \ker A^* = \bigvee \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \sqrt{2} \end{bmatrix} \right\}$$

and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \sqrt{2} \end{bmatrix} \right\}$ are linearly independent, hence $\ker A \cap \ker A^* = \{0\}$, a contradiction. On the other hand, if A is unitarily similar to $A_2 \oplus A_3$, where $A_2, A_3 \in M_2$, then the fact that 0 is the only eigenvalue of A implies $A_2^2 = A_3^2 = 0$. This guarantees that $A^2 = 0$, a contradiction. Hence, we conclude that A is unitarily irreducible.

We are now ready to prove Theorem 2.9. For convenience, let $A \cong B$ denote that the matrix A is unitarily similar to the matrix B. Furthermore, let $\sum_{j=1}^{p} \oplus B_j$ stand for the direct sum of the matrices B_j, $j = 1, 2, \ldots, p$.

Proof of Theorem 2.9 (a) and (b) follow directly from Proposition 2.1, [8, Theorem 2.2] and [8, Corollary 4.7 (a)]. The sufficiency of (c) is trivial. We need only prove the necessity of (c).

Suppose $n/3 < k < n/2$ and $W(A) = \Lambda_k(A)$. If A is a scalar matrix, then the desired decomposition always holds. Hence, we assume that A is not a scalar matrix. Since $W(A)$ is either a line segment or an elliptic disc by Lemma 2.10, after suitable translation, rotation and scaling, we may assume that $W(A)$ centres at the origin, its axes lie on $\mathbb{R}$ and $i\mathbb{R}$, the length of the former is 2 and the length of the latter is $2b$, where $0 \leq b \leq 1$. Since $W(A) = \Lambda_k(A)$, then 1 (respectively, -1) is the maximal (respectively, minimal) eigenvalue of $\operatorname{Re} A$ with multiplicity at least k. From [13, Theorem 2.7], we obtain that A is unitarily similar to the matrix

$$\begin{bmatrix} I_k & E & * \\ -E^* & -I_k & * \\ * & * & * \end{bmatrix} \quad \text{on } \mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^{n-2k},$$

where $E \in M_k$. Let $A' = \begin{bmatrix} I_k & E \\ -E^* & -I_k \end{bmatrix} \in M_{2k}$. Notice that, as mentioned in the paragraph after Theorem 2.2, we have $W(A) = \Lambda_{3k-n}(A') = W(A')$. Let $E = U\Sigma V^*$ be the singular value decomposition of E, where U and V are k-by-k unitary matrices and $\Sigma = \text{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_k)$ for some $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 0$, and let $W = U^* \oplus V^*$. We obtain that

$$\begin{aligned} WA'W^* &= \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} I_k & E \\ -E^* & -I_k \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} I_k & \Sigma \\ -\Sigma & -I_k \end{bmatrix} \\ &\cong \sum_{j=1}^{k} \oplus \begin{bmatrix} 1 & \alpha_j \\ -\alpha_j & -1 \end{bmatrix} = \sum_{j=1}^{k} \oplus B_j, \end{aligned}$$

where $B_j = \begin{bmatrix} 1 & \alpha_j \\ -\alpha_j & -1 \end{bmatrix} \in M_2$. Therefore, $W(B_j) \subseteq W(A') \subseteq W(A)$ for all $j = 1, 2, \ldots, k$. Moreover, $W(\text{Im}\, B_j) \subseteq W(\text{Im}\, A) = [-b, b]$ implies that $\alpha_j \leq b$ for all $j = 1, 2, \ldots, k$. In addition, the fact $W(A) = \Lambda_{3k-n}(A') = W(A') = \bigcup_{j=1}^{k} W(B_j)$ ensures that $\alpha_1 = \alpha_2 = \cdots = \alpha_{3k-n} = b$, $B_1 = B_2 = \cdots = B_{3k-n}$ and $W(B_1) = W(A)$. Consequently, from [13, Theorem 2.7], we deduce that

$$A \cong \left[\begin{array}{ccc|c} B_1 & & & \\ & \ddots & & D \\ & & B_1 & \\ \hline & -D^* & & C \end{array}\right],$$

where $C \in M_{3n-6k}$, D is a $(6k-2n)$-by-$(3n-6k)$ matrix, and B_1 appears $3k-n$ times. Since $W(B_1) = W(A)$, by Corollary 2.12, we obtain that $D = 0$ and $A \cong \left(\sum_{j=1}^{3k-n} \oplus B_1\right) \oplus C$. Among other things, since $\partial W(A) = \partial \Lambda_k(A) = \partial W(B_1)$ and $B_1 \in M_2$, from the proof of Theorem 2.9, we have $p_{B_1}^k$ divides p_A. Moreover, $p_A = p_C \cdot p_{B_1}^{3k-n}$ implies that $p_{B_1}^{n-2k}$ is a factor of p_C. It follows that

$$W(B_1) \subseteq \Lambda_{n-2k}(C) \subseteq W(C) \subseteq W(A) = W(B_1),$$

hence the inclusions are indeed equalities. □

We end this paper by remarking that, in Theorem 2.9, the number $n/3$ is sharp for the reducibility of A, that is, we cannot replace it with any smaller integer, because there exists a $3k$-by-$3k$ unitarily irreducible matrix A which satisfies $W(A) = \Lambda_k(A)$. For example, let

$$E = \begin{bmatrix} 0 & \sqrt{2} & 0 & \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, E is unitarily irreducible and $W(E) = W_2(E) = \Lambda_2(E)$ is the closed unit disc (cf. [8, Theorem 3.2]). As a result, the matrix C in Theorem 2.9 (c) may be unitarily irreducible and the decomposition of A is the best representation. For example, let

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus E \in M_{10}.$$

Then, $W(A) = \Lambda_4(A)$ is the closed unit disc and $\Lambda_5(A) = \{z \in \mathbb{C} : |z| \le 1/(2\sqrt{2})\}$. It is clear that E is unitarily irreducible and $3k - n = 3 \cdot 4 - 10 = 2$.

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