Advances in Mathematics 256 (2014) 46–103

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Zeta functions of complexes arising from PGL(3) ☆

Ming-Hsuan Kang [a,*], Wen-Ching Winnie Li [b,c]

[a] Department of Mathematics, National Chiao-Tung University, Hsinchu, Taiwan
[b] Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
[c] National Center for Theoretical Sciences, Mathematics Division, National Tsing Hua University, Hsinchu 30013, Taiwan

ARTICLE INFO

*Article history:*
Received 25 September 2012
Accepted 30 December 2013
Available online 19 February 2014
Communicated by Roman Bezrukavnikov

*MSC:*
primary 22E35
secondary 11F70

*Keywords:*
Bruhat–Tits building
Zeta functions
Discrete cocompact subgroups

ABSTRACT

In this paper we obtain a closed form expression of the zeta function $Z(X_\Gamma, u)$ of a finite quotient $X_\Gamma$ of the Bruhat–Tits building of $\mathrm{PGL}_3$ over a nonarchimedean local field $F$ by a discrete cocompact torsion-free subgroup $\Gamma$ of $\mathrm{PGL}_3$. Analogous to a graph zeta function, $Z(X_\Gamma, u)$ is a rational function with two different expressions and it satisfies the Riemann hypothesis if and only if $X_\Gamma$ is a Ramanujan complex.



## 1. Introduction

*1.1.* First introduced by Ihara [9] for groups and later reformulated by Serre for regular graphs, the zeta function of a finite, connected, undirected graph $X$ is defined as

☆ This research was supported in part by the DARPA grant HR0011-06-1-0012 and the NSF grants DMS-0457574 and DMS-0801096 (both authors), NSF grant DMS-1101368 (W.-C.W. Li), and also the NSC grant 100-2115-M-009-008-MY2 (M. Kang). Part of the research was performed while both authors were visiting the National Center for Theoretical Sciences, Mathematics Division, in Hsinchu, Taiwan. They would like to thank the Center for its support and hospitality.

* Corresponding author.
*E-mail addresses:* mhkang@math.nctu.edu.tw (M.-H. Kang), wli@math.psu.edu (W.-C.W. Li).

0001-8708/$ – see front matter 
http://dx.doi.org/10.1016/j.aim.2013.12.033

$$Z(X,u)=\prod_{[C]}\bigl(1-u^{l([C])}\bigr)^{-1},$$

where the product is over equivalence classes $[C]$ of geodesic tailless primitive cycles $C$, and $l([C])$ is the length of a cycle in $[C]$. In this paper we adopt the convention that a cycle is an oriented closed path with a starting point and possible repetition of vertices. Two cycles are equivalent if one is obtained from the other by changing the starting vertex. A geodesic path on a graph means no backtracking. A cycle is tailless if all cycles equivalent to it are geodesic; it is primitive if it is not a repetition of a shorter cycle more than once. Taking the logarithmic derivative of $Z(X,u)$, one gets

$$Z(X,u)=\exp\biggl(\sum_{n\geqslant 1}\frac{N_n(X)}{n}u^n\biggr),$$

where $N_n(X)$ counts the number of geodesic tailless cycles in $X$ of length $n$.

Not only defined analogous to the zeta function of a curve over a finite field, the zeta function of a graph is also a rational function. This can be seen in two ways. The first is the result of Ihara:

**Theorem 1.1.1.** *(See Ihara [9].) Let $X$ be a $(q+1)$-regular graph. Then its zeta function is a rational function of the form*

$$Z(X,u)=\frac{(1-u^2)^{\chi(X)}}{\det(I-Au+qu^2I)},$$

*where $\chi(X)$ is the Euler characteristic of $X$ and $A$ is the adjacency matrix of $X$.*

This theorem is extended to irregular graphs in [1,7,20,8]. The reader is referred to [20] and the references therein for the history and various zeta functions attached to a graph.

Endow two opposite orientations on each edge of $X$. Define the out-neighbor of the directed edge $u\to v$ to be the edges $v\to w$ with $w\neq u$. The (directed) edge adjacency matrix $A_e$ has its rows and columns indexed by the directed edges $e$ of $X$ such that the $ee'$ entry records the number of times $e'$ is an out-neighbor of $e$. Hashimoto [6] observed that $N_n(X)=\operatorname{Tr}A_e^n$ so that

$$Z(X,u)=\frac{1}{\det(I-A_eu)}. \tag{1.1}$$

This gives the second proof of the rationality of the graph zeta function.

A $(q+1)$-regular graph $X$ is called *Ramanujan* if all eigenvalues $\lambda$ of its adjacency matrix $A$ other than $\pm(q+1)$ satisfy $|\lambda|\leqslant 2\sqrt{q}$ (cf. [14]). The Ramanujan graphs are optimal expanders with extremal spectral property. It is easily checked that $X$ is Ramanujan if and only if its zeta function $Z(X,u)$ satisfies the Riemann hypothesis, that is, the poles of $Z(X,u)$ other than $\pm 1$ and $\pm q^{-1}$ (called nontrivial poles) have the same absolute value $q^{-1/2}$ (cf. [20]).

*1.2.* When $q$ is a prime power, the universal cover of a $(q+1)$-regular graph can be identified with the $(q+1)$-regular tree associated to $\mathrm{PGL}_2(F)$ for a nonarchimedean local field $F$ with $q$ elements in its residue field. Denote by $\mathcal{O}_F$ its ring of integers and let $\pi$ be a uniformizer of $F$. The vertices of the tree can be parametrized by the right cosets of the standard maximal compact subgroup $\mathrm{PGL}_2(\mathcal{O}_F)$ and the directed edges by the right cosets of the Iwahori subgroup $\mathcal{I}$ of $\mathrm{PGL}_2(\mathcal{O}_F)$. Moreover, the (vertex) adjacency operator $A$ on the tree is the Hecke operator given by the double coset $\mathrm{PGL}_2(\mathcal{O}_F)\,\mathrm{diag}(1,\pi)\mathrm{PGL}_2(\mathcal{O}_F)$ and the edge adjacency operator $A_e$ is the Iwahori–Hecke operator given by the double coset $\mathcal{I}\,\mathrm{diag}(1,\pi)\mathcal{I}$. One obtains a $(q+1)$-regular graph $X_{\tilde{\Gamma}}$ by taking a left quotient by a torsion-free discrete cocompact subgroup $\tilde{\Gamma}$ of $\mathrm{PGL}_2(F)$.

This set-up has a higher dimensional extension to the Bruhat–Tits building $\mathcal{B}_n$ associated to $\mathrm{PGL}_n(F)$, which is a contractable $(n-1)$-dimensional simplicial complex. Like graphs, one obtains finite complexes $X_\Gamma$ by taking quotients of $\mathcal{B}_n$ by torsion-free discrete cocompact subgroups $\Gamma$ of $\mathrm{PGL}_n(F)$. The concept of Ramanujan complexes was introduced in [13], called Ramanujan hypergraphs there. Three explicit constructions of infinite families of Ramanujan complexes were given in [13], [15] and [18], respectively, using deep results on the Ramanujan conjecture over function fields for automorphic representations of the multiplicative group of a division algebra by Laumon, Rapoport and Stuhler [12] and of $\mathrm{GL}_n$ by Lafforgue [11]. Further, the paper [16] discusses what kind of $\Gamma$ would fail to yield a Ramanujan complex.

To extend zeta functions from graphs to complexes, one seeks a similarly defined zeta function counting closed geodesic tailless cycles in $X_\Gamma$ with the following properties:

(a) it is a rational function with a closed form expression;
(b) it captures both topological and spectral information of $X_\Gamma$; and
(c) it satisfies the Riemann hypothesis if and only if $X_\Gamma$ is a Ramanujan complex.

The purpose of this paper is to present zeta functions with such properties for 2-dimensional complexes which are finite quotients of $\mathcal{B}_3$. This was previously considered in [2] by Deitmar and Hoffman. The zeta functions there were defined differently, and they were not shown to possess the properties (a)–(c). Recently, Fang, Li and Wang in [3] obtained zeta functions for 2-dimensional complexes arising from finite quotients of the building associated to $\mathrm{Sp}_4(F)$.

*1.3.* In what follows, we fix a local field $F$ with $q$ elements in its residue field as before. Let $\mathcal{B}$ denote the Bruhat–Tits building $\mathcal{B}_3$ associated to $\mathrm{PGL}_3(F)$, which is a 2-dimensional contractable simplicial complex. Write $G$ for the group $\mathrm{GL}_3(F)$, $Z$ for its center, $K$ for its standard maximal compact subgroup $\mathrm{GL}_3(\mathcal{O}_F)$, and $B$ for the standard Iwahoric subgroup of $K$. Denote by $E$ the parahoric subgroup of $K$ whose elements modulo $\pi\mathcal{O}_F$ are of the form $\left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right)$. Similar to the case of $\mathrm{PGL}_2(F)$, the vertices, type 1 edges, and pointed chambers of the building $\mathcal{B}$ can be parametrized by the right $KZ$-, $EZ$-, and $BZ$-cosets of $G$, respectively. The Hecke operators $A_1$ and $A_2$ associated to the double cosets $K\,\mathrm{diag}(1,1,\pi)KZ$ and $K\,\mathrm{diag}(1,\pi,\pi)KZ$ describe the type 1 and type 2 out-neighbors of a vertex, the operator $L_E$ associated to the double coset $E\,\mathrm{diag}(1,1,\pi)EZ$ describes the type 1 out-neighbors of a type 1 edge, and the out-neighbors of a pointed chamber are given by the operator $L_B$ associated to the double coset $B\left(\begin{smallmatrix} & 1 & \\ & & 1 \\ \pi & & \end{smallmatrix}\right)BZ$. Details are given in Section 3.

All 1-dimensional paths in $\mathcal{B}$ considered in this paper are contained in the 1-skeleton of $\mathcal{B}$. A 1-geodesic between two vertices in $\mathcal{B}$ is a shortest path in the 1-skeleton of $\mathcal{B}$. As $\mathcal{B}$ is the union of apartments and each apartment is a Euclidean plane, there is a metric on $\mathcal{B}$ so that a geodesic in $\mathcal{B}$ is a straight line contained in an apartment. Thus a 1-geodesic in $\mathcal{B}$ is a geodesic if and only if it consists of edges of the same type. Let $\Gamma$ be a discrete torsion-free cocompact-mod-center subgroup of $G$ satisfying $\mathrm{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$. Denote by $X_\Gamma$ the (finite) quotient $\Gamma\backslash\mathcal{B}$. A 1-geodesic in $X_\Gamma$ is called a geodesic if all of its liftings in $\mathcal{B}$ are geodesics.

*1.4.* For $i = 1$ or 2, the type $i$ edge zeta function of $X_\Gamma$ is defined as

$$Z_{1,i}(X_\Gamma, u) = \prod_{[C]} \left(1 - u^{l_A([C])}\right)^{-1},$$

where $[C]$ runs through the equivalence classes of tailless primitive closed geodesics $C$ in $X_\Gamma$ consisting of edges of type $i$, and $l_A([C])$ is the algebraic length of any geodesic in $[C]$ defined in Section 5.3. The first main result below follows from Proposition 6.1.1 and Theorem 8.5.1. It extends Hashimoto's identity (1.1) to type $i$ edge zeta functions, and gives an explicit formula in terms of conjugacy classes of $\Gamma$ for the number of tailless closed geodesics in $X_\Gamma$ of a given length.

**Theorem A.** *The edge zeta functions are rational functions in $u$ with the following expressions*:

$$Z_{1,i}(X_\Gamma, u) = \frac{1}{\det(1 - L_E u^i)} = \exp\left(\sum_{n \geqslant 1} \frac{N_n(X_\Gamma)}{n} u^{in}\right), \quad i = 1, 2,$$

*where $N_n(X_\Gamma)$ counts the number of closed tailless geodesics of algebraic length $n$ using only type 1 edges in $X_\Gamma$; it is given by*

$$N_n(X_\Gamma) = \sum_{\gamma \in [\Gamma],\ [\gamma] \text{ of type } (n,0)} \mathrm{vol}\big([\gamma]\big)\omega_{[\gamma]}. \tag{1.2}$$

*Here* $[\Gamma]$ *is a set of representatives of conjugacy classes of* $\Gamma$, $[\gamma]$ *is a set of closed geodesics defined by* (4.1), $\operatorname{vol}([\gamma])$ *is given in* (5.4), *and* $\omega_{[\gamma]}$ *is as in Theorem* 7.2.1 *and Proposition* 8.4.4.

In addition to paths formed by directed edges, we also consider paths formed by edge-adjacent chambers, called galleries. The type 1 chamber zeta function of $X_\Gamma$ is defined similar to the type 1 edge zeta function:

$$Z_{2,1}(X_\Gamma, u) = \prod_{[C]} \bigl(1 - u^{l([C])}\bigr)^{-1},$$

where $[C]$ runs through the equivalence classes of primitive closed tailless galleries $C$ in $X_\Gamma$ of type 1, and $l([C])$ is the length of any gallery in $[C]$. (See Section 9.2 for definitions.) Our second main result is a detailed description of $Z_{2,1}(X_\Gamma, u)$, obtained from Proposition 10.1.1 and Corollary 10.2.2.

**Theorem B.** *The type* 1 *chamber zeta function is a rational function with the following expressions*:

$$Z_{2,1}(X_\Gamma, u) = \frac{1}{\det(I - L_B u)} = \exp\biggl(\sum_{n\geqslant 1} \frac{M_n(X_\Gamma)}{n} u^n\biggr), \tag{1.3}$$

*where the number* $M_n(X_\Gamma)$ *of closed tailless galleries in* $X_\Gamma$ *of type* 1 *and length* $n$ *is given below*:

(1) *If* $n = 2m+1$ *is odd, then*

$$M_n(X_\Gamma) = \sum_{\substack{\gamma\in[\Gamma] \text{ ramified rank-one split},\\ [\gamma] \text{ of type } (1,m)}} \operatorname{vol}\bigl([\gamma]\bigr).$$

(2) *If* $n = 2m$ *is even, then*

$$\begin{aligned} M_n(X_\Gamma) = {} & \sum_{\substack{\gamma\in[\Gamma] \text{ split},\\ [\gamma] \text{ of type } (0,m)}} \operatorname{vol}\bigl([\gamma]\bigr)\omega_{[\gamma]} + \sum_{\substack{\gamma\in[\Gamma] \text{ irregular},\\ [\gamma] \text{ of type } (0,m)}} \operatorname{vol}\bigl([\gamma]\bigr) q \\ & + \sum_{\substack{\gamma\in[\Gamma] \text{ unramified rank-one split},\\ [\gamma] \text{ of type } (0,m)}} \operatorname{vol}\bigl([\gamma]\bigr)(\omega_{[\gamma]} - 2) \\ & + \sum_{\substack{\gamma\in[\Gamma] \text{ ramified rank-one split},\\ [\gamma] \text{ of type } (0,m)}} \operatorname{vol}\bigl([\gamma]\bigr)(\omega_{[\gamma]} - 1). \end{aligned}$$

*Here* $[\Gamma]$, $[\gamma]$, $\operatorname{vol}([\gamma])$ *and* $\omega_{[\gamma]}$ *are as in Theorem* A.

The elements of $\Gamma$ are classified in Section 4.2 according to their eigenvalues. It is interesting to compare the above two theorems with the zeta function of the finite regular graph $X_{\tilde{\Gamma}}$ in Section 1.2. The equivalence classes of primitive closed tailless geodesics in $X_{\tilde{\Gamma}}$ correspond bijectively to the conjugacy classes of primitive elements in $\tilde{\Gamma}$ such that corresponding classes have the same length. Thus the zeta function of $X_{\tilde{\Gamma}}$ can be rewritten as

$$Z(X_{\tilde{\Gamma}}, u) = \prod_{[\tilde{\gamma}]} \frac{1}{1 - u^{l(\tilde{\gamma})}} = \exp\left( \sum_{n \geqslant 1} \frac{N_n(X_{\tilde{\Gamma}})}{n} u^n \right), \tag{1.4}$$

where $[\tilde{\gamma}]$ runs through conjugacy classes of primitive elements in $\tilde{\Gamma}$. The number $N_n(X_{\tilde{\Gamma}})$ of tailless geodesic cycles in $X_{\tilde{\Gamma}}$ with length $n$ is equal to

$$N_n(X_{\tilde{\Gamma}}) = \sum_{[\tilde{\gamma}] \text{ primitive},\ l([\tilde{\gamma}])|n} l(\tilde{\gamma}).$$

All nontrivial elements in $\tilde{\Gamma}$ are hyperbolic, analogous to the “split” elements in $\Gamma$. One has $l(\tilde{\gamma}) = \mathrm{vol}([\tilde{\gamma}]) = \mathrm{vol}([\tilde{\gamma}^m]) = \max(\mathrm{ord}_\pi a/b, \mathrm{ord}_\pi b/a)$, where $a$ and $b$ are eigenvalues of $\tilde{\gamma}$, and $\omega_{[\tilde{\gamma}]} = \omega_{[\tilde{\gamma}^m]} = 1$ for all $m \neq 0$. Therefore the formulas for $N_n(X_\Gamma)$ and $M_n(X_\Gamma)$ generalize that for $N_n(X_{\tilde{\Gamma}})$. On the other hand, since both $\mathrm{vol}([\gamma^m])$ and $\omega_{[\gamma^m]}$ vary with the exponent $m$ in a complicated way, there are no simple expressions for the edge and chamber zeta functions of $X_\Gamma$ as Euler products over conjugacy classes in $\Gamma$, similar to (1.4) for graphs.

The zeta function of $X_\Gamma$ is defined as

$$Z(X_\Gamma, u) = Z_{1,1}(X_\Gamma, u) Z_{1,2}(X_\Gamma, u).$$

The explicit expressions of the edge and chamber zeta functions above lead to a new expression for the zeta function $Z(X_\Gamma, u)$, which can be viewed as a 2-dimensional analogue of Theorem 1.1.1.

**Theorem C.** *The zeta function of the finite complex $X_\Gamma = \Gamma \backslash \mathcal{B}$ can be expressed as*

$$Z(X_\Gamma, u) = \frac{(1 - u^3)^{\chi(X_\Gamma)}}{\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I) \det(I + L_B u)}, \tag{1.5}$$

*in which $\chi(X_\Gamma)$ is the Euler characteristic of $X_\Gamma$, $A_1$ and $A_2$ are operators on vertices, and $L_B$ is the operator on pointed chambers in $X_\Gamma$ introduced above.*

Combining Theorems A and B, and noting that the transpose $(L_E)^t$ of $L_E$ is the edge adjacency operator of type 2 edges in $X_\Gamma$, we rephrase the identity (1.5) in terms of the operators on $X_\Gamma$ as

$$\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I-A_1u+qA_2u^2-q^3u^3I)}=\frac{\det(I+L_Bu)}{\det(I-L_Eu)\det(I-(L_E)^tu^2)}. \tag{1.6}$$

Compared to the parallel identity of operators on a $(q+1)$-regular graph $X$:

$$\frac{(1-u^2)^{\chi(X)}}{\det(I-Au+qu^2I)}=\frac{1}{\det(I-A_eu)},$$

the similarity is reminiscent of the zeta functions attached to a surface and a curve over a finite field. It is likely that the identity (1.6) expressed in terms of the operators on the finite complex is a prototype of complex zeta functions in general. Indeed, the identity on zeta functions in [3] for the $GSp_4(F)$ case is formulated after this. Theorem C was proved in [10] from representation-theoretical viewpoint by comparing the eigenvalues of the operators in (1.6), while the proof in this paper explores the combinatorial and group-theoretic viewpoints of the identity.

*1.5.* Our $Z(X_\Gamma,u)$ clearly has properties (a) and (b). Now we discuss its connection with the Riemann hypothesis. The trivial zeros of $\det(I-A_1u+qA_2u^2-q^3u^3I)$ arise from the trivial eigenvalues of $A_1$ and $A_2$ on $X_\Gamma$; they are the roots of $(1-u^3)(1-q^3u^3)(1-q^6u^3)$. We say that $Z(X_\Gamma,u)$ satisfies the Riemann hypothesis if the nontrivial zeros of $\det(I-A_1u+qA_2u^2-q^3u^3I)$ have the same absolute value $q^{-1}$, which is equivalent to $X_\Gamma$ being Ramanujan (cf. [13]).

The zeros of each determinant in (1.6) are computed in [10]; they give rise to a description of the Ramanujan condition in terms of the operators on each dimension.

**Theorem 1.5.1.** *(See [10], Theorem 2.) The following four statements on $X_\Gamma$ are equivalent.*

(1) *$X_\Gamma$ is a Ramanujan complex;*
(2) *The nontrivial zeros of* $\det(I-A_1u+qA_2u^2-q^3u^3I)$ *have absolute value $q^{-1}$;*
(3) *The nontrivial zeros of* $\det(I-L_Eu)$ *have absolute values $q^{-1}$ and $q^{-1/2}$; and*
(4) *The nontrivial zeros of* $\det(I+L_Bu)$ *have absolute values $1$, $q^{-1/2}$ and $q^{-1/4}$.*

Thus the Riemann hypothesis for $Z(X_\Gamma,u)$ is actually a statement concerning the nontrivial zeros of each determinant in (1.6), analogous to the Riemann hypothesis for a surface zeta function.

When $F$ is the completion of a function field $M$ at a place $v$, as constructed in [13], there are infinitely many $\Gamma$ arising from a suitable central division algebra of dimension 9 over $M$ unramified at $v$ such that the polynomial $\det(I-A_1u+qA_2u^2-q^3u^3I)/(1-u^3)(1-q^3u^3)(1-q^6u^3)$ from $X_\Gamma$ agrees with the portion of the zeta function coming from the second $\ell$-adic cohomology of a moduli surface studied in [12]. Computations in [10] imply that

$$\bigl(1-u^3\bigr)\bigl(1-q^6u^3\bigr)\det(I-L_Eu)/\det\bigl(I-A_1u+qA_2u^2-q^3u^3I\bigr)$$

is a polynomial whose zeros have the same absolute value $q^{-1/2}$. It would be interesting to know whether there is any geometric interpretation for suitable choices of $\Gamma$.

*1.6.* We sketch the main ingredients of the proofs of Theorems A, B and C. Each $\gamma \in \Gamma$ has an associated rational form $r_\gamma$, constructed from the eigenvalues of $\gamma$. The base point free homotopy classes of closed 1-geodesics in $X_\Gamma$ are partitioned into sets indexed by the conjugacy classes $[\gamma]$ of $\Gamma$. Theorem 5.4.1 asserts that the 1-geodesics in $[\gamma]$ achieving minimal algebraic (resp. geometric) length, called algebraically (resp. geometrically) minimal, have the same algebraic (resp. geometric) length as $r_\gamma$. Further, the set $[\gamma]$ contains tailless geodesic cycles of type 1 or 2 if and only if $r_\gamma$ has type 1 or 2. In this case, by Proposition 5.7.1, there is no distinction among tailless geodesic, geometrically minimal, and algebraically minimal cycles. Algebraically minimal cycles in $[\gamma]$ afford an explicit algebraic characterization, as shown in Section 7 for $\gamma$ split or irregular and in Section 8 for $\gamma$ rank-one split, and hence are more amenable to computation. In Section 7.2 and Section 8.4 we enumerate the number of cycles in $[\gamma]$ with given algebraic length, along with those of type 1. These numbers establish Theorem A, and they are also used in the proof of Theorem C.

The chamber zeta function defined above counts closed tailless galleries in $X_\Gamma$ of type 1. In Section 9 we first convert this to counting closed pointed galleries in $X_\Gamma$ with respect to the adjacency defined by the operator $L_B$ (Proposition 9.3.1). Then we characterize the closed pointed galleries in Section 9.4. Using this criterion we prove Theorem B by comparing the logarithmic derivatives of the chamber zeta function and the type 2 edge zeta function.

The proof of Theorem C given in Section 11 results from comparing the logarithmic derivatives of both sides of (1.6). More precisely, that of the left hand side counts the number of type 1 tailless closed geodesics in $X_\Gamma$, as given by the logarithmic derivative of $1/\det(I - L_E u)$, and some extra terms arising from sets represented by irregular and rank-one split $\gamma$'s. These extra terms are shown to equal to the logarithmic derivative of $\det(I + L_B u)/\det(I - L_E^t u^2)$ by comparing $N_n(X_\Gamma)$ in Theorem A and $M_n(X_\Gamma)$ in Theorem B. It should be pointed out that while the edge zeta functions count only tailless cycles of types 1 and 2, to prove the identity, we actually consider all cycles, with and without tails. This is similar in spirit to the proof of Theorem 1.1.1 given in [9].

## 2. Hecke operators on $\mathbf{PGL_3}(F)$

### *2.1. Hecke operators*

By the elementary divisor theorem, the group $G$ is equal to the disjoint union of the $KZ$-double cosets

$$T_{n,m} = K \ \operatorname{diag}\bigl(1, \pi^m, \pi^{m+n}\bigr) KZ$$

as $m, n$ run through all non-negative integers. We shall also regard each $T_{n,m}$ as the Hecke operator acting on functions $f \in L^2(G/KZ)$ via

$$T_{n,m} f(gKZ) = \sum_{\alpha KZ \in T_{n,m}/KZ} f(g\alpha KZ).$$

In particular, set

$$A_1 = T_{1,0} \quad \text{and} \quad A_2 = T_{0,1}.$$

*2.2. Recursive relations*

It is well-known that each Hecke operator is a polynomial in $A_1$ and $A_2$. Tamagawa [21] obtained a recursive relation on Hecke operators for $\mathrm{GL}_n(F)$. We prove a different recursive formula adapted for our needs.

**Theorem 2.2.1.**

$$\begin{aligned} q\sum_{k=1}^{\infty} T_{k,0} u^k - (q-1)\left(\sum_{k=1}^{\infty} \sum_{n+2m=k} T_{n,m} u^k\right) \frac{1-q^2u^3}{1-u^3} \\ = u\frac{d}{du} \log \frac{(1-u^3)^r I}{I - A_1 u + A_2 q u^2 - q^3 u^3 I}, \end{aligned} \tag{2.1}$$

*where* $r = \frac{(q+1)(q-1)^2}{3}$.

**Proof.** The Hecke algebra for $G/Z$ is isomorphic to the polynomial ring $\mathbb{C}[z_1, z_2, z_3]^{S_3}/\langle z_1 z_2 z_3 - 1\rangle$, denoted by $H$, under the Satake isomorphism $\psi$ (cf. [19]). Our strategy is to show that the identity holds after applying the Satake isomorphism. For this, we need to compute the values of $\psi$ on $\{T_{n,m}\}$. Using $z_1, z_2, z_3 \in H$ we define a quasi-character $\chi$ on the Borel subgroup $P$ of $G$ by

$$\chi\left(\begin{bmatrix} b_1 & * & * \\ & b_2 & * \\ & & b_3 \end{bmatrix}\right) = z_1^{\mathrm{ord}_\pi(b_1)} z_2^{\mathrm{ord}_\pi(b_2)} z_3^{\mathrm{ord}_\pi(b_3)},$$

and regard it as a map from $P/Z$ to $\mathbb{C}[z_1, z_2, z_3]/\langle z_1 z_2 z_3 - 1\rangle$. Denote by $\delta_P$ the modular character on $P/Z$. Let $\phi$ be the function on $G/Z$ given by

$$\phi(bk) = \chi(b)\delta_P^{1/2}(b) \quad (b \in P,\ k \in K).$$

Then the value of the Satake isomorphism at $T_{n,m}$ is

$$\psi(T_{n,m}) = \sum_{g \in I_{n,m}} \phi(g),$$

where $T_{n,m} = \bigsqcup_{g \in I_{n,m}} gKZ$.

Direct computations give $\psi(A_1) = q(z_1 + z_2 + z_3)$ and $\psi(A_2) = q(z_1z_2 + z_2z_3 + z_3z_1)$ so that

$$\psi\bigl(I - A_1u + qA_2u^2 - q^3u^3I\bigr) = (1 - qz_1u)(1 - qz_2u)(1 - qz_3u),$$

which allows us to get the value of the right hand side of the identity under $\psi$.

For $k \geqslant 1$, let $T_k = \sum_{n+2m=k} T_{n,m}$, and set

$$\sigma_{k,1}(z_1, z_2, z_3) = z_1^k + z_2^k + z_3^k,$$
$$\sigma_{k,2}(z_1, z_2, z_3) = \sum_{1\leqslant a\leqslant k-1} z_1^a z_2^{k-a} + z_2^a z_3^{k-a} + z_3^a z_1^{k-a},$$

and

$$\sigma_{k,3}(z_1, z_2, z_3) = \sum_{a,b,c\geqslant 1,\ a+b+c=k} z_1^a z_2^b z_3^c.$$

For the left hand side of the identity, we compute the coefficient of $z_1^{a_1} z_2^{a_2} z_3^{a_3}$ in $\psi(T_k)$ with $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$ and $a_1 + a_2 + a_3 = k$, then use symmetry to determine $\psi(T_k)$.

It is straightforward to check that the number of cosets $gKZ$ in $\bigsqcup_{n+2m=k} T_{n,m}$ mapped to $z_1^{a_1} z_2^{a_2} z_3^{a_3}$ by $\chi$ is equal to $q^{2a_1+a_2}$ if $a_3 = 0$, and $(q^3 - 1)q^{2a_1+a_2-3}$ if $a_3 > 0$. Moreover, for such $gKZ$ we have $\delta_P(gKZ)^{1/2} = q^{a_3-a_1}$. Therefore the coefficient of $z_1^{a_1} z_2^{a_2} z_3^{a_3}$ in $\psi(T_k)$ is equal to $q^{a_1+a_2+a_3}$ or $q^{a_1+a_2+a_3-3}(q^3 - 1)$ according to $a_3 = 0$ or $a_3 > 0$. By symmetry, this gives rise to

$$\psi(T_k) = q^k\left(\sigma_{k,1} + \sigma_{k,2} + \frac{q^3 - 1}{q^3}\sigma_{k,3}\right).$$

Noting that

$$\sum_{k=1}^{\infty} \sigma_{k,3}u^k = \bigl((z_1z_2z_3)u^3 + (z_1z_2z_3)^2u^6 + \cdots\bigr) \sum_{k=0}^{\infty}(1 + \sigma_{k,1} + \sigma_{k,2})u^k$$
$$= \frac{u^3}{1 - u^3} \sum_{k=0}^{\infty}(1 + \sigma_{k,1} + \sigma_{k,2})u^k,$$

we obtain

$$\psi\left(\sum_{k=1}^{\infty} T_k u^k\right) = \sum_{k=1}^{\infty}\left(\sigma_{k,1} + \sigma_{k,2} + \frac{q^3 - 1}{q^3}\sigma_{k,3}\right)(qu)^k$$
$$= \frac{(q^3 - 1)u^3}{1 - q^3u^3} + \frac{1 - u^3}{1 - q^3u^3} \sum_{k=1}^{\infty}(\sigma_{k,1} + \sigma_{k,2})(qu)^k.$$

On the other hand, put $G_0 = \bigsqcup_{k=1}^{\infty} T_{k,0}$. One verifies that the number of elements in $G_0/KZ$ mapped to $z_1^{a_1} z_2^{a_2} z_3^{a_3}$ by $\chi$ is $q^{2a_1}$ if $a_2 = a_3 = 0$, $(q-1)q^{2a_1+a_2-1}$ if $a_2 > a_3 = 0$, and $(q-1)^2 q^{2a_1+a_2-2}$ if $a_2 \geqslant a_3 > 0$. Therefore,

$$\begin{aligned}
\psi\left(\sum_{k=1}^{\infty} T_{k,0} u^k\right) &= \sum_{k=1}^{\infty}\left(\sigma_{k,1} + \frac{q-1}{q}\sigma_{k,2} + \frac{(q-1)^2}{q^2}\sigma_{k,3}\right)(qu)^k \\
&= \frac{q(q-1)^2 u^3}{1-q^3u^3} + \frac{1+qu^3-2q^2u^3}{1-q^3u^3}\sum_{k=1}^{\infty}\sigma_{k,1}(qu)^k \\
&\quad + \frac{(q-1)(1-q^2u^3)}{q(1-q^3u^3)}\sum_{k=1}^{\infty}\sigma_{k,2}(qu)^k.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\psi\left(q\left(\sum_{k=1}^{\infty} T_{k,0}u^k\right) - (q-1)\left(\sum_{k=1}^{\infty} T_k u^k\right)\frac{1-q^2u^3}{1-u^3}\right) \\
&= \sum_{k=0}^{\infty}\sigma_{k,1}(qu)^k + \frac{(q-1)(q^2-1)u^3}{1-u^3} \\
&= \frac{z_1qu}{1-z_1qu} + \frac{z_2qu}{1-z_2qu} + \frac{z_2qu}{1-z_2qu} - \frac{3ru^3}{1-u^3} \\
&= u\frac{d}{du}\log\frac{(1-u^3)^r}{(1-z_1qu)(1-z_2qu)(1-z_3qu)} \\
&= \psi\left(u\frac{d}{du}\log\frac{(1-u^3)^r}{I - A_1u + A_2qu^2 - q^3u^3 I}\right),
\end{aligned}$$

where $r = \frac{(q+1)(q-1)^2}{3}$. □

## 3. Parametrizations of simplices in $\mathcal{B}$ and operators

### 3.1. Simplicial complex structure on $\mathcal{B}$

The vertices of $\mathcal{B}$ are homothety classes of rank-3 $\mathcal{O}_F$-lattices in $F^3$. Two distinct vertices $[L]$ and $[L']$ are adjacent if they are represented by lattices $L$ and $L'$ such that $\pi L \subset L' \subset L$ (and hence $\pi L' \subset \pi L \subset L'$). Note that $\pi L$ has index $q^3$ in $L$, $L'$ has index $q^i$ in $L$, and $\pi L$ has index $q^{3-i}$ in $L'$, where $i = 1$ or 2. Call $[L']$ a type $i$ out-neighbor of $[L]$ and the directed edge $([L],[L'])$ of type $i$; its opposite $([L'],[L])$ has type $3-i$. An ordered triple $([L],[L'],[L''])$ of three distinct vertices form a pointed chamber if the vertices are represented by lattices $L, L', L''$ such that $\pi L \subset L'' \subset L' \subset L$. Thus $([L'],[L''],[L])$ and $([L''],[L],[L'])$ are also pointed chambers. The unordered triple $\langle[L],[L'],[L'']\rangle$ is called a chamber. Hence a chamber yields three pointed chambers. This describes the simplices in $\mathcal{B}$.

### 3.2. Parametrization of simplices in $\mathcal{B}$

Each element $g \in G$ gives rise to a rank-3 lattice $L_g$ with $\mathcal{O}_F$-basis the three columns of $g$, and all rank-3 lattices over $\mathcal{O}_F$ arise this way. Changing basis of $L_g$ amounts to right multiplication of $g$ by elements in $K$, and lattices equivalent to $L_g$ result from multiplying $g$ by the center $Z$. Thus the assignment $gKZ \to [L_g]$ yields a parametrization of the vertices of $\mathcal{B}$ by $G/KZ$. Note that for each vertex $gKZ$, the number $\operatorname{ord}_\pi \det g \bmod 3$ is well-defined, called the type of $gKZ$.

The group $G$ acts transitively on vertices of $\mathcal{B}$ by left translations. It is straightforward to verify that this action preserves adjacency, the type of edges, and pointed chambers. Moreover, the actions on directed edges and pointed chambers are both transitive. Let $\sigma = \begin{pmatrix} & 1 & \\ & & 1 \\ \pi & & \end{pmatrix}$. It is easy to see that $F_0 := (KZ, \sigma KZ, \sigma^2 KZ)$ is a pointed chamber of $\mathcal{B}$, whose boundary contains the directed type 1 edge $E_0 := (KZ, \sigma KZ)$. Then $E := K \cap \sigma K \sigma^{-1}$ is the standard parahoric subgroup and $B := K \cap \sigma K \sigma^{-1} \cap \sigma^2 K \sigma^{-2}$ is the standard Iwahori subgroup of $K$. As $EZ$ is the stabilizer of $E_0$ and $BZ$ the stabilizer of $F_0$ in $G$, so $G/EZ$ parametrizes all type 1 (and also all type 2) edges and $G/BZ$ parametrizes all pointed chambers of $\mathcal{B}$.

Write $\mathbb{F}_q$ for the residue field of $F$. Counting the number of lines and planes in $\mathbb{F}_q^3$, we see that each vertex has $q^2+q+1$ type 1 neighbors and $q^2+q+1$ type 2 neighbors. Further, the opposite of a type $i$ directed edge has type $3-i$.

### 3.3. Operators on vertices $G/KZ$

The $q^2+q+1$ type 1 neighbors of $gKZ$ are $g\alpha KZ$, where $\alpha KZ$ are the $KZ$-cosets contained in the double coset of the Hecke operator

$$\begin{aligned} A_1 = T_{1,0} &= K \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} KZ \\ &= \bigcup_{a,b \in \mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} \pi & a & b \\ & 1 & \\ & & 1 \end{pmatrix} KZ \bigcup_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} 1 & & \\ & \pi & c \\ & & 1 \end{pmatrix} KZ \cup \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} KZ. \end{aligned}$$

This is because modulo $\pi\mathcal{O}_F$, the columns of these coset representatives generate the distinct 2-dimensional subspaces of $\mathbb{F}_q^3$. The $q^2+q+1$ type 2 neighbors of $gKZ$ can be similarly described using the $KZ$-coset representatives of $A_2 = T_{0,1}$:

$$\begin{pmatrix} \pi & & b \\ & \pi & c \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} \pi & a & \\ & 1 & \\ & & \pi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & \pi & \\ & & \pi \end{pmatrix}, \quad \text{where } a, b, c \in \mathcal{O}_F/\pi\mathcal{O}_F.$$

### *3.4. Operator on type 1 edges $G/EZ$*

Define the out-neighbors of a type 1 edge $(g_1KZ, g_2KZ)$ to be the type 1 edges $(g_2KZ, g_3KZ)$ such that $(g_1KZ, g_2KZ, g_3KZ)$ is not a pointed chamber. Since each line in $\mathbb{F}_q^3$ is contained in $q+1$ planes, among the $q^2+q+1$ type 1 neighbors $g_3KZ$ of $g_2KZ$, exactly $q+1$ of them will form a pointed chamber $(g_1KZ, g_2KZ, g_3KZ)$. Hence a type 1 edge has $q^2$ out-neighbors. Expressed in terms of $EZ$-cosets, the out-neighbors of a type 1 edge $gEZ$ are given by $g\alpha EZ$, where $\alpha EZ$ are the $EZ$-cosets occurring in the double coset

$$L_E = E\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} EZ = \coprod_{x,y\in\mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} 1 & & \\ & 1 & \\ x\pi & y\pi & \pi \end{pmatrix} EZ.$$

It suffices to check the out-neighbors of $EZ$; the rest will follow from the action by $G$. For an $EZ$-coset representative $\alpha = \begin{pmatrix} 1 & & \\ & 1 & \\ x\pi & y\pi & \pi \end{pmatrix}$ of $L_E$, we have $\alpha = \operatorname{diag}(1,1,\pi)k$ for some $k \in K$ so that $\alpha KZ = \sigma KZ$. On the other hand, from $\sigma^{-1}\alpha\sigma = \begin{pmatrix} \pi & x & y \\ & 1 & \\ & & 1 \end{pmatrix} =: \beta$ we see that $\alpha\sigma KZ = \sigma\beta KZ$ is a type 1 neighbor of $\sigma KZ$ not adjacent to $KZ$ and $\alpha EZ = (\alpha KZ, \alpha\sigma KZ) = (\sigma KZ, \sigma\beta KZ)$ runs through all out-neighbors of $EZ = (KZ, \sigma KZ)$ as $\alpha$ varies.

Similar to $A_1$ and $A_2$, $L_E$ may be regarded as the parahoric operator on $L^2(G/EZ)$ sending a function $f \in L^2(G/EZ)$ to the function $L_Ef$ given by

$$L_Ef(gEZ) = \sum_{x,y\in\mathcal{O}_F/\pi\mathcal{O}_F} f\left(g\begin{pmatrix} 1 & & \\ & 1 & \\ x\pi & y\pi & \pi \end{pmatrix} EZ\right).$$

### *3.5. Operator on pointed chambers $G/BZ$*

Define the out-neighbors of a pointed chamber $(g_1KZ, g_2KZ, g_3KZ)$ to be $(g_2KZ, g_3KZ, g_4KZ)$ with $g_4KZ \neq g_1KZ$. As remarked above, there are $q+1$ choices of vertices $g_4KZ$ to make $(g_2KZ, g_3KZ, g_4KZ)$ a pointed chamber, so a pointed chamber has $q$ out-neighbors. In terms of $BZ$ cosets, the out-neighbors of a pointed chamber $gBZ$ are $g\alpha BZ$, where $\alpha BZ$ are the $BZ$-cosets occurring in the Iwahori–Hecke operator

$$L_B = B\begin{pmatrix} 1 & & \\ & & 1 \\ & \pi & \end{pmatrix} BZ = \coprod_{x\in\mathcal{O}_F/\pi\mathcal{O}_F} \begin{pmatrix} 1 & & \\ & & 1 \\ \pi x & \pi & \end{pmatrix} BZ.$$

To see this, given a $BZ$-coset representative $\alpha = \begin{pmatrix} 1 & & \\ & & 1 \\ \pi x & \pi & \end{pmatrix}$ of $L_B$, it is straightforward to check that left multiplication by $\alpha$ sends $BZ = (KZ, \sigma KZ, \sigma^2 KZ)$ to $\alpha BZ =$

$(\alpha KZ, \alpha\sigma KZ, \alpha\sigma^2 KZ) = (\sigma KZ, \sigma^2 KZ, \alpha\sigma^2 KZ)$, where $\alpha\sigma^2 KZ = \begin{pmatrix} & & 1 \\ & \pi & \\ \pi^2 & & \pi x \end{pmatrix} KZ \neq KZ$, so that $\alpha BZ$ runs through different out-neighbors of $BZ$ as $\alpha$ varies.

Similar to the previous cases, $L_B$ may be interpreted as an operator on $L^2(G/BZ)$ which sends a function $f \in L^2(G/BZ)$ to

$$L_B f(gBZ) = \sum_{x \in \mathcal{O}_F/\pi\mathcal{O}_F} f\left(g \begin{pmatrix} 1 & & \\ & & 1 \\ \pi x & \pi & \end{pmatrix} BZ\right).$$

## 4. Finite quotients of $\mathcal{B}$

### *4.1. The group $\Gamma$ and the quotient $X_\Gamma$*

Let $\Gamma$ be a discrete torsion-free subgroup of $G$ such that $\Gamma\backslash G/Z$ is compact. Then $\Gamma$ intersects any compact subgroup of $G$ trivially. Assume that $\Gamma$ intersects any conjugate of $KZ$ trivially and $\operatorname{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$. For instance we may choose $\Gamma$ to be a subgroup of $\mathrm{SL}_3(F)$. See [18, §3] for some examples of such $\Gamma$. The action of $\Gamma$ on $\mathcal{B}$ by left translation is free of fixed points and preserves the types of vertices. The quotient $X_\Gamma = \Gamma\backslash\mathcal{B}$ is a finite connected 2-dimensional simplicial complex, whose vertices are the double cosets $\Gamma\backslash G/KZ$. Since the vertices in an edge or a chamber of $\mathcal{B}$ have different types, each edge or chamber of $\mathcal{B}$ in the quotient remains an edge or a chamber of $X_\Gamma$. Therefore the type 1 (and also type 2) edges in $X_\Gamma$ are parametrized by $\Gamma\backslash G/EZ$, and pointed chambers by $\Gamma\backslash G/BZ$.

The operators $A_1$ and $A_2$ on $G/KZ$, $L_E$ on $G/EZ$ and $L_B$ on $G/BZ$ defined in the previous section induce operators on vertices, types 1 edges and pointed chambers of $X_\Gamma$, respectively. They will be denoted by the same notation. Since $X_\Gamma$ has finitely many vertices, edges and chambers, these operators can also be interpreted combinatorially. More precisely, $A_i$, $i = 1, 2$, is the matrix parametrized by the vertices $v$ of $X_\Gamma$ such that the $vv'$ entry is the number of type $i$ edges from $v$ to $v'$. As such, $A_2$ is the transpose of $A_1$. Similarly $L_E$ has its rows and columns parametrized by the type 1 edges $e$ of $X_\Gamma$ so that the $ee'$ entry denotes the number of times when $e'$ is an out-neighbor of $e$. Since type 2 edges are the opposite of type 1 edges, the transpose $L_E^t$ of $L_E$ describes adjacency relation among type 2 edges of $X_\Gamma$. Likewise, $L_B$ can be viewed as the matrix recording the adjacency relation among the pointed chambers of $X_\Gamma$.

### *4.2. Classification of elements in $\Gamma$*

Observe that every element in $\Gamma$ has an eigenvalue in $F$. Indeed, if $\gamma \in \Gamma$ has no eigenvalues in $F$, then the characteristic polynomial of $\gamma$ is irreducible over $F$. As $\operatorname{ord}_\pi(\det\gamma) = 3m$ for some integer $m$, the eigenvalues of $\gamma' := \pi^{-m}\gamma$ are units in a cubic extension of $F$, which implies that $\gamma$ lies in the intersection of $\Gamma$ with a conjugate of $KZ$, and hence is the identity element. Together with the fact that every element in a discrete cocompact-mod-center lattice is semisimple (see [17], Theorem 1.12), we arrive at

**Theorem 4.2.1** *(Classification of elements in $\Gamma$). Every element $\gamma$ of $\Gamma$ falls in one of the following types*:

1) *$\gamma$ is the identity*;
2) *$\gamma$ is split, that is, it has three distinct eigenvalues in $F^\times$*;
3) *$\gamma$ is ramified/unramified rank-one split, that is, $\gamma$ has three distinct eigenvalues and the field $F\langle\gamma\rangle$ obtained by $F$ joining eigenvalues of $\gamma$ is a ramified/unramified quadratic extension of $F$*;
4) *$\gamma$ is irregular, that is, its eigenvalues are in $F^\times$ and one eigenvalue has multiplicity two.*

The following conclusion on $\Gamma$ shown in [10] results from the closed form expression of the zeta function identity of $X_\Gamma$.

**Proposition 4.2.2.** *(See [10], Corollary 4.) $\Gamma$ contains rank-one split elements.*

## *4.3. Rational form*

Let $\gamma$ be a non-identity element in $\Gamma$ and $L = F\langle\gamma\rangle$ be the field over $F$ generated by the eigenvalues of $\gamma$. If $L = F$, then there is a scalar $z \in Z$ such that $\gamma z$ is conjugate to $r_\gamma := \operatorname{diag}(1, a, b)$ where $1, a, b \in F^\times$ satisfy $\operatorname{ord}_\pi b \geqslant \operatorname{ord}_\pi a \geqslant 0$. If $L$ is a quadratic extension of $F$, fix a generator $\lambda$ so that it is a unit if $L$ is unramified over $F$ and it is a uniformizing element if $L$ is ramified over $F$. Let $x^2 - bx - c$ be the irreducible polynomial of $\lambda$ over $F$ and let $\bar\lambda$ be the Galois conjugate of $\lambda$. Then $\operatorname{ord}_\pi c = 0$ or $1$ according as $L$ is unramified or ramified over $F$ and $\operatorname{ord}_\pi b \geqslant \frac{1}{2}\operatorname{ord}_\pi c$. There are elements $a, e, d \in \mathcal{O}_F$ with at least one of them a unit such that $a$, $e + d\lambda$ and $e + d\bar\lambda$ are the eigenvalues of $\gamma z$ for some scalar $z \in Z$. Consequently, up to a scalar multiple, $\gamma$ is conjugate to $r_\gamma := \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$. Call $r_\gamma$ a *rational form of* $\gamma$. It is unique modulo scalars in $\mathcal{O}_F^\times$, and it depends only on the conjugacy class of $\gamma$.

## *4.4. Homotopy classes of closed paths in $X_\Gamma$*

A cycle in $X_\Gamma$ is a closed path starting at a vertex of $X_\Gamma$ and contained in the 1-skeleton of $X_\Gamma$. Repetition of vertices is allowed. A 1-geodesic between two vertices of $\mathcal{B}$ is a path in the 1-skeleton which uses the minimal number of edges. A cycle in $X_\Gamma$ is called 1-geodesic (resp. geodesic) if it can be lifted to a path in $\mathcal{B}$ which is 1-geodesic (resp. geodesic).

A 1-geodesic cycle in $X_\Gamma$ starting at the vertex $\Gamma gKZ$ can be lifted to a 1-geodesic in $\mathcal{B}$ starting at $gKZ$ and ending at $\gamma gKZ$ for some $\gamma \in \Gamma$. Two such 1-geodesic cycles in $X_\Gamma$ are homotopic in $X_\Gamma$ if and only if their liftings in $\mathcal{B}$ to two 1-geodesics starting at $gKZ$ have the same ending vertex. Denote by $\kappa_\gamma(gKZ)$ the homotopy class of the 1-geodesics from $gKZ$ to $\gamma gKZ$ in $\mathcal{B}$. When projected to $X_\Gamma$, these 1-geodesics become homotopic closed 1-geodesics which use least number of edges among all cycles in its homotopy class in $X_\Gamma$. By abuse of notation, $\kappa_\gamma(gKZ)$ also denotes the homotopy class of its projection in $X_\Gamma$. Thus the fundamental group of $X_\Gamma$ based at $\Gamma gKZ$ is

$$\pi_1(X_\Gamma, \Gamma gKZ) = \{\kappa_\gamma(gKZ)\colon\ \gamma \in \Gamma\}.$$

Since $\Gamma$ has no fixed points, all $\kappa_\gamma(gKZ)$ are distinct and $\pi_1(X_\Gamma, \Gamma gKZ)$ is isomorphic to $\Gamma$.

We shall take all base points into account, but regroup the homotopy classes $\kappa_\gamma(gKZ)$ with respect to the conjugacy classes of $\Gamma$. For each conjugacy class of $\Gamma$ fix a representative $\gamma$ and denote that class by $\langle\gamma\rangle_\Gamma$. Let $[\Gamma] = \{\gamma\}$ be the set of chosen representatives of conjugacy classes. Denote by $C_\Gamma(\gamma)$ the centralizer of $\gamma$ in $\Gamma$. Given $\gamma \in \Gamma$, the map $h \mapsto h^{-1}\gamma h$ is a bijection from $C_\Gamma(\gamma)\backslash\Gamma$ to the conjugacy class $\langle\gamma\rangle_\Gamma$. So $\Gamma = \coprod_{\gamma\in[\Gamma]}\langle\gamma\rangle_\Gamma$ corresponds bijectively to $\coprod_{\gamma\in[\Gamma]} C_\Gamma(\gamma)\backslash\Gamma$. Letting, for each $\gamma \in [\Gamma]$,

$$[\gamma] = \{\kappa_\gamma(gKZ) \mid g \in C_\Gamma(\gamma)\backslash G/KZ\}, \tag{4.1}$$

we obtain the following partition of all vertex-based homotopy classes of closed 1-geodesics in $X_\Gamma$:

$$\begin{aligned}
&\coprod_{\Gamma gKZ\in\Gamma\backslash G/KZ} \pi_1(X_\Gamma, \Gamma gKZ) \\
&\quad = \{\kappa_\gamma(gKZ)\colon\ \gamma \in \Gamma,\ g \in \Gamma\backslash G/KZ\} \\
&\quad = \{\kappa_{h^{-1}\gamma h}(gKZ)\colon\ \gamma \in [\Gamma],\ h \in C_\Gamma(\gamma)\backslash\Gamma,\ g \in \Gamma\backslash G/KZ\}.
\end{aligned}$$

Note that $\kappa_{h^{-1}\gamma h}(gKZ)$ consists of 1-geodesics from $gKZ$ to $h^{-1}\gamma hgKZ$; left multiplication by $h$ yields a bijection from $\kappa_{h^{-1}\gamma h}(gKZ)$ to $\kappa_\gamma(hgKZ)$. When $h \in \Gamma$, both $\kappa_\gamma(hgKZ)$ and $\kappa_{h^{-1}\gamma h}(gKZ)$ project to the same homotopy class of 1-geodesic cycles in $X_\Gamma$. Hence we rewrite

$$\begin{aligned}
\coprod_{\Gamma gKZ\in\Gamma\backslash G/KZ} \pi_1(X_\Gamma, \Gamma gKZ) &= \{\kappa_\gamma(gKZ)\colon\ \gamma \in [\Gamma],\ g \in C_\Gamma(\gamma)\backslash G/KZ\} \\
&= \coprod_{\gamma\in[\Gamma]} [\gamma]
\end{aligned}$$

since $\Gamma$ intersects conjugates of $KZ$ trivially.

## 5. Type and lengths

### 5.1. Algebraic length and canonical algebraic length

Given $g$ in $G$, there is a scalar $z \in F^\times$ such that $g' = zg$ is a matrix in $M_3(\mathcal{O}_F) \smallsetminus \pi M_3(\mathcal{O}_F)$; call $g'$ a minimally integral matrix associated to $g$. It is unique up to multiplication by $\mathcal{O}_F^\times$. Define the *algebraic length* of $g$ to be

$$l_A(g) = \operatorname{ord}_\pi\bigl(\det(g')\bigr).$$

Thus we always have $l_A(g_1 g_2) \leqslant l_A(g_1) + l_A(g_2)$ for $g_1, g_2 \in G$. Extend the definition of algebraic length to elements $g \in \mathrm{GL}_3(L)$ for any finite extension $L$ over $F$ by

$$l_A(g) = \frac{1}{[L:F]} \operatorname{ord}_\pi\bigl(N_{L/F} \circ \det(g')\bigr),$$

where $g'$ is a minimally integral matrix in $M_3(\mathcal{O}_L)$ associated to $g$. Note that $l_A(g)$ is independent of the choice of the field $L$ containing entries of $g$, and multiplication by scalars. Analogous to canonical heights, define the *canonical algebraic length* of $g$ to be

$$L_A(g) = \lim_{n\to\infty} \frac{1}{n} l_A\bigl(g^n\bigr)$$

provided that the limit exists. We exhibit some properties of the canonical algebraic length.

**Proposition 5.1.1.** *Let $g$ be a semisimple element in $G$ and let $d_g$ be a minimally integral diagonal matrix in $\mathrm{GL}_3(L)$ conjugate to $g$ up to a scalar multiple in a finite extension $L$ of $F$. Then*

1. *$L_A(g)$ exists and is equal to $l_A(d_g) = L_A(d_g)$, hence it is invariant under conjugation*;
2. *$L_A(g^n) = nL_A(g)$ for all integers $n \geqslant 1$*;
3. *$L_A(g) \leqslant l_A(g)$.*

**Proof.** By assumption, there is a scalar $z \in L^\times$ and $h \in \mathrm{GL}_3(L)$ such that $zg = hd_g h^{-1}$. Since $d_g$ is a minimally integral diagonal matrix, so is $d_g^n$ and $l_A(d_g^n) = nl_A(d_g)$ for all integers $n > 0$. Thus $L_A(d_g)$ exists and is equal to $l_A(d_g)$. The relation $z^n g^n = hd_g^n h^{-1}$ implies

$$l_A\bigl(d_g^n\bigr) - l_A(h) - l_A\bigl(h^{-1}\bigr) \leqslant l_A\bigl(g^n\bigr) \leqslant l_A\bigl(d_g^n\bigr) + l_A(h) + l_A\bigl(h^{-1}\bigr)$$

for all $n > 0$. Thus $L_A(g)$ also exists and equals to $L_A(d_g)$. The remaining assertions are clear. $\square$

Note that for $g \in T_{n,m}$, its algebraic length is equal to $l_A(g) = n + 2m$. In this case, we say it has *type* $(n, m)$ and *geometric length* $l_G(g) = n + m$.

*5.2. Geodesics and lengths in* $\mathcal{B}$

The building $\mathcal{B}$ is the union of its apartments, and each apartment is a Euclidean plane. It can be shown that all 1-geodesics between two vertices $g_1KZ$ and $g_2KZ$ with $g_1^{-1}g_2 \in T_{n,m}$ lie in the same apartment, and they use $n$ type 1 edges and $m$ type 2 edges. We say that they have type $(n, m)$ (the same type as $g_1^{-1}g_2$), geometric length $n+m = l_G(g_1^{-1}g_2)$ and algebraic length $n+2m = l_A(g_1^{-1}g_2)$. When $m = 0$ (resp. $n = 0$), the path is said to have *type* 1 (resp. *type* 2) for short. Note that the same path traveled backwards is of type $(m, n)$ and has algebraic length $m+2n$. Further, when the path has type 1 or 2, there is only one 1-geodesic between the two vertices, and it is a geodesic in the building $\mathcal{B}$, called a geodesic of type 1 or 2 accordingly.

*5.3. The type and lengths of a homotopy class*

The type, geometric length and algebraic length of a homotopy class $\kappa_\gamma(gKZ)$ of $X_\Gamma$ are those of $\kappa_\gamma(gKZ)$ in $\mathcal{B}$. In other words, If $g^{-1}\gamma g \in T_{n,m}$, then $\kappa_\gamma(gKZ)$ has algebraic length $l_A(\kappa_\gamma(gKZ)) = n+2m$, geometric length $l_G(\kappa_\gamma(gKZ)) = n+m$, and type $(n, m)$. Moreover, $\kappa_\gamma(gKZ)$ is of type 1 if $m = 0$ and type 2 if $n = 0$. By assumption, $\kappa_\gamma(gKZ)$ has positive length if and only if $\gamma$ is not identity.

*5.4. The type and lengths of* $[\gamma]$

Let $\gamma \in [\Gamma]$ be non-identity, and let $r_\gamma$ be a rational form of $\gamma$ defined in Section 4.3. Fix a choice of $P_\gamma \in G$ such that $r_\gamma = (P_\gamma)^{-1}\gamma P_\gamma z_\gamma$ for some $z_\gamma \in Z$. As the centralizers of $\gamma$ and $r_\gamma$ in $G$ are related by $C_G(\gamma) = P_\gamma C_G(r_\gamma)P_\gamma^{-1}$, we have $C_\Gamma(\gamma)P_\gamma = P_\gamma C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$, and $[\gamma]$ may be expressed in two ways:

$$\begin{aligned}[\gamma] &= \{\kappa_\gamma(gKZ) \mid g \in C_\Gamma(\gamma)\backslash G/KZ\} \\ &= \{\kappa_\gamma(P_\gamma gKZ) \mid g \in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash G/KZ\}. \qquad (5.1)\end{aligned}$$

The second expression will facilitate our computations later.

Suppose $r_\gamma \in T_{n,m}$. We say that $[\gamma]$ has type $(n, m)$, algebraic length $l_A([\gamma]) = n+2m$ and geometric length $l_G([\gamma]) = n + m$. For brevity, call $[\gamma]$ of type 1 or 2 according as $m = 0$ or $n = 0$. We shall prove

**Theorem 5.4.1.** *Let* $\gamma \in [\Gamma]$ *and* $\gamma \neq id$. *Then*

$$l_A([\gamma]) = \min_{\kappa_\gamma(gKZ)\in[\gamma]} l_A(\kappa_\gamma(gKZ)) \quad \text{and} \quad l_G([\gamma]) = \min_{\kappa_\gamma(gKZ)\in[\gamma]} l_G(\kappa_\gamma(gKZ)).$$

*Moreover, for $g \in C_G(r_\gamma)$, we have $l_A(\kappa_\gamma(P_\gamma gKZ)) = l_A([\gamma])$, $l_G(\kappa_\gamma(P_\gamma gKZ)) = l_G([\gamma])$ and the type of $\kappa_\gamma(P_\gamma gKZ)$ coincides with the type of $[\gamma]$.*

The second assertion is obvious because $(P_\gamma g)^{-1}\gamma P_\gamma g z_\gamma = g^{-1} r_\gamma g = r_\gamma$ for $g \in C_G(r_\gamma)$. The proof of the first assertion is contained in Theorem 7.1.1 for $\gamma$ split or irregular, and Theorem 8.3.1 for $\gamma$ rank-one split.

Note that $l_A(\kappa_\gamma(gKZ)) \equiv \operatorname{ord}_\pi \det \gamma \pmod 3$, hence $l_A(\kappa_\gamma(gKZ)) = l_A([\gamma]) + 3m$ for some non-negative integer $m$.

### *5.5. Algebraically minimal and geometrically minimal cycles*

In view of Theorem 5.4.1, a homotopy class $\kappa_\gamma(gKZ)$ is called *algebraically minimal* if its algebraic length agrees with $l_A([\gamma])$. Likewise, it is called *geometrically minimal* if its geometric length is $l_G([\gamma])$.

**Proposition 5.5.1.** *$\kappa_\gamma(gKZ)$ is geometrically minimal if and only if $\kappa_\gamma(gKZ)$ and $[\gamma]$ have the same type. Moreover, if $\kappa_\gamma(gKZ)$ is geometrically minimal, then it is also algebraically minimal.*

**Proof.** Suppose $[\gamma]$ is of type $(n,m)$ and $\kappa_\gamma(gKZ)$ is of type $(i,j)$. Applying Theorem 5.4.1 to both $[\gamma]$ and $[\gamma^{-1}]$, we have

$$n + 2m \leqslant i + 2j \qquad \text{and} \qquad 2n + m \leqslant 2i + j.$$

If $\kappa_\gamma(gKZ)$ is geometrically minimal, then $n + m = i + j$. Together with the above inequalities, we conclude that $(i,j) = (n,m)$. On the other hand, if $\kappa_\gamma(gKZ)$ and $[\gamma]$ have the same type, then they obviously have the same algebraic and geometric lengths. Therefore, $\kappa_\gamma(gKZ)$ is both geometrically and algebraically minimal. □

**Corollary 5.5.2.** *If $L_A(\gamma) = l_A(\kappa_\gamma(gKZ))$ and $L_A(\gamma^{-1}) = l_A(\kappa_{\gamma^{-1}}(gKZ))$, then $\kappa_\gamma(gKZ)$ and $[\gamma]$ have the same type. Consequently $\kappa_\gamma(gKZ)$ is geometrically and algebraically minimal.*

**Proof.** Recall that $l_A(\kappa_\gamma(gKZ)) = l_A(g^{-1}\gamma g)$ by definition. It follows from Theorem 5.4.1 and Proposition 5.1.1 that, for $\gamma \in \Gamma$,

$$L_A(\gamma) = L_A(r_\gamma) \leqslant l_A(r_\gamma) = l_A\bigl([\gamma]\bigr) \leqslant l_A\bigl(\kappa_\gamma(gKZ)\bigr) = L_A(\gamma). \tag{5.2}$$

Therefore $l_A([\gamma]) = l_A(\kappa_\gamma(gKZ))$. By the same argument, $l_A([\gamma^{-1}]) = l_A(\kappa_{\gamma^{-1}}(gKZ))$. Suppose $\kappa_\gamma(gKZ)$ is of type $(m,n)$ and $[\gamma]$ is of type $(m',n')$, then we have $2m + n = 2m' + n'$ and $m + 2n = m' + 2n'$, which implies $(m,n) = (m',n')$. □

### 5.6. Tailless cycles

Recall that a 1-geodesic cycle is tailless if it remains 1-geodesic when the starting vertex is changed. We give useful criteria for 1-geodesic type 1 cycles to be tailless.

**Proposition 5.6.1.** *Let $\kappa_\gamma(gKZ)$ be a 1-geodesic cycle of type 1 and geometric length $n > 1$ in $X_\Gamma$. The following statements are equivalent:*

1. *$\kappa_\gamma(gKZ)$ is tailless;*
2. *$\kappa_\gamma(gKZ)$ repeated $m$-times is type 1 geodesic for all $m > 0$;*
3. *$\kappa_\gamma(gKZ)$ is geometrically minimal.*

**Proof.** ($1 \Rightarrow 2$) Suppose $\kappa_\gamma(gKZ)$ is tailless. Let $g_0KZ \to \cdots \to g_{2n}KZ$ be a lifting in $\mathcal{B}$ of $\kappa_\gamma(gKZ)$ repeated 2 times. Then $g_{n+i}KZ = \gamma g_iKZ$ for $0 \leqslant i \leqslant n$. The path $g_iKZ \to g_{i+1}KZ \to \cdots \to g_{i+n}KZ$ is a geodesic for $i = 0, \ldots, n-1$ by assumption. Hence $g_0KZ \to \cdots \to g_{2n}KZ$ is a 1-geodesic in $\mathcal{B}$ and thus $\kappa_\gamma(gKZ)$ repeated twice is a type 1 geodesic cycle in $X_\Gamma$, and so are $\kappa_\gamma(gKZ)$ repeated $m$ times for $m > 0$.

($2 \Rightarrow 3$) Suppose $\kappa_\gamma(gKZ)$ repeated $m$-times is a type 1 geodesic of length $nm$ for all $m > 0$. Then $g^{-1}\gamma^m g \in T_{nm,0}$ for all $m \geqslant 0$ and, by Proposition 5.1.1,

$$L_A(\gamma) = L_A\big(g^{-1}\gamma g\big) = \lim_{m\to\infty} \frac{1}{m} l_A\big(g^{-1}\gamma^m g\big)$$
$$= \lim_{m\to\infty} \frac{1}{m} m l_A\big(g^{-1}\gamma g\big) = n = l_A\big(\kappa_\gamma(gKZ)\big).$$

As $\kappa_{\gamma^{-1}}(gKZ)$ is $\kappa_\gamma(gKZ)$ traveled backwards, it is a 1-geodesic cycle of type 2 and algebraic length $2n$. Further $\kappa_{\gamma^{-1}}(gKZ)$ repeated $m$ times is a type 2 geodesic for all $m > 0$. A similar argument gives $L_A(\gamma^{-1}) = 2n = l_A(\kappa_{\gamma^{-1}}(gKZ))$. We conclude from Corollary 5.5.2 that $\kappa_\gamma(gKZ)$ is geometrically minimal.

($3 \Rightarrow 1$) Suppose $\kappa_\gamma(gKZ)$ is geometrically minimal. Let $C: g_0KZ \to \cdots \to g_{2n}KZ = \gamma^2 g_0KZ$ be a lifting in $\mathcal{B}$ of $\kappa_\gamma(gKZ)$ repeated twice. If we change the starting vertex of $\kappa_\gamma(gKZ)$ to obtain a new cycle, then a lifting in $\mathcal{B}$ of this new cycle is contained in $C$. Thus it suffices to show that $C$ is a 1-geodesic. By Proposition 5.5.1 and the assumption on $\kappa_\gamma(gKZ)$, $[\gamma]$ has type $(n, 0)$ and $r_\gamma$ is of the form

$$\begin{pmatrix} 1 & & \\ & a & \\ & & \pi^n b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \pi^n & \\ & M \end{pmatrix},$$

where $a, b \in \mathcal{O}_F^\times$ and $M \in GL_2(\mathcal{O}_F)$. In both cases we find $[\gamma^2]$ of type $(2n, 0)$ and $l_G([\gamma^2]) = 2n$. As $C$ has geometric length $2n$ and it is homotopic to a 1-geodesic from $g_0KZ$ to $\gamma^2 g_0KZ$, combined with Theorem 5.4.1, we get $2n \geqslant l_G(\kappa_{\gamma^2}(gKZ)) \geqslant l_A([\gamma^2]) = 2n$. This shows that $C$ is a 1-geodesic, as desired. □

**Corollary 5.6.2.** *If $[\gamma]$ contains a tailless geodesic cycle of type $i \in \{1,2\}$, then $[\gamma]$ is of type $i$. In this case the tailless geodesic cycles in $[\gamma]$ are those which are geometrically minimal.*

### 5.7. The number of tailless cycles in $[\gamma]$ and the volume of $[\gamma]$

Since $X_\Gamma$ is finite, it contains only finitely many 1-geodesic cycles with a given algebraic or geometric length. Hence for each $\gamma \in [\Gamma]$, there are only finitely many cycles $\kappa_\gamma(gKZ)$ in $[\gamma]$ with given algebraic or geometric length. Let

$$\Delta_A([\gamma]) = \{gKZ \in G/KZ \mid l_A(\kappa_\gamma(P_\gamma gKZ)) = l_A([\gamma])\}. \tag{5.3}$$

As noted before, $\Delta_A([\gamma]) \supset C_G(r_\gamma)K/KZ$ and is invariant under left multiplication by $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$. Define the *volume* of $[\gamma]$ to be

$$\mathrm{vol}([\gamma]) = \#(C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)). \tag{5.4}$$

It follows from (5.1) and Theorem 5.4.1 that the number of algebraically minimal cycles in $[\gamma]$ is the cardinality of $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$, which is at least $\mathrm{vol}([\gamma])$. This in particular implies the finiteness of $\mathrm{vol}([\gamma])$.

Set

$$\Delta_G([\gamma]) = \{gKZ \in G/KZ \mid l_G(\kappa_\gamma(P_\gamma gKZ)) = l_G([\gamma])\}. \tag{5.5}$$

By Theorem 5.4.1 and Proposition 5.5.1, geometrically minimal cycles in $[\gamma]$ have the same type as $[\gamma]$, and they are also algebraically minimal. Thus $\Delta_G([\gamma]) \subseteq \Delta_A([\gamma])$. The cardinality of $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_G([\gamma])$ counts the number of geometrically minimal cycles in $[\gamma]$. The first statement below for $\gamma$ of type 1 follows from Corollary 7.1.2 for $\gamma$ split or irregular and Corollary 8.3.2 for $\gamma$ rank-one split, hence it also holds for $\gamma$ of type 2. The second statement is from Corollary 5.6.2.

**Proposition 5.7.1.** *Suppose $\gamma \in [\Gamma]$ has type 1 or 2. Then $\Delta_A([\gamma]) = \Delta_G([\gamma])$, i.e., in $[\gamma]$ there is no distinction among algebraically minimal, geometrically minimal, and tailless cycles.*

## 6. Edge zeta functions of $X_\Gamma$

### 6.1. Type 1 and type 2 edge zeta functions of $X_\Gamma$

Recall that a type 1 or 2 tailless 1-geodesic is a geodesic in $X_\Gamma$. A cycle is primitive if it is not a repetition of a shorter cycle. Note that every 1-geodesic cycle $C$ is a repetition of a primitive 1-geodesic cycle $C'$ and the number of 1-geodesic cycles equivalent to $C$ is the geometric length of $C'$, which is equal to the algebraic length of $C'$ if $C$ is of type 1.

Denote by $N_n(X_\Gamma)$ the number of geodesic type 1 tailless cycles in $X_\Gamma$ of length $n$. In terms of the operator $L_E$ described in Section 4.1, we have $N_n(X_\Gamma) = \operatorname{Tr} L_E^n$ for all integers $n \geqslant 1$. For $i = 1, 2$, define the type $i$ edge zeta function of $X_\Gamma$ to be

$$Z_{1,i}(X_\Gamma, u) = \prod_{[C]} \bigl(1 - u^{l_A([C])}\bigr)^{-1}, \tag{6.1}$$

where $[C]$ runs through the equivalence classes of tailless primitive geodesic cycles of type $i$ in $X_\Gamma$, and $l_A([C])$ is the algebraic length of any cycle in $[C]$. Similar to Hashimoto's result [6] for graphs, we have

**Proposition 6.1.1.** *For $i \in \{1, 2\}$ the type $i$ edge zeta function has the following expressions*:

$$Z_{1,i}(X_\Gamma, u) = \exp\left( \sum_{n \geqslant 1} \frac{N_n(X_\Gamma)}{n} u^{in} \right) = \frac{1}{\det(I - L_E u^i)}.$$

**Proof.** Since type 2 edges are the opposite of type 1 edges but with twice algebraic length, the number $N_n(X_\Gamma)$ also counts tailless geodesic cycles in $X_\Gamma$ using only type 2 edges and with algebraic length $2n$. It suffices to prove the case $i = 1$. Taking the logarithmic derivative of (6.1) yields

$$u \frac{d}{du} \log Z_{1,1}(X_\Gamma, u) = \sum_{[C]} \sum_{m \geqslant 1} l_A([C]) u^{l_A([C])m} = \sum_{C} \sum_{m \geqslant 1} u^{l_A(C)m}$$

since each primitive class $[C]$ consists of $l_A([C])$ cycles. Here $C$ runs through all tailless primitive geodesic cycles in $X_\Gamma$ of type 1. Clearly any such $C$ repeated $m$ times is a tailless geodesic cycle with algebraic length $l_A([C])m$, and we obtain all tailless geodesic cycles of type 1 this way. So the last sum can be rewritten as

$$\sum_{n \geqslant 1} N_n(X_\Gamma) u^n = \sum_{n \geqslant 1} \operatorname{Tr} L_E^n u^n,$$

which, by Lemma 3 of [20], is equal to

$$u \frac{d}{du} \det(I - L_E u)^{-1}.$$

This proves the proposition up to constant multiples. Finally noting that, as formal power series in $u$, all three expressions have the same constant term, we conclude the equality. □

In the next two sections, we shall enumerate $N_n(X_\Gamma)$ by relating them to conjugacy classes of $\Gamma$.

## 7. Homotopy cycles in $[\gamma]$ for $\gamma$ split or irregular

Let $|\ |$ be the valuation on $F$ such that $|\pi| = q^{-1}$. In this section we fix a split or irregular $\gamma \in [\Gamma]$ with rational form $r_\gamma = \mathrm{diag}(1, a, b)$, where $\mathrm{ord}_\pi b \geqslant \mathrm{ord}_\pi a \geqslant 0$.

### *7.1. Minimal lengths of homotopy cycles in $[\gamma]$*

We begin by proving the first assertion of Theorem 5.4.1 for the split and irregular cases.

**Theorem 7.1.1.** *Suppose $\gamma \in \Gamma$ is split or irregular with $r_\gamma = \mathrm{diag}(1, a, b)$, where $\mathrm{ord}_\pi b \geqslant \mathrm{ord}_\pi a \geqslant 0$. Then*

(1) $l_A([\gamma]) = \mathrm{ord}_\pi a + \mathrm{ord}_\pi b = \min_{\kappa_\gamma(gKZ)\in[\gamma]} l_A(\kappa_\gamma(gKZ))$, *and*
(2) $l_G([\gamma]) = \mathrm{ord}_\pi b = \min_{\kappa_\gamma(gKZ)\in[\gamma]} l_G(\kappa_\gamma(gKZ))$.

**Proof.** For $\gamma$ split, the centralizer $C_G(r_\gamma)$ consists of the diagonal matrices in $G$ so that, by Iwasawa decomposition, $G = C_G(r_\gamma)UK$, where

$$U = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \,\middle|\, x, y, z \in F \text{ modulo } \mathcal{O}_F \right\}.$$

It suffices to consider the lengths of $\kappa_\gamma(P_\gamma gKZ)$ with $g \in U$. Write $g = \left(\begin{smallmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{smallmatrix}\right)$. Then

$$\begin{aligned}(P_\gamma g)^{-1}\gamma P_\gamma g z_\gamma = g^{-1} r_\gamma g &= \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & & \\ & a & \\ & & b \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x(1-a) & y(1-b) + xz(b-a) \\ & a & z(a-b) \\ & & b \end{pmatrix} \\ &\in K \begin{pmatrix} \pi^{e_1} & & \\ & \pi^{e_2} & \\ & & \pi^{e_3} \end{pmatrix} K\end{aligned}$$

for some integers $e_1 \leqslant e_2 \leqslant e_3$. In fact, for $1 \leqslant i \leqslant 3$, $e_1 + \cdots + e_i = \min_y\{\mathrm{ord}_\pi y\}$ where $y$ runs through the determinant of all $i \times i$ minors of $g^{-1} r_\gamma g$. Consequently,

$$e_1 = \min\bigl\{0, \mathrm{ord}_\pi x(1-a), \mathrm{ord}_\pi z(a-b), \mathrm{ord}_\pi\bigl(y(1-b) + xz(b-a)\bigr)\bigr\} \leqslant 0, \tag{7.1}$$

$$\begin{aligned} e_1 + e_2 = {} & \min\bigl\{\mathrm{ord}_\pi a, \mathrm{ord}_\pi\bigl[x(1-a)z(a-b) - a\bigl(y(1-b) + xz(b-a)\bigr)\bigr]\bigr\} \\ & \leqslant \mathrm{ord}_\pi a, \end{aligned} \tag{7.2}$$

and

$$e_1 + e_2 + e_3 = \operatorname{ord}_\pi a + \operatorname{ord}_\pi b. \tag{7.3}$$

In particular, $e_3 \geqslant \operatorname{ord}_\pi b$ from the last two inequalities. Moreover, we have, for any $g \in G$,

$$\begin{aligned} l_A\big(\kappa_\gamma(P_\gamma gKZ)\big) &= e_3 + e_2 + e_1 - 3e_1 = \operatorname{ord}_\pi a + \operatorname{ord}_\pi b - 3e_1 \\ &\geqslant \operatorname{ord}_\pi a + \operatorname{ord}_\pi b = l_A\big([\gamma]\big) \end{aligned} \tag{7.4}$$

and

$$l_G\big(\kappa_\gamma(P_\gamma gKZ)\big) = e_3 - e_1 \geqslant \operatorname{ord}_\pi b - e_1 \geqslant \operatorname{ord}_\pi b = l_G\big([\gamma]\big). \tag{7.5}$$

As noted before, the equalities in (7.4) and (7.5) hold for $g \in C_G(r_\gamma)$. Therefore

$$l_A\big([\gamma]\big) = \min_{\kappa_\gamma(gKZ)\in[\gamma]} l_A\big(\kappa_\gamma(gKZ)\big) \qquad \text{and} \qquad l_G\big([\gamma]\big) = \min_{\kappa_\gamma(gKZ)\in[\gamma]} l_G\big(\kappa_\gamma(gKZ)\big).$$

For $\gamma$ irregular, we have either $a = b$ or $a = 1$, and the centralizer $C_G(r_\gamma)$ is isomorphic to $\mathrm{GL}_2(F) \times Z$ and $G = C_G(r_\gamma)U_0K$, where $U_0$ consists of the elements in $U$ with $z = 0$ (when $a = b$) or $x = 0$ (when $a = 1$). The above argument still holds. This proves the theorem. □

The proof above shows that if $\kappa_\gamma(P_\gamma gKZ)$ is algebraically minimal, then $e_1 = 0$; and it is geometrically minimal if the additional condition $e_1 + e_2 = \operatorname{ord}_\pi a$ is satisfied. By (7.2), this obviously holds when $\operatorname{ord}_\pi a = 0$, i.e., $\gamma$ has type 1. The proof above also shows that for $\gamma$ irregular of type 1, a tailless $\kappa_\gamma(P_\gamma gKZ)$ has $g \in C_G(r_\gamma)K$. We record this in

**Corollary 7.1.2.** *Suppose $[\gamma]$ has type 1. Then algebraically minimal cycles in $[\gamma]$ are geometrically minimal, hence they agree with the tailless cycles in $[\gamma]$. Moreover, if $\gamma$ is irregular and has type 1, then the tailless cycles in $[\gamma]$ are $\kappa_\gamma(P_\gamma gKZ)$ with $g \in C_G(r_\gamma)K$.*

### *7.2. Counting homotopy cycles in $[\gamma]$ in algebraic length*

As discuss in Section 5.7, the number of algebraically minimal cycles in $[\gamma]$ is the cardinality of $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ with $\Delta_A([\gamma])$ defined by (5.3). We showed in the previous section that for $\gamma$ irregular, $\Delta_A([\gamma]) = C_G(r_\gamma)K/KZ$ so that the number of algebraically tailless cycles in $[\gamma]$ is equal to $\mathrm{vol}([\gamma])$ given by (5.4).

The following theorem, stated in terms of a formal power series, counts the number of homotopy cycles in $[\gamma]$ with given algebraic length.

**Theorem 7.2.1.** *Suppose $\gamma \in [\Gamma]$ is split or irregular with $r_\gamma = \operatorname{diag}(1, a, b)$. Then*

$$\#\big(C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A\big([\gamma]\big)\big) = \operatorname{vol}\big([\gamma]\big)\omega_{[\gamma]}$$

*and*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(gKZ))} = \begin{cases} \operatorname{vol}([\gamma]) \cdot \omega_{[\gamma]} \cdot u^{l_A([\gamma])} \frac{1-u^3}{1-q^3u^3} & \text{if } \gamma \text{ splits}, \\ \operatorname{vol}([\gamma]) \cdot \omega_{[\gamma]} \cdot u^{l_A([\gamma])} \frac{1-u^3}{1-q^2u^3} & \text{if } \gamma \text{ is irregular}. \end{cases}$$

*Here $\operatorname{vol}([\gamma])$ is given by (5.4), $\omega_{[\gamma]} = (|1-a||a-b||b-1|)^{-1}$ for $\gamma$ split, and $\omega_{[\gamma]} = 1$ for $\gamma$ irregular.*

**Proof.** If $\gamma$ is split, the stabilizer $C_G(r_\gamma)$ is the diagonal subgroup of $G$ and $G = C_G(r_\gamma)UKZ$; while if $\gamma$ is irregular, the stabilizer $C_G(r_\gamma)$ is $\mathrm{GL}_2(F)Z$, where $\mathrm{GL}_2(F)$ is imbedded in $G$ as diagonal block $\mathrm{GL}_2(F) \times \{1\}$ (when $a = 1$) or $\{1\} \times \mathrm{GL}_2(F)$ (when $a = b$), and $G = C_G(r_\gamma)U_0KZ$. Here $U$ and $U_0$ are as in the proof of Theorem 7.1.1. Put $W = U$ or $U_0$ according as $\gamma$ split or irregular. Then $(C_G(r_\gamma) \cap KZ)WKZ = WKZ$. Suppose $S$ represents the double cosets in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma) \cap KZ)$, then $S$ has cardinality $\operatorname{vol}([\gamma])$ by (5.4), and

$$G = \bigcup_{h\in S} C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)h\big(C_G(r_\gamma) \cap KZ\big)WKZ = \bigcup_{h\in S} C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)hWKZ.$$

**Lemma 7.2.2.** *For $\gamma \in \Gamma$ split or irregular, the elements $hu$ with $h \in S$ and $u \in W$, where $S$ and $W$ are defined above, are double coset representatives of $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash G/KZ$.*

**Proof.** Suppose $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)huKZ = C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)h'u'KZ$ for $h, h' \in S$ and $u, u' \in W$. Then there is some $c \in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$ such that $huKZ = ch'u'KZ$, i.e., $u^{-1}h^{-1}ch'u' \in KZ$. This together with the definition of $W$ implies that $h^{-1}ch' \in C_G(r_\gamma) \cap KZ$. Therefore $h$ and $h'$ in $S$ represent the same double coset of $C_G(r_\gamma)$, hence $h = h'$. On the other hand, since $\Gamma$ intersects $gZKg^{-1}$ trivially for all $g \in G$ by assumption, the same holds for its conjugate $h^{-1}P_\gamma^{-1}\Gamma P_\gamma h$. Now $h^{-1}ch \in (h^{-1}P_\gamma^{-1}\Gamma P_\gamma h) \cap KZ$, hence is equal to the identity in $G$. So $c = id$ and consequently $uKZ = u'KZ$. This implies $u = u'$ by definition of $W$, as desired. □

Since $\kappa_\gamma(P_\gamma hgKZ)$ and $\kappa_\gamma(P_\gamma gKZ)$ have the same algebraic length for $h \in C_G(r_\gamma)$ and $g \in G$, we get

$$\sum_{\kappa_\gamma(P_\gamma gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(P_\gamma gKZ))} = \operatorname{vol}\big([\gamma]\big) \sum_{v\in W} u^{l_A(\kappa_\gamma(P_\gamma vKZ))},$$

where $W = U$ or $U_0$ according to $\gamma$ split or irregular.

To proceed, we compute the sum on the right hand side. First assume $\gamma$ split so that $W = U$. Given $v \in U$, write $v = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}$. As computed in the proof of Theorem 7.1.1,

$$(P_\gamma v)^{-1}\gamma P_\gamma v z_\gamma = v^{-1} r_\gamma v = \begin{pmatrix} 1 & x(1-a) & y(1-b)+xz(b-a) \\ & a & z(a-b) \\ & & b \end{pmatrix} = (v_{i,j}).$$

For fixed $m \geqslant 0$, we count the number of $v$'s such that $l_A(\kappa_\gamma(P_\gamma vKZ)) \leqslant l_A([\gamma]) + 3m$. By (7.4), the constraints are $|v_{ij}| \leqslant q^m$ for all $1 \leqslant i, j \leqslant 3$. In other words,

$$\bigl|x(1-a)\bigr| \leqslant q^m, \qquad \bigl|z(a-b)\bigr| \leqslant q^m \quad \text{and} \quad \bigl|y(1-b)+xz(b-a)\bigr| \leqslant q^m. \tag{7.6}$$

This implies

$$|x| \leqslant q^m|1-a|^{-1} \quad \text{and} \quad |z| \leqslant q^m|a-b|^{-1}$$

so that the numbers of $x$ and $z$ in $F/\mathcal{O}_F$ are $q^m|1-a|^{-1}$ and $q^m|a-b|^{-1}$, respectively. Further, for chosen $x$ and $z$, there are $q^m|1-b|^{-1}$ choices of $y$ satisfying the above constraint. We have shown

$$\#\bigl\{v \in U \bigm| l_A\bigl(\kappa_\gamma(P_\gamma vKZ)\bigr) = l_A\bigl([\gamma]\bigr)\bigr\} = \bigl(|1-a||a-b||b-1|\bigr)^{-1} = \omega_{[\gamma]} \tag{7.7}$$

and, for $m > 0$,

$$\#\bigl\{v \in U \bigm| l_A\bigl(\kappa_\gamma(P_\gamma vKZ)\bigr) = l_A\bigl([\gamma]\bigr) + 3m\bigr\} = \bigl(q^{3m} - q^{3m-3}\bigr)\omega_{[\gamma]}. \tag{7.8}$$

Put together, this gives

$$\begin{aligned} \sum_{v \in U} u^{l_A(\kappa_\gamma(P_\gamma vKZ))} &= \omega_{[\gamma]} u^{l_A([\gamma])}\Bigl(1 + \sum_{m \geqslant 1}\bigl(q^{3m} - q^{3m-3}\bigr)u^{3m}\Bigr) \\ &= \omega_{[\gamma]} u^{l_A([\gamma])}\Bigl(\frac{1-u^3}{1-q^3u^3}\Bigr). \end{aligned}$$

Next consider the case $\gamma$ irregular so that $W = U_0$. Recall that $U_0$ consists of elements in $U$ with $z = 0$ (when $a = b$) or $x = 0$ (when $a = 1$). Note that $\operatorname{ord}_\pi b > 0$, for otherwise $\gamma$ would lie in the intersection of $\Gamma$ with a conjugate of $K$, which is trivial. Consequently, $1 - b$ is a unit in $\mathcal{O}_F$ so that $|1-b| = 1$. The argument above restricted to elements in $U_0$ goes through as before, but the three inequalities in (7.6) are reduced to two with either $x(1-a) = 0$ or $z(a-b) = 0$. This then shows that the number of nonzero $x$ or $z$ is $q^m - 1$ and the number of $y$ is $q^m$. Hence we obtain

$$\#\bigl\{v \in U_0 \bigm| l_A\bigl(\kappa_\gamma(P_\gamma vKZ)\bigr) = l_A\bigl([\gamma]\bigr) + 3m\bigr\} = q^{2m} - q^{2m-2}, \tag{7.9}$$

which in turn gives

$$\sum_{v\in U_0} u^{l_A(\kappa_\gamma(P_\gamma vKZ))} = u^{l_A([\gamma])}\left(1+\sum_{m\geqslant 1}\left(q^{2m}-q^{2m-2}\right)u^{3m}\right)$$
$$= \omega_{[\gamma]}u^{l_A([\gamma])}\left(\frac{1-u^3}{1-q^2u^3}\right). \qquad \square$$

## 7.3. *Counting homotopy cycles of type* 1 *in* $[\gamma]$

The theorem below gives the number of type 1 homotopy cycles in $[\gamma]$ of given algebraic length. The result depends on the type of $[\gamma]$.

**Theorem 7.3.1.** *With the same notation as in Theorem 7.2.1, we have:*

(A) *If* $[\gamma]$ *splits and is not of type* 1*, then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))} = \operatorname{vol}\left([\gamma]\right)\omega_{[\gamma]}u^{l_A([\gamma])}\left(1-q^{-1}\right)\left(\frac{1-q^2u^3}{1-q^3u^3}\right).$$

*Moreover, no type* 1 *cycles in* $[\gamma]$ *are geometrically minimal.*

(B) *If* $[\gamma]$ *splits and has type* 1*, then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))} = \operatorname{vol}\left([\gamma]\right)\omega_{[\gamma]}u^{l_A([\gamma])}\left(q^{-1}+\left(1-q^{-1}\right)\left(\frac{1-q^2u^3}{1-q^3u^3}\right)\right).$$

(C) *Suppose* $\gamma\in\Gamma$ *is irregular. Then* $[\gamma]$ *contains no cycles of type* 1 *if* $[\gamma]$ *is not of type* 1*; while if* $[\gamma]$ *has type* 1*, then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))} = \operatorname{vol}\left([\gamma]\right)\omega_{[\gamma]}u^{l_A([\gamma])}.$$

**Proof.** For $\gamma\in\Gamma$ split or irregular, we have $r_\gamma=\operatorname{diag}(1,a,b)$, so $[\gamma]$ has type $(\operatorname{ord}_\pi b-\operatorname{ord}_\pi a, \operatorname{ord}_\pi a)$ and $l_A([\gamma])=\operatorname{ord}_\pi b+\operatorname{ord}_\pi a$. It has type 1 if and only of $\operatorname{ord}_\pi a=0$. The argument is similar to the proof of Theorem 7.2.1; the difference is that we only need to consider those $v\in W$ such that $\kappa_\gamma(P_\gamma vKZ)$ has type 1. Here $W=U$ or $U_0$ according as $\gamma$ split or irregular. So we determine the cardinality of the set

$$\left\{v\in W \mid l_G\left(\kappa_\gamma(P_\gamma vKZ)\right)=l_A\left(\kappa_\gamma(P_\gamma vKZ)\right)=l_A\left([\gamma]\right)+3m=\operatorname{ord}_\pi b+\operatorname{ord}_\pi a+3m\right\}$$

for each $m\geqslant 0$. As before, writing $v$ as $\left(\begin{smallmatrix}1&x&y\\&1&z\\&&1\end{smallmatrix}\right)$ and following the proofs of Theorem 7.2.1 and Theorem 7.1.1, we arrive at the following constraints on $x,y,z\in F/\mathcal{O}_F$:

(1) $\min\{0,\operatorname{ord}_\pi x(1-a),\operatorname{ord}_\pi z(a-b),\operatorname{ord}_\pi(y(1-b)+xz(b-a))\}=-m$, and
(2) $\min\{\operatorname{ord}_\pi a,\operatorname{ord}_\pi[x(1-a)z(a-b)-a(y(1-b)+xz(b-a))]\}=-2m$.

For $m > 0$, the two constraints are equivalent to

(3) $\operatorname{ord}_\pi x(1-a) = -m = \operatorname{ord}_\pi z(a-b)$ and $\operatorname{ord}_\pi(y(1-b)+xz(b-a)) \geqslant -m$.

First assume $\gamma$ splits. The number of $x$ is $(1-q^{-1})q^m|1-a|^{-1}$, the number of $z$ is $(1-q^{-1})q^m|a-b|^{-1}$, and the number of $y$ is $q^m|1-b|^{-1}$ so that the total number of $v$ is $(1-q^{-1})^2q^{3m}\omega_{[\gamma]}$. For $m = 0$ and $\operatorname{ord}_\pi a > 0$, the same constraint (3) holds. In this case the number of $x$ is $|1-a|^{-1} = 1$, the number of $y$ is $|1-b|^{-1} = 1$ and the number of $z$ is $(1-q^{-1})|a-b|^{-1}$ so that the total number of $v$ is $(1-q^{-1})\omega_{[\gamma]}$. Finally, when $m = \operatorname{ord}_\pi a = 0$, the constraints (1) and (2) are equivalent to

(4) $\operatorname{ord}_\pi x(1-a) \geqslant 0$, $\operatorname{ord}_\pi z(a-b) \geqslant 0$ and $\operatorname{ord}_\pi(y(1-b)+xz(b-a)) \geqslant 0$.

Hence the numbers of $x$, $y$ and $z$ are $|1-a|^{-1}$, $|1-b|^{-1}$ and $|a-b|^{-1}$, respectively, so that the number of $v$ is $\omega_{[\gamma]}$. Note that $y = z = 0$ in this case.

Since $\operatorname{vol}([\gamma])\omega_{[\gamma]}$ is present in all cases, it suffices to compute

$$\frac{1}{\operatorname{vol}([\gamma])\omega_{[\gamma]}} \sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))}.$$

In case $\operatorname{ord}_\pi a > 0$, namely $[\gamma]$ does not have type 1, this sum is equal to

$$u^{l_A([\gamma])}\left(1-q^{-1}+\sum_{m\geqslant 1}(1-q^{-1})^2q^{3m}u^{3m}\right) = u^{l_A([\gamma])}(1-q^{-1})\left(\frac{1-q^2u^3}{1-q^3u^3}\right),$$

and in case $\operatorname{ord}_\pi a = 0$, namely $[\gamma]$ has type 1, it is equal to

$$u^{l_A([\gamma])}\left(1+\sum_{m\geqslant 1}(1-q^{-1})^2q^{3m}u^{3m}\right) = u^{l_A([\gamma])}\left(q^{-1}+(1-q^{-1})\left(\frac{1-q^2u^3}{1-q^3u^3}\right)\right).$$

This proves (A) and (B).

When $\gamma$ is irregular, either $a = 1$ or $a = b$, so (3) never holds and there are no cycles in $[\gamma]$ of type 1 and algebraic length $> l_A([\gamma])$. Further, there are $\operatorname{vol}([\gamma])$ cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ and they have the same type as $[\gamma]$. This proves the assertion (C). □

Contained in the proof above is the following statement.

**Corollary 7.3.2.** *Suppose $\gamma \in \Gamma$ is split or irregular with $r_\gamma = \operatorname{diag}(1,a,b)$. Assume that $\gamma$ has type 1, $a \in \mathcal{O}_F^\times$ and $n = \operatorname{ord}_\pi b$. Let $\delta = \delta([\gamma]) = \operatorname{ord}_\pi(1-a)$ for $\gamma$ split, and $\delta = 0$ for $\gamma$ irregular. Then*

$$\Delta_A\big([\gamma]\big) = \Bigg\{ hv_x KZ \;\Big|\; h \in C_G(r_\gamma)/\big(C_G(r_\gamma)\cap KZ\big),\ v_x = \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \textit{with } x \in \pi^{-\delta}\mathcal{O}_F/\mathcal{O}_F \Bigg\}.$$

## 8. Homotopy cycles in $[\gamma]$ for $\gamma$ rank-one split

In this section we fix a rank-one split $\gamma \in [\Gamma]$ whose eigenvalues generate a quadratic extension $L = F(\lambda)$ of $F$. Here $\lambda$ is a unit or uniformizer in $L$ according as $L$ is unramified or ramified over $F$, i.e., $\gamma$ is unramified or ramified rank-one split. Let $r_\gamma = \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$ be a rational form of $\gamma$ as in Section 4.3. Fix a matrix $P_\gamma$ so that $P_\gamma^{-1}\gamma P_\gamma z_\gamma = r_\gamma$ for some $z_\gamma \in Z$.

### *8.1. The centralizers of $r_\gamma$ for $\gamma$ rank-one split*

Embed $L^\times$ in $\mathrm{GL}_2(F)$ as the subgroup

$$\left\{ \begin{pmatrix} u & vc \\ v & u+vb \end{pmatrix} \;\Big|\; u, v \in F,\ \text{not both zero} \right\}, \tag{8.1}$$

which is further imbedded in $\mathrm{GL}_3(F)$ as $\left\{ \begin{pmatrix} 1 & & \\ & u & vc \\ & v & u+vb \end{pmatrix} \right\}$. Note that $C_G(r_\gamma) = L^\times Z$. Further $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)$ has cardinality $\mathrm{vol}([\gamma])$ by (5.4).

The group of units $\mathcal{U}_L$ of $L^\times$ is contained in $K$. If $L$ is unramified over $F$, then $L^\times = \langle\pi\rangle\mathcal{U}_L$ so that $C_G(r_\gamma)K/KZ$ is represented by the vertices $\mathrm{diag}(\pi^n, 1, 1)KZ$, $n \in \mathbb{Z}$, on a line in $\mathcal{B}$, and $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)$ represented by $\mathrm{diag}(\pi^n, 1, 1)KZ$, $n \bmod \mathrm{vol}([\gamma])$. If $L$ is ramified over $F$, then $L^\times = \langle\pi_L\rangle\mathcal{U}_L$, where the uniformizer $\pi_L$ does not lie in $F$ and $\pi_L^2$ differs from $\pi$ by a unit multiple. In this case $C_G(r_\gamma)K/KZ$ is represented by the vertices $\mathrm{diag}(\pi^n, 1, 1)KZ$ and $\mathrm{diag}(\pi^n, 1, 1)\pi_L KZ$, $n \in \mathbb{Z}$, lying on two lines in $\mathcal{B}$. There are two possibilities for $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$:

Case (i). The vertices in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)KZ/KZ$ are contained in the line $\mathrm{diag}(\pi^n, 1, 1)KZ$, $n \in \mathbb{Z}$. Then $\mathrm{vol}([\gamma])$ is even so that $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)$ is represented by the vertices $\mathrm{diag}(\pi^n, 1, 1)KZ$ and $\mathrm{diag}(\pi^n, 1, 1)\pi_L KZ$, $n \bmod \mathrm{vol}([\gamma])/2$.

Case (ii). $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)KZ/KZ$ contains a vertex on the line $\mathrm{diag}(\pi^n, 1, 1)\pi_L KZ$, $n \in \mathbb{Z}$. Let $y \in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$ be such that $yKZ = \mathrm{diag}(\pi^N, 1, 1)\pi_L KZ$ has the least non-negative $N$. Then $y$ generates the group $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)$, $y^2 KZ = \mathrm{diag}(\pi^{2N-1}, 1, 1)KZ$, $\mathrm{vol}([\gamma]) = 2N - 1$ is odd, and $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)$ is represented by the vertices $\mathrm{diag}(\pi^n, 1, 1)KZ$, $0 \leqslant n \leqslant N - 1 = (\mathrm{vol}([\gamma]) - 1)/2$, and $\mathrm{diag}(\pi^n, 1, 1)\pi_L KZ$, $0 \leqslant n \leqslant N - 2 = (\mathrm{vol}([\gamma]) - 3)/2$.

*8.2. Double coset representatives of $C_G(r_\gamma)\backslash G/KZ$*

**Proposition 8.2.1.** *The double cosets in $C_G(r_\gamma)\backslash G/KZ$ are represented by elements in*

$$S = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & 0 \\ & & \pi^n \end{pmatrix} \;\middle|\; x, y \in F/\mathcal{O}_F,\ n \geqslant 0 \right\}.$$

**Proof.** Write an element $g \in G$ as $wk$ for some upper triangular $w$ and some $k \in K$. Since $C_G(r_\gamma) = L^\times Z$, modulo the center $Z$, we may assume that $w = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & \pi^n \end{pmatrix}$, where $x, y, z \in F/\mathcal{O}_F$ and $n \in \mathbb{Z}$. We are reduced to proving

$$\mathrm{GL}_2(F) = \coprod_{n \geqslant 0} L^\times \begin{pmatrix} 1 & \\ & \pi^n \end{pmatrix} GL_2(\mathcal{O}_F), \tag{8.2}$$

where $L^\times$ is given by (8.1) (cf. [4], Lemma 1 on p. 30).

First we check the disjoint union. Suppose otherwise. Then there exist $m \neq n$ and $g$ satisfying

$$g \in L^\times \cap \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_F) \begin{pmatrix} 1 & \\ & \pi^{-n} \end{pmatrix}.$$

Replacing $g$ by its inverse if necessary, we may assume $m > n$. Write $g = \begin{pmatrix} x & y\pi^{-n} \\ \pi^m z & w\pi^{m-n} \end{pmatrix} = \begin{pmatrix} u & vc \\ v & u+vb \end{pmatrix}$ for some $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$ and $u, v \in F$. Comparing entries, we find $y\pi^{-n} = cz\pi^m$ and $w\pi^{m-n} = x + zb\pi^m$. Since $x, y, z, w, b, c$ are all integral, we conclude that $x$ is a nonunit and hence $z$ and $y$ should both be units, but then $y\pi^{-n} = cz\pi^m$ cannot hold by checking the order of both sides.

Next we prove equality. Let $w = \begin{pmatrix} 1 & z \\ & \pi^m \end{pmatrix} \in \mathrm{GL}_2(F)$. Observe that for $m \geqslant 0$,

$$\begin{pmatrix} 0 & c \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi^m \end{pmatrix} = \begin{pmatrix} 0 & c\pi^m \\ 1 & b\pi^m \end{pmatrix} = \begin{pmatrix} c\pi^m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b\pi^m \end{pmatrix},$$

showing that $\begin{pmatrix} 1 & 0 \\ 0 & \pi^m \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \pi^{-m-\mathrm{ord}_\pi c} \end{pmatrix}$ represent the same double coset. Since $\mathrm{ord}_\pi c = 0$ or 1, only such diagonal matrices with $m \geqslant 0$ are needed as double coset representatives. Thus we assume $\mathrm{ord}_\pi z < 0$. It suffices to reduce $w$ to a diagonal matrix via left multiplication by elements in $L^\times$ and right multiplication by elements in $\mathrm{GL}_2(\mathcal{O}_F)$.

Case (I). $0 > \mathrm{ord}_\pi z \geqslant m + \mathrm{ord}_\pi c$. Choose $v \in \mathcal{O}_F$ with $\mathrm{ord}_\pi v + m + \mathrm{ord}_\pi c = \mathrm{ord}_\pi z$ and $u$ a unit in $\mathcal{O}_F$ satisfying $uz = -cv\pi^m$. Then $\begin{pmatrix} u & vc \\ v & u+vb \end{pmatrix} w = \begin{pmatrix} u & 0 \\ v & vz+(u+vb)\pi^m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi^m \end{pmatrix} k$ for some $k \in \mathrm{GL}_2(\mathcal{O}_F)$. Here we used the fact that $u(u+vb) - v^2c$ is a unit. It is obvious if $v$ or $c$ (and hence $b$) is not a unit; when $v$ and $c$ are both units, this results from the irreducibility of $x^2 - bx - c$.

Case (II). $m + \operatorname{ord}_\pi c > \operatorname{ord}_\pi z$. Choose $u \in \mathcal{O}_F$ with $\operatorname{ord}_\pi u + \operatorname{ord}_\pi z = m + \operatorname{ord}_\pi c$ and $v$ a unit such that $uz = -vc\pi^m$. Then $\begin{pmatrix} u & vc \\ v & u+vb \end{pmatrix} w = \begin{pmatrix} u & 0 \\ v & vz+(u+vb)\pi^m \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & z \end{pmatrix} k$ for some $k \in \mathrm{GL}_2(\mathcal{O}_F)$.

In both cases we have shown that $w$ lies in the right hand side of (8.2), therefore (8.2) holds. This proves the proposition. □

*8.3. Minimal lengths of cycles in $[\gamma]$*

The type of $[\gamma]$, as defined in Section 5.4, is $(n,m)$ such that $r_\gamma \in T_{n,m} = K \operatorname{diag}(1, \pi^m, \pi^{n+m})KZ$. Observe that $\operatorname{ord}_\pi \det \gamma \equiv \operatorname{ord}_\pi \det r_\gamma \equiv \operatorname{ord}_\pi a(e + d\lambda)(e + d\bar{\lambda}) \equiv 0 \bmod 3$ by the assumption on $\Gamma$. Hence if $e + d\lambda$ is a unit in $L$, then at least one of $e, d$ is a unit and $a$ is not a unit. Consequently, $[\gamma]$ has type $(\operatorname{ord}_\pi a, 0)$. Next assume $e + d\lambda$ is not a unit. We distinguish two cases. If $L$ is unramified over $F$ (hence $\lambda$ is a unit), then both $e$ and $d$ are non-units and $a$ is a unit; in this case $[\gamma]$ has type $(0, \min(\operatorname{ord}_\pi e, \operatorname{ord}_\pi d))$. If $L$ is ramified over $F$ (hence $\lambda$ is a uniformizer of $L$), then there are two possibilities:

(i) $\operatorname{ord}_\pi(e + d\lambda)(e + d\bar{\lambda}) = 1$. This happens if and only if $e$ is a non-unit, $d$ is a unit, and $\operatorname{ord}_\pi a \geqslant 2$; in this case $[\gamma]$ has type $(\operatorname{ord}_\pi a - 1, 1)$.

(ii) $\operatorname{ord}_\pi(e + d\lambda)(e + d\bar{\lambda}) > 1$. This happens if and only if both $e$ and $d$ are non-units and $a$ is a unit; in this case $[\gamma]$ has type $(0, \operatorname{ord}_\pi e)$ if $\operatorname{ord}_\pi e \leqslant \operatorname{ord}_\pi d$, and type $(1, \operatorname{ord}_\pi d)$ if $\operatorname{ord}_\pi e > \operatorname{ord}_\pi d$.

This proves the first assertion of

**Theorem 8.3.1.** *Let $\gamma$ be a rank-one split element in $[\Gamma]$ with rational form $r_\gamma = \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$. Suppose that $r_\gamma \in K \operatorname{diag}(1, \pi^m, \pi^{m+n})KZ$. Then*

(1) *The type $(n,m)$ of $[\gamma]$ is as follows.*
 (1.i) *If $\operatorname{ord}_\pi c = 0$, then $(n,m) = (\operatorname{ord}_\pi a, \min\{\operatorname{ord}_\pi e, \operatorname{ord}_\pi d\})$.*
 (1.ii) *If $\operatorname{ord}_\pi c = 1$, then $(n,m) = (\operatorname{ord}_\pi a, \operatorname{ord}_\pi e)$ provided that $\operatorname{ord}_\pi e \leqslant \operatorname{ord}_\pi d$, otherwise $(n,m) = (\max\{\operatorname{ord}_\pi a - 1, 1\}, \max\{\operatorname{ord}_\pi d, 1\})$.*

(2) *$l_A([\gamma]) = \min_{\kappa_\gamma(gKZ) \in [\gamma]} l_A(\kappa_\gamma(gKZ)) = \operatorname{ord}_\pi a(e^2 + edb - cd^2) = n + 2m$.*

(3) *$l_G([\gamma]) = \min_{\kappa_\gamma(gKZ) \in [\gamma]} l_G(\kappa_\gamma(gKZ)) = n + m$.*

This theorem combined with Theorem 7.1.1 completes the proof of Theorem 5.4.1.

**Remark.** If $\gamma$ is ramified rank-one split and $[\gamma]$ has type $(n,1)$, then $[\gamma^2]$ has type $(2n+1, 0)$.

**Proof.** It remains to show that the algebraic and geometric lengths of the cycles in $[\gamma]$ are at least those of $[\gamma]$ since, as observed before, the cycles $\kappa_\gamma(P_\gamma gKZ)$ with $g \in C_G(r_\gamma)$ have the same algebraic and geometric lengths as $[\gamma]$. By Proposition 8.2.1, it suffices to compute $(P_\gamma g)^{-1}\gamma P_\gamma g z_\gamma = g^{-1} r_\gamma g$ for $g \in S$. Let $g = \left(\begin{smallmatrix} 1 & x & y \\ & 1 & 0 \\ & & \pi^i \end{smallmatrix}\right)$, where $x, y \in F/\mathcal{O}_F$ and $i \geqslant 0$. Then

$$\begin{aligned} g^{-1} r_\gamma g &= \begin{pmatrix} 1 & -x & -y\pi^{-i} \\ & 1 & 0 \\ & & \pi^{-i} \end{pmatrix} \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & 0 \\ & & \pi^i \end{pmatrix} \\ &= \begin{pmatrix} a & (a-e)x - dy\pi^{-i} & (a-e-db)y - cdx\pi^i \\ & e & dc\pi^i \\ & d\pi^{-i} & e+db \end{pmatrix} \\ &\in K \begin{pmatrix} \pi^{e_1} & & \\ & \pi^{e_2} & \\ & & \pi^{e_3} \end{pmatrix} K. \end{aligned}$$

Here $e_1 \leqslant e_2 \leqslant e_3$, and as in the proof of Theorem 7.1.1, we have

$$e_1 \leqslant \min\{\operatorname{ord}_\pi a, -i + \operatorname{ord}_\pi d, \operatorname{ord}_\pi e\} \leqslant \min\{\operatorname{ord}_\pi a, \operatorname{ord}_\pi d, \operatorname{ord}_\pi e\} = 0, \tag{8.3}$$

$$\begin{aligned} e_1 + e_2 &\leqslant \min\bigl\{\operatorname{ord}_\pi ae, -i + \operatorname{ord}_\pi ad, \operatorname{ord}_\pi\bigl(e^2 + bed - cd^2\bigr)\bigr\} \\ &\leqslant \min\bigl\{\operatorname{ord}_\pi ae, \operatorname{ord}_\pi ad, \operatorname{ord}_\pi\bigl(e^2 + bed - cd^2\bigr)\bigr\} = m, \end{aligned} \tag{8.4}$$

and

$$e_1 + e_2 + e_3 = \operatorname{ord}_\pi a\bigl(e^2 + bed - cd^2\bigr) = n + 2m, \tag{8.5}$$

in which the last upper bound for $e_1+e_2$ can be verified using the statement (1). Therefore $l_A(\kappa_\gamma(P_\gamma gKZ)) = e_1 + e_2 + e_3 - 3e_1 \geqslant e_1 + e_2 + e_3 = n + 2m = l_A([\gamma])$ since $e_1 \leqslant 0$. The inequalities (8.4) and (8.5) together give the lower bound $e_3 \geqslant n + 2m - m = n + m$, which in turn implies $l_G(\kappa_\gamma(P_\gamma gKZ)) = e_3 - e_1 \geqslant n + m$. This proves the theorem. □

As shown in the proof above, if $[\gamma]$ has type 1, i.e. $m = 0$, then an algebraically minimal cycle in $[\gamma]$ satisfies $e_1 = 0$, which implies $e_1 + e_2 \geqslant 0$ and hence $e_1 + e_2 = 0$ by (8.4) and $e_3 = n$ by (8.5). This proves

**Corollary 8.3.2.** *Suppose $\gamma \in [\Gamma]$ is rank-one split. If $[\gamma]$ has type 1, then the algebraically minimal cycles in $[\gamma]$ coincide with the geometrically minimal (hence tailless) cycles in $[\gamma]$.*

### *8.4. Counting the number of cycles in $[\gamma]$ in algebraic length*

As observed before, given $s \in S$, the cycles $\kappa_\gamma(P_\gamma gKZ)$ have the same algebraic length for all $gKZ \in C_G(r_\gamma)sK/KZ$. Since $S$ represents the double coset $C_G(r_\gamma)\backslash G/KZ$, to count the number of cycles in $[\gamma]$ of a given length, we need to determine the cardinality of $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)sK/KZ$ for $s \in S$. For this, we may take as representatives the product of representatives of $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)$ (independent of $s$) by the representatives of $(C_G(r_\gamma)\cap KZ)sK/KZ$. The number of the former representatives is $\mathrm{vol}([\gamma])$ by (5.4).

It remains to compute the cardinality of the latter. Recall that $L^\times \cap K$ consists of the units in $L^\times$, which are identified with the matrices

$$\mathcal{U}_L = \left\{ \begin{pmatrix} u & vc \\ v & u+vb \end{pmatrix} \;\middle|\; u, v \in \mathcal{O}_F,\ u^2 + uvb - cv^2 \text{ is a unit} \right\}.$$

Denote by $K'$ the group $GL_2(\mathcal{O}_F)$. As analyzed in the proof of Proposition 8.2.1, we are reduced to counting, for each $m \geqslant 0$, the cardinality of $\mathcal{U}_L \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} K'/K'$.

**Proposition 8.4.1.**

$$\#\left[\mathcal{U}_L \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} K'/K'\right] = \begin{cases} 1 & \textit{when } m = 0, \\ q^m & \textit{when } m \geqslant 1 \textit{ and } \mathrm{ord}_\pi c = 1, \\ q^m + q^{m-1} & \textit{when } m \geqslant 1 \textit{ and } \mathrm{ord}_\pi c = 0. \end{cases}$$

**Proof.** It is clear that the cardinality is 1 when $m = 0$. Thus assume $m \geqslant 1$.

Case (I). $\mathrm{ord}_\pi c = 1$. Then any $\begin{pmatrix} u & vc \\ v & u+vb \end{pmatrix} \in \mathcal{U}_L$ satisfies $u \in \mathcal{O}_F^\times$. For $n \geqslant 0$, let

$$\mathcal{U}_L(n) = \left\{ \begin{pmatrix} u & vc\pi^n \\ v\pi^n & u+vb\pi^n \end{pmatrix} \in \mathcal{U}_L \;\middle|\; u, v \in \mathcal{O}_F^\times \right\}$$

so that

$$\mathcal{U}_L = \mathcal{U}_L(\infty) \bigcup_{n\geqslant 0} \mathcal{U}_L(n),$$

where

$$\mathcal{U}_L(\infty) = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \;\middle|\; u \in \mathcal{O}_F^\times \right\}.$$

One verifies that

$$\mathcal{U}_L(n) \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} K' = \bigcup_{u \in \mathcal{O}_F^\times/\pi^{m-n}\mathcal{O}_F} \begin{pmatrix} \pi^{m-n} & u \\ & \pi^n \end{pmatrix} K'$$

for $0 \leqslant n < m$, and

$$\mathcal{U}_L(n) \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} K' = \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} K'$$

for $n \geqslant m$ and $n = \infty$. Therefore

$$\#\left[\mathcal{U}_L\begin{pmatrix}1 & \\ & \pi^m\end{pmatrix}K'/K'\right] = 1 + \sum_{0\leqslant n<m}\left(q^{m-n} - q^{m-n-1}\right) = q^m.$$

Case (II). $\operatorname{ord}_\pi c = 0$. Let

$$\mathcal{U}_L' = \left\{\begin{pmatrix}u & vc \\ v & u+vb\end{pmatrix} \in \mathcal{U}_L \;\middle|\; u \in \mathcal{O}_F^\times\right\}$$

and

$$\mathcal{U}_L'' = \left\{\begin{pmatrix}u & vc \\ v & u+vb\end{pmatrix} \in \mathcal{U}_L \;\middle|\; u \in \pi\mathcal{O}_F\right\}$$

so that

$$\mathcal{U}_L = \mathcal{U}_L' \cup \mathcal{U}_L''.$$

As in Case (I), we have

$$\mathcal{U}_L'\begin{pmatrix}1 & \\ & \pi^m\end{pmatrix}K' = \bigcup_{\substack{m\geqslant n\geqslant 0\\ u\in\mathcal{O}_F^\times/\pi^{m-n}\mathcal{O}_F}}\begin{pmatrix}\pi^{m-n} & u \\ & \pi^n\end{pmatrix}K'.$$

One checks that

$$\mathcal{U}_L''\begin{pmatrix}1 & \\ & \pi^m\end{pmatrix}K' = \bigcup_{z\in\pi\mathcal{O}_F/\pi^m\mathcal{O}_F}\begin{pmatrix}\pi^m & z \\ & 1\end{pmatrix}K'.$$

Therefore

$$\#\left[\mathcal{U}_L\begin{pmatrix}1 & \\ & \pi^m\end{pmatrix}K'/K'\right] = q^m + q^{m-1}$$

for $m \geqslant 1$. □

We summarize the above discussion in

**Corollary 8.4.2.** *For each* $s = \begin{pmatrix}1 & x & y \\ & 1 & 0 \\ & & \pi^n\end{pmatrix} \in S$, *the cardinality of* $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)sK/KZ$ *is*

$$\operatorname{vol}([\gamma])\begin{cases}1 & \text{when } n=0,\\ q^n & \text{when } n\geqslant 1 \text{ and } \gamma \text{ is ramified rank-one split},\\ q^n+q^{n-1} & \text{when } n\geqslant 1 \text{ and } \gamma \text{ is unramified rank-one split.}\end{cases}$$

Now we are ready to state the main result of this section.

**Theorem 8.4.3.** *Suppose* $\gamma \in [\Gamma]$ *is rank-one split with rational form* $r_\gamma = \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$. *Set* $\delta = \delta([\gamma]) = \mathrm{ord}_\pi d$ *and* $\mu = \mu([\gamma]) = \mathrm{ord}_\pi((a-e)^2 - db(a-e) - cd^2)$.

(A) *If* $\gamma$ *is unramified rank-one split, then the following hold.*

(A1)

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(\frac{q^{\delta+1}+q^\delta-2}{q-1} + \frac{(q+1)q^{\delta+2}u^3}{1-q^3u^3}\right)\left(\frac{1-u^3}{1-q^2u^3}\right).$$

(A2) *If* $[\gamma]$ *does not have type* 1, *then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type 1}} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(q^\delta + q^{\delta-1} + \frac{(q^2-1)q^{\delta+1}u^3}{1-q^3u^3}\right).$$

(A3) *If* $[\gamma]$ *has type* 1, *then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type 1}} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(\frac{q^{\delta+1}+q^\delta-2}{q-1} + \frac{(q^2-1)q^{\delta+1}u^3}{1-q^3u^3}\right).$$

(B) *If* $\gamma$ *is ramified rank-one split, then the following hold.*

(B1)

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)q^\mu u^{l_A([\gamma])}\left(\frac{q^{\delta+1}-1}{q-1} + \frac{q^{\delta+3}u^3}{1-q^3u^3}\right)\frac{1-u^3}{1-q^2u^3}.$$

(B2) *If* $[\gamma]$ *does not have type* 1, *then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type 1}} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(q^\delta\big(q^\mu-\mu\big) + \frac{(q-1)q^{\delta+\mu+2}u^3}{1-q^3u^3}\right).$$

(B3) *If* $[\gamma]$ *has type* 1, *then*

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type 1}} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(\frac{q^{\delta+1}-1}{q-1} + \frac{(q-1)q^{\delta+2}u^3}{1-q^3u^3}\right).$$

*Moreover, in each case, if $[\gamma]$ does not have type 1, none of the type 1 cycles in $[\gamma]$ are geometrically minimal.*

**Remarks.** 1. $\mu = 0$ unless $a, e, c$ are all nonunits, in which case it is 1 and $\delta = 0$.
2. $\mu = 0$ when $[\gamma]$ has type 1.
3. $\delta > 0$ in case (A2), while $\delta$ may be zero in case (A3).

**Proof.** Recall that the algebraic length of a cycle in $[\gamma]$ is equal to $l_A([\gamma]) + 3m$ for some $m \geqslant 0$. We shall follow the same notation and computation as in the proof of Theorem 8.3.1, letting $g$ run through all elements in the double coset representatives $S$ and computing, for each $m \geqslant 0$, the number of cycles $\kappa_\gamma(P_\gamma gKZ)$ with $l_A(\kappa_\gamma(P_\gamma gKZ)) \leqslant l_A([\gamma]) + 3m$ using Corollary 8.4.2. As $g = \begin{pmatrix} 1 & x & y \\ & 1 & 0 \\ & & \pi^i \end{pmatrix}$, this amounts to computing the number of $x, y \in F/\mathcal{O}_F$ and $i \geqslant 0$ such that

$$\begin{aligned} e_1 = \min\bigl\{&\operatorname{ord}_\pi\bigl((a-e)x - d\pi^{-i}y\bigr), \operatorname{ord}_\pi\bigl(-cd\pi^i x + (a-e-db)y\bigr), -i + \operatorname{ord}_\pi d\bigr\} \\ &\geqslant -m. \end{aligned}$$

This is equivalent to $0 \leqslant i \leqslant m + \operatorname{ord}_\pi d$, $(a-e)x - d\pi^{-i}y \in \pi^{-m}\mathcal{O}_F$ and $-cd\pi^i x + (a-e-db)y \in \pi^{-m}\mathcal{O}_F$. Denote $\operatorname{ord}_\pi d$ by $\delta$ for short. So for each $0 \leqslant i \leqslant m + \delta$, we solve the following system of linear equations

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a-e & -d\pi^{-i} \\ -cd\pi^i & a-e-db \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \tag{8.6}$$

for $\alpha, \beta \in \pi^{-m}\mathcal{O}_F$ and count the distinct pairs $(x, y) \in F/\mathcal{O}_F \times F/\mathcal{O}_F$. Recall that $a, e, d$ are integral, at least one of them is a unit, and $a$ and $e$ cannot be both units since $\operatorname{ord}_\pi \det r_\gamma > 0$. Let

$$\mu := \operatorname{ord}_\pi \det M = \operatorname{ord}_\pi\bigl((a-e)^2 - db(a-e) - cd^2\bigr),$$

which is 0 unless $a$, $e$ and $c$ are all nonunits, in which case it is 1. Put

$$\varepsilon := \min\bigl\{\operatorname{ord}_\pi(a-e), -i+\delta, \operatorname{ord}_\pi(a-e-bd)\bigr\},$$

which is equal to $-i + \delta$ if $\delta \leqslant i \leqslant m + \delta$, and 0 if $0 \leqslant i < \delta$. Then the coefficient matrix $M = k_1 \operatorname{diag}(\pi^\varepsilon, \pi^{\mu-\varepsilon})k_2$ for some $k_1, k_2 \in GL_2(\mathcal{O}_F)$. Thus system (8.6) has the same number of solutions as the system

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \pi^\varepsilon & \\ & \pi^{\mu-\varepsilon} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{8.7}$$

for $\alpha, \beta \in \pi^{-m}\mathcal{O}_F$ and $(x, y) \in F/\mathcal{O}_F \times F/\mathcal{O}_F$. We get the solutions $x \in \pi^{-m-\varepsilon}\mathcal{O}_F/\mathcal{O}_F$ and $y \in \pi^{-m-\mu+\varepsilon}\mathcal{O}_F/\mathcal{O}_F$ so that there are $q^{2m+\mu}$ different pairs $(x, y)$ for each $0 \leqslant i \leqslant m + \delta$. To proceed, we distinguish two cases.

Case (A) $\mathrm{ord}_\pi c = 0$, that is, $\gamma$ is unramified rank-one split. Then $\mu = 0$. By Corollary 8.4.2, the number of classes in $[\gamma]$ with algebraic length at most $l_A([\gamma]) + 3m$ is

$$\mathrm{vol}\big([\gamma]\big)q^{2m}\Bigg(1 + \sum_{1\leqslant n\leqslant m+\delta} q^n + q^{n-1}\Bigg) = \mathrm{vol}\big([\gamma]\big)q^{2m}\bigg(\frac{q^{m+\delta}-1}{q-1} + \frac{q^{m+\delta+1}-1}{q-1}\bigg)$$
$$= \frac{\mathrm{vol}([\gamma])}{q-1}\big(q^{3m+\delta+1} + q^{3m+\delta} - 2q^{2m}\big).$$

Therefore

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(gKZ))} = \sum_{\kappa_\gamma(P_\gamma gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(P_\gamma gKZ))}$$
$$= \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\frac{1}{q-1}\Bigg(q^{\delta+1} + q^\delta - 2$$
$$+ \sum_{m\geqslant 1}\big(q^{3m+\delta+1} + q^{3m+\delta} - 2q^{2m} - q^{3m+\delta-2} - q^{3m+\delta-3} + 2q^{2m-2}\big)u^{3m}\Bigg)$$
$$= \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\frac{1}{q-1}\bigg(\frac{q^{\delta+1}+q^\delta}{1-q^3u^3} - \frac{2}{1-q^2u^3}\bigg)\big(1-u^3\big)$$
$$= \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\bigg(\frac{q^{\delta+1}+q^\delta-2}{q-1} + \frac{(q+1)q^{\delta+2}u^3}{1-q^3u^3}\bigg)\bigg(\frac{1-u^3}{1-q^2u^3}\bigg).$$

Among the cycles with $l_A(\kappa_\gamma(P_\gamma gKZ)) = l_A([\gamma]) + 3m$, we compute the number of those with type 1. First consider the case $m \geqslant 1$. In order that $l_A(\kappa_\gamma(P_\gamma gKZ)) = l_A([\gamma]) + 3m$ and $\kappa_\gamma(P_\gamma gKZ)$ has type 1, two conditions must be satisfied:

$$e_1 = \min\big\{\mathrm{ord}_\pi\big((a-e)x - d\pi^{-i}y\big), \mathrm{ord}_\pi\big(-cd\pi^i x + (a-e-db)y\big), -i+\delta\big\} = -m,$$

and

$$e_1 + e_2 = \mathrm{ord}_\pi\big[\big((a-e)x - d\pi^{-i}y\big)(e+db) - d\pi^{-i}\big(-cd\pi^i x + (a-e-db)y\big)\big] = -2m.$$

These two conditions are equivalent to $i = \delta + m$, $\mathrm{ord}_\pi(-cd\pi^i x + (a-e-db)y) = -m$, and $\mathrm{ord}_\pi((a-e)x - d\pi^{-i}y) \geqslant -m$. This amounts to solving system (8.6) with $\alpha \in \pi^{-m}\mathcal{O}_F$ and $\beta \in \pi^{-m}\mathcal{O}_F^\times$, hence we obtain $(q-1)q^{2m-1}$ distinct pairs $(x,y)$. Combined with Corollary 8.4.2, we see that the number of type 1 cycles $\kappa_\gamma(P_\gamma gKZ)$ with $l_A(\kappa_\gamma(P_\gamma gKZ)) = l_A([\gamma]) + 3m$ is $\mathrm{vol}([\gamma])(q-1)q^{2m-1}(q^{\delta+m} + q^{\delta+m-1})$.

Next consider the case $m = 0$. Under the assumption $\mathrm{ord}_\pi c = 0$, we know from Theorem 8.3.1 that $[\gamma]$ has type $(\mathrm{ord}_\pi a, \min\{\mathrm{ord}_\pi e, \mathrm{ord}_\pi d\})$. Therefore it has type 1 if and only if $\mathrm{ord}_\pi a > 0$, in which case all cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ have type 1, and the number of such cycles is $\mathrm{vol}([\gamma])\frac{q^{\delta+1}+q^\delta-2}{q-1}$, as computed above. If $[\gamma]$ does not have type 1, then $\delta = \mathrm{ord}_\pi d > 0$; the condition $e_1 = e_2 = 0$ implies $i = \delta$

and only one solution $(x, y) = (0, 0)$. In this case the number of type 1 cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ is $q^\delta + q^{\delta-1}$ by Corollary 8.4.2. Put together, we have shown the following:

If $[\gamma]$ has type 1, then

$$\begin{aligned}&\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))}\\&\quad= \operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\left(\frac{q^{\delta+1}+q^\delta-2}{q-1}+\sum_{m\geqslant 1}(q-1)q^{2m-1}\bigl(q^{\delta+m}+q^{\delta+m-1}\bigr)u^{3m}\right)\\&\quad= \operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\left(\frac{q^{\delta+1}+q^\delta-2}{q-1}+\frac{(q^2-1)q^{\delta+1}u^3}{1-q^3u^3}\right),\end{aligned}$$

while if $[\gamma]$ does not have type 1, then

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))} = \operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\left(q^\delta+q^{\delta-1}+\frac{(q^2-1)q^{\delta+1}u^3}{1-q^3u^3}\right).$$

Case (B) $\operatorname{ord}_\pi c = 1$, that is, $\gamma$ is ramified rank-one split. Then $\mu = 0$ or $1$. The same computation as in Case (A) together with Corollary 8.4.2 shows that the number of classes in $[\gamma]$ with algebraic length at most $l_A([\gamma]) + 3m$ is

$$\begin{aligned}\operatorname{vol}\bigl([\gamma]\bigr)q^{2m+\mu}\sum_{0\leqslant n\leqslant m+\delta} q^n &= \operatorname{vol}\bigl([\gamma]\bigr)q^{2m+\mu}\frac{q^{m+\delta+1}-1}{q-1}\\&= \operatorname{vol}\bigl([\gamma]\bigr)\frac{q^\mu}{q-1}\bigl(q^{3m+\delta+1}-q^{2m}\bigr).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{\kappa_\gamma(gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(gKZ))}\\&\quad= \operatorname{vol}\bigl([\gamma]\bigr)\frac{q^\mu}{q-1}u^{l_A([\gamma])}\left(\sum_{m\geqslant 0}\bigl(q^{3m+\delta+1}-q^{2m}\bigr)u^{3m}-\sum_{m\geqslant 1}\bigl(q^{3m+\delta-2}-q^{2m-2}\bigr)u^{3m}\right)\\&\quad= \operatorname{vol}\bigl([\gamma]\bigr)\frac{q^\mu}{q-1}u^{l_A([\gamma])}\left(\frac{q^{\delta+1}}{1-q^3u^3}-\frac{1}{1-q^2u^3}\right)\bigl(1-u^3\bigr)\\&\quad= \operatorname{vol}\bigl([\gamma]\bigr)q^\mu u^{l_A([\gamma])}\left(\frac{q^{\delta+1}-1}{q-1}+\frac{q^{\delta+3}u^3}{1-q^3u^3}\right)\frac{1-u^3}{1-q^2u^3}.\end{aligned}$$

Now we compute the number of type 1 cycles $\kappa_\gamma(P_\gamma gKZ)$ with algebraic length $l_A(\kappa_\gamma(P_\gamma gKZ)) = l_A([\gamma]) + 3m$. First consider the case $m \geqslant 1$. Following the same argument as in Case (A) and applying Corollary 8.4.2, we see that the number of such cycles is $\operatorname{vol}([\gamma])(q-1)q^{2m+\mu-1}q^{\delta+m}$.

Next we discuss the remaining case $m = 0$. By Theorem 8.3.1, $[\gamma]$ has type 1 if and only if $\mathrm{ord}_\pi a > 0$ and $\mathrm{ord}_\pi e = 0$, in which case all cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ are of type 1, and the number of such cycles is $\mathrm{vol}([\gamma])q^\mu \frac{q^{\delta+1}-1}{q-1}$. When $[\gamma]$ does not have type 1, we have $\mathrm{ord}_\pi e > 0$; the condition $e_1 = e_2 = 0$ implies $i = \delta$. Moreover, if $\mu = 0$, in which case $a$ is a unit, then there is only one pair $(x, y) = (0, 0)$; while if $\mu = 1$, in which case $a$ is not a unit, then there are $q-1$ pairs $(x, y) = (0, y)$ with $y \in \pi^{-1}\mathcal{O}_F^\times/\mathcal{O}_F$ so that $\mathrm{ord}_\pi(-cd\pi^i x + (a - e - db)y) = 0$. Consequently, when $[\gamma]$ does not have type 1, the number of type 1 cycles in $[\gamma]$ with algebraic length equal to $l_A([\gamma])$ is $\mathrm{vol}([\gamma])q^\delta$ if $\mu = 0$, and $\mathrm{vol}([\gamma])(q-1)q^\delta$ if $\mu = 1$. In other words, it is $\mathrm{vol}([\gamma])q^\delta(q^\mu - \mu)$. Summing up, we have proved the following:

If $[\gamma]$ has type 1, then

$$\begin{aligned}
&\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))} \\
&\quad = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}q^\mu\left(\frac{q^{\delta+1}-1}{q-1} + \sum_{m\geqslant 1}(q-1)q^{3m+\delta-1}u^{3m}\right) \\
&\quad = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}q^\mu\left(\frac{q^{\delta+1}-1}{q-1} + \frac{(q-1)q^{\delta+2}u^3}{1-q^3u^3}\right),
\end{aligned}$$

while if $[\gamma]$ does not have type 1, then

$$\sum_{\kappa_\gamma(gKZ)\in[\gamma],\ \text{type } 1} u^{l_A(\kappa_\gamma(gKZ))} = \mathrm{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(q^\delta\big(q^\mu - \mu\big) + \frac{(q-1)q^{\delta+\mu+2}u^3}{1-q^3u^3}\right).$$

This completes the proof of the theorem. □

Contained in the proofs of Corollary 8.4.2 and Theorem 8.4.3 is the proposition below, in which

$$g_{i,j,u} = \begin{pmatrix} 1 & & \\ & \pi^{i-j} & u \\ & & \pi^j \end{pmatrix} \quad \text{and} \quad g_{i,z} = \begin{pmatrix} 1 & & \\ & \pi^i & z \\ & & 1 \end{pmatrix}. \tag{8.8}$$

**Proposition 8.4.4.** *Let $\gamma \in [\Gamma]$ be rank-one split with $r_\gamma = \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$. Set $\delta = \delta([\gamma]) = \mathrm{ord}_\pi d$. Suppose that $[\gamma]$ has type 1 with $n = \mathrm{ord}_\pi a$. If $\gamma$ is ramified rank-one split, then*

$$\begin{aligned}
\Delta_A\big([\gamma]\big) = \big\{hg_{i,j,u}KZ \bigm| h \in C_G(r_\gamma)/\big(C_G(r_\gamma)\cap KZ\big),\ 0 \leqslant j \leqslant i \leqslant \delta, \\
u \in \mathcal{O}_F^\times/\pi^{i-j}\mathcal{O}_F \text{ for } j < i,\ \text{and } u = 0 \text{ for } j = i\big\};
\end{aligned}$$

*while if $\gamma$ is unramified rank-one split, then*

$$\Delta_A\big([\gamma]\big) = \{hg_{i,j,u}KZ \mid h \text{ and } g_{i,j,u} \text{ as above}\} \cup \big\{hg_{i,z}KZ \bigm| h \text{ as above},\ 1 \leqslant i \leqslant \delta,\ z \in \pi\mathcal{O}_F/\pi^i\mathcal{O}_F\big\}.$$

*Consequently, the number of algebraically minimal cycles in* $[\gamma]$ *is*

$$\#\big(C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A\big([\gamma]\big)\big) = \operatorname{vol}\big([\gamma]\big)\omega_{[\gamma]},$$

*where*

$$\omega_{[\gamma]} = \begin{cases} \frac{q^{\delta+1}+q^\delta-2}{q-1} & \text{if } [\gamma] \text{ is unramified rank-one split,} \\ \frac{q^{\delta+1}-1}{q-1} & \text{if } [\gamma] \text{ is ramified rank-one split.} \end{cases}$$

We end this subsection by comparing $\Delta_G([\gamma])$ and $\Delta_G([\gamma^2])$, where $\Delta_G([\gamma])$ is defined by (5.5). Suppose $[\gamma]$ is of type $(m,n)$. If $[\gamma^2]$ is of type $(2m,2n)$, then a geometrically minimal 1-geodesic $\kappa_\gamma(P_\gamma gKZ)$ repeated twice is still geometrically minimal, hence $\Delta_G([\gamma]) \subseteq \Delta_G([\gamma^2])$.

If $[\gamma]$ is not of type $(2m,2n)$, then $\gamma$ is ramified rank-one split of type $(n,1)$ or $(1,n)$. Assume first that $\gamma$ is of type $(n,1)$ so that $\mu = 1$ and $\delta = 0$ in Theorem 8.4.3. In this case, there are $q \cdot \operatorname{vol}([\gamma])$ algebraically minimal geodesics in $[\gamma]$. Among these, $(q-1)\operatorname{vol}([\gamma])$ of them are of type 1, and $\operatorname{vol}([\gamma])$ of them are of type $(n,1)$. The latter ones are also geometrically minimal. On the other hand $\kappa_\gamma(P_\gamma gKZ)$ is geometrically minimal for all $g \in C_G(r_\gamma)$. We conclude that $\kappa_\gamma(P_\gamma gKZ)$ is geometrically minimal if and only if $g \in C_G(r_\gamma)$. As $C_G(r_\gamma) \subseteq C_G(r_\gamma^2)$, any $g \in C_G(r_\gamma)$ gives rise to a geometrically minimal cycle $\kappa_{\gamma^2}(P_\gamma gKZ)$. This shows $\Delta_G([\gamma]) \subseteq \Delta_G([\gamma^2])$ if $\gamma$ is of type $(n,1)$.

Finally, note that $\kappa_{\gamma^{-1}}(P_\gamma gKZ)$ and $\kappa_\gamma(P_\gamma gKZ)$ have the same geometric length but opposite types, so the same conclusion holds for $\gamma$ of type $(1,n)$. We have shown

**Proposition 8.4.5.** *For* $\gamma \in \Gamma$ *we have* $\Delta_G([\gamma]) \subseteq \Delta_G([\gamma^2])$.

### 8.5. Counting the number of tailless cycles in $X_\Gamma$ of given algebraic length

Recall that $N_n(X_\Gamma)$ counts the number of tailless cycles of type 1 in $X_\Gamma$ with algebraic length $n$. These cycles fall in the disjoint union of $[\gamma]$ as $[\gamma]$ runs through type 1 conjugacy classes of $\Gamma$, and Theorem 7.2.1 and Proposition 8.4.4 give the number of such cycles in each $[\gamma]$. Combined with Proposition 6.1.1 we obtain the following explicit expressions of the edge zeta functions.

**Theorem 8.5.1.** *For* $i = 1, 2$ *we have*

$$u\frac{d}{du}\log Z_{1,i}(X_\Gamma, u) = \sum_{\gamma\in[\Gamma],\ [\gamma] \text{ of type } 1} i \operatorname{vol}\big([\gamma]\big)\omega_{[\gamma]}u^{il_A([\gamma])},$$

*where* $\operatorname{vol}([\gamma])$ *is defined by* (5.4) *and* $\omega_{[\gamma]}$ *is as in Theorem* 7.2.1 *and Proposition* 8.4.4.

## 9. Galleries and pointed galleries in $X_\Gamma$

### 9.1. Iwahori–Hecke algebra on $\mathcal{B}$

Recall that the pointed chambers on $\mathcal{B}$ are parametrized by cosets in $G/BZ$ with the Iwahori subgroup $B$, admitting the action of $G$ by left translation. The matrices

$$t_1 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ \pi & & \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

generate the Weyl group $W$ of $\mathrm{SL}_3(F)$ subject to the relations $t_i^2 = Id$ and $(t_i t_j)^3 = Id$ for $i \neq j$.

The extended affine Weyl group of $G$ is $W \ltimes \langle\sigma\rangle$, where $\sigma = \left(\begin{smallmatrix} & 1 & \\ & & 1 \\ \pi & & \end{smallmatrix}\right)$, as in Section 3.2, so that

$$G = \coprod_{w \in W \ltimes \langle\sigma\rangle} BwBZ.$$

Each element $w \in W \ltimes \langle\sigma\rangle$ defines an operator $L_w$ on $L^2(G/BZ)$ by sending a function $f$ to $L_w f$ given by

$$L_w f(gBZ) = \sum_{w_i BZ \in BwBZ/BZ} f(gw_i BZ) \quad \text{for all } gBZ.$$

They form a generalized Iwahori–Hecke algebra satisfying the following relations (cf. [5]):

1. $L_{t_i} \cdot L_{t_i} = (q-1)L_{t_i} + qId$,
2. $L_{t_i} \cdot L_{t_j} = L_{t_i t_j}$ for $i \neq j$,
3. $L_{t_i} \cdot L_w = L_{t_i w}$ if the length of $t_i w$ is 1 plus the length of $w$,
4. $L_\sigma \cdot L_{t_i} = L_{\sigma t_i} = L_{t_{i+1}\sigma}$ for $i = 1, 2, 3$.

As explained in Section 3.5, the operator

$$L_B = L_{t_1\sigma} \tag{9.1}$$

describes out-neighbors of a pointed chamber. The above relations imply $(L_B)^n = L_{t_1 t_2 \cdots t_n \sigma^n}$ for $n \geqslant 1$. Here the indices are read modulo 3.

### 9.2. Galleries in $\mathcal{B}$

Two chambers are adjacent if they share a common edge. Paths formed by adjacent chambers are called galleries. A gallery between two chambers is called a *geodesic* *gallery* if it contains the least number of intermediate chambers. Let $\tilde{B}$ be the stabilizer in $G$ of the chamber with vertices $KZ$, $\sigma KZ$ and $\sigma^2 KZ$. Thus it is generated by $B, Z$ and $\sigma$. Since $G$ acts transitively on all chambers, we can parametrize chambers by $G/\tilde{B}$. Notice that a chamber $g\tilde{B}$ gives rise to three pointed chambers: $gBZ$, $g\sigma BZ$ and $g\sigma^2 BZ$.

To get a geodesic gallery from $g_1\tilde{B}$ to $g_2\tilde{B}$, we find an element $w \in W$ such that $g_1^{-1}g_2 \in \tilde{B}w\tilde{B}$ and write $w = t_{i_1} \cdots t_{i_n}$ as a word using the least number of reflections $t_1, t_2, t_3$; call $n$ the *length* of the gallery. Note that $w$ is unique up to conjugation by some power of $\sigma$. Write $g_1^{-1}g_2 = bwb'$ for some $b, b' \in \tilde{B}$. Since $g_1\tilde{B} = g_1 b\tilde{B}$, we may assume $b$ is the identity so that

$$g_1\tilde{B} \to g_1 t_{i_1}\tilde{B} \to \cdots \to g_1 t_{i_1} \cdots t_{i_n}\tilde{B} = g_2\tilde{B}$$

represents a geodesic gallery from $g_1\tilde{B}$ to $g_2\tilde{B}$. Moreover, since $\sigma \in \tilde{B}$ and $\sigma t_i \sigma^{-1} = t_{i+1}$ for all $i$, replacing $g_1$ by $g_1\sigma^{1-i_1}$ if necessary, we may assume that $t_{i_1} = t_1$.

All geodesic galleries from $g_1\tilde{B}$ to $g_2\tilde{B}$ have length $n$; different galleries arise from different expressions of $w$ as a product of generators, and they are regarded as *homotopic*. Like the case of paths, given two distinct chambers $g_1\tilde{B}$ and $g_2\tilde{B}$, there is only one homotopy class of geodesic galleries in $\mathcal{B}$ from $g_1\tilde{B}$ to $g_2\tilde{B}$.

Observe that a geodesic gallery arising from $w = t_{i_1} \cdots t_{i_n}$ is a straight strip if and only if the difference $i_{k+1} - i_k$ remains the same mod 3 for $1 \leqslant k \leqslant n-1$. It is said to have type 1 or 2 according to the common difference being 1 or 2. Note that the homotopy class of a gallery of type 1 or 2 contains only one geodesic gallery, thus we shall drop the word "homotopy" in this case. Further, a geodesic gallery of type 1 can always be represented by

$$g_1\tilde{B} \to g_1 t_1\tilde{B} \to g_1 t_1 t_2\tilde{B} \to \cdots \to g_1 t_1 \cdots t_n\tilde{B} = g_2\tilde{B}.$$

### *9.3. Closed galleries and pointed galleries in $X_\Gamma$*

A closed gallery in $X_\Gamma$ starting at the chamber $\Gamma g\tilde{B}$ of $X_\Gamma$ can be lifted to a gallery in $\mathcal{B}$ starting at $g\tilde{B}$ and ending at $\gamma g\tilde{B}$ for some $\gamma \in \Gamma$. Denote by $\kappa_\gamma(g\tilde{B})$ the homotopy class of geodesic galleries in $\mathcal{B}$ from $g\tilde{B}$ to $\gamma g\tilde{B}$. By abuse of notation, it also represents a homotopy class of closed geodesic galleries in $X_\Gamma$ starting at $\Gamma g\tilde{B}$. A closed geodesic gallery is *tailless* if it remains a geodesic when the starting chamber is changed.

Recall that a pointed chamber $g_2BZ = (g_2KZ, g_2\sigma KZ, g_2\sigma^2 KZ)$ is an out-neighbor of $g_1BZ = (g_1KZ, g_1\sigma KZ, g_1\sigma^2 KZ)$ if and only if $g_1\sigma KZ = g_2KZ, g_1\sigma^2 KZ = g_2\sigma KZ$ and $g_1KZ \neq g_2\sigma^2 KZ$, or equivalently $g_1^{-1}g_2 \in L_B Z$.

A sequence $\Gamma g_0BZ \to \Gamma g_1BZ \to \cdots \to \Gamma g_nBZ = \Gamma g_0BZ$ of pointed chambers in $X_\Gamma$ is called a closed pointed gallery of length $n$ if there is a lifting $g_0BZ \to \cdots \to g_nBZ$ in $\mathcal{B}$ so that $g_{i+1}BZ$ is an out-neighbor of $g_iBZ$ for $0 \leqslant i \leqslant n-1$ and $g_nBZ = \gamma g_0BZ$ for some $\gamma$ in $\Gamma$. Denote this pointed gallery by $\kappa_\gamma(g_0BZ)$ for short. Note that there is a pointed gallery from $g_0BZ$ to $\gamma g_0BZ$ in $\mathcal{B}$ of length $n$ if and only if $g_0^{-1}\gamma g_0 \in (L_B)^n$. In this case, $\Gamma g_0\tilde{B} \to \Gamma g_1\tilde{B} \to \cdots \to \Gamma g_n\tilde{B}$ is the gallery $\kappa_\gamma(g_0\tilde{B})$ and we say the gallery $\kappa_\gamma(g_0\tilde{B})$ admits the pointed gallery $\kappa_\gamma(g_0BZ)$. Note that if a gallery admits a pointed gallery, then this pointed gallery is unique.

Analogous to the case of 1-geodesics, we have several descriptions of tailless galleries:

**Proposition 9.3.1.** *For a type* 1 *closed geodesic gallery* $\kappa_\gamma(g\tilde{B})$, *the following are equivalent*:

1. $\kappa_\gamma(g\tilde{B})$ *is tailless.*
2. $\kappa_\gamma(g\tilde{B})$ *repeated* $m$*-times is a type* 1 *geodesic gallery for all* $m > 0$.
3. $\kappa_\gamma(g\tilde{B})$ *admits a closed pointed gallery* $\kappa_\gamma(g_0BZ)$ *for a unique* $g_0BZ$ *such that* $g_0\tilde{B} = g\tilde{B}$.

Consequently, the map sending $\kappa_\gamma(g\tilde{B})$ to $\kappa_\gamma(g_0BZ)$ is a length preserving bijection from the set of tailless type 1 closed geodesic galleries to the set of closed pointed galleries in $X_\Gamma$.

**Proof.** $(1 \Rightarrow 2)$ Suppose $\kappa_\gamma(g\tilde{B})$ is tailless. Let $g_0\tilde{B} \to \cdots \to g_{mn}\tilde{B}$ be a lifting of $\kappa_\gamma(g\tilde{B})$ repeated $m$-times in $\mathcal{B}$. Then there is a word $w = t_{i_1}\cdots t_{i_{mn}}$ so that $g_j^{-1}g_k \in \tilde{B}t_{i_{j+1}}t_{i_{j+2}}\cdots t_{i_k}\tilde{B}$ for all $0 \leqslant j < k \leqslant mn$. By the tailless assumption, $g_j\tilde{B} \to \cdots \to g_{j+n}\tilde{B}$ is geodesic of type 1 for $j = 0,\ldots,n(m-1)$, so we have $t_{i_{(j+1)}} = t_{i_j+1}$ for $j = 0,\ldots,mn-1$. This shows that $w$ is a reduced word and $\kappa_\gamma(g\tilde{B})$ repeated $m$-times is a type 1 geodesic gallery.

$(2 \Rightarrow 3)$ Let $g\tilde{B} = g_0\tilde{B} \to \cdots \to g_{2n}\tilde{B}$ be a lifting of $\kappa_\gamma(g\tilde{B})$ repeated twice. Since it is a type 1 geodesic gallery, as noted before, we may assume that $g_i\tilde{B} = gt_1\cdots t_i\tilde{B}$ and $g_i = gt_1\cdots t_i\sigma^i$ for $i = 0,\ldots,2n$. Then $g_iBZ$ is a pointed chamber of $g_i\tilde{B}$ and $g_i^{-1}g_{i+1} = \sigma^{-i}t_{i+1}\sigma^{i+1} = t_1\sigma \in L_B$. Therefore $g_0BZ \to \cdots \to g_{2n}BZ$ is a pointed gallery. It remains to show that $\gamma g_0BZ = g_nBZ$ so that $g_0BZ \to \cdots \to g_nBZ$ is a lifting of a closed pointed gallery of $X_\Gamma$. From $\gamma g_0\tilde{B} = g_n\tilde{B}$ and $\operatorname{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$ by assumption we conclude $n \in 3\mathbb{Z}$ and $\gamma g_0BZ = g_n\sigma^iBZ$ for some $i \in \{0,1,2\}$. Comparing determinants of both sides gives $i = 0$ since $\det\sigma = \pi$. This proves $\gamma g_0BZ = g_nBZ$.

$(3 \Rightarrow 1)$ Let $g_0BZ \to \cdots \to g_{2n}BZ$ be a lifting in $\mathcal{B}$ of the pointed gallery admitted by $\kappa_\gamma(g\tilde{B})$ repeated twice. Thus $g_i^{-1}g_{i+1} \in L_BZ$ for $i = 0,\ldots,2n-1$. Note that every gallery obtained by changing the starting chamber of $\kappa_\gamma(g\tilde{B})$ has a lifting contained in $C : g_0\tilde{B} \to \cdots \to g_{2n}\tilde{B}$, so it suffices to show that $C$ is a geodesic gallery. This is because

$$g_0^{-1}g_{2n} = \left(g_0^{-1}g_1\right)\cdots\left(g_{2n-1}^{-1}g_{2n}\right) \in (L_BZ)^{2n} \subset \tilde{B}t_1\cdots t_{2n}\tilde{B}$$

and $t_1\cdots t_{2n}$ is a reduced word of length $2n$. $\square$

*9.4. Characterizing closed pointed galleries in $X_\Gamma$*

We begin by extracting information on the starting vertex and the type of a closed pointed gallery.

**Proposition 9.4.1.** *Let $\gamma \in \Gamma$. Suppose $\kappa_\gamma(gBZ)$ is a closed pointed gallery in $X_\Gamma$ of length $n$. Then vertices $g\sigma^i KZ$ for $i = 0, 1, 2$ all lie in $\Delta_G([\gamma])$ defined by (5.5). Moreover, $[\gamma]$ is of type $(0, n/2)$ for $n$ even, and $(1, (n-1)/2)$ for $n$ odd. In the latter case, $\gamma$ is ramified rank-one split.*

**Proof.** Observe that

$$(t_1\sigma)^{2m} = \begin{pmatrix} 1 & & \\ & \pi^m & \\ & & \pi^m \end{pmatrix} \quad \text{and} \quad (t_1\sigma)^{2m+1} = \begin{pmatrix} 1 & & \\ & & \pi^m \\ & \pi^{m+1} & \end{pmatrix}.$$

Since $\kappa_\gamma(gBZ)$ is a pointed gallery of length $n$, we have $g^{-1}\gamma g \in (L_B)^n \subset K(t_1\sigma)^n KZ$. To find the type of $\kappa_\gamma(g\sigma^i KZ)$, write $\sigma = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$. Using the fact $\sigma B = B\sigma$, we have

$$\begin{aligned} \sigma^{-1}g^{-1}\gamma g\sigma \in \sigma^{-1}(L_B)^n\sigma &= B\sigma^{-1}(t_1\sigma)^n\sigma BZ \\ &\subset K\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi^{-1} \end{pmatrix}(t_1\sigma)^n\begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix}KZ \\ &= K(t_1\sigma)^n KZ. \end{aligned}$$

By the same argument, we also have $\sigma g^{-1}\gamma g\sigma^{-1} \in K(t_1\sigma)^n KZ$. Therefore, if $n = 2m$, $\kappa_\gamma(g\sigma^i KZ)$ has type $(0, m)$; if $n = 2m+1$, $\kappa_\gamma(g\sigma^i KZ)$ has type $(1, m)$ for all $i$. It remains to show that $[\gamma]$ has the same type as $\kappa_\gamma(g\sigma^i KZ)$, so that they have the same geometric length.

Since $g^{-1}\gamma^{2k}g \subset K(t_1\sigma)^{2nk}KZ = T_{0,nk}$, we have, by Proposition 5.1.1,

$$L_A(\gamma) = L_A\big(g^{-1}\gamma g\big) = \lim_{k\to\infty}\frac{1}{k}l_A\big(g^{-1}\gamma^k g\big) = \lim_{2k\to\infty}\frac{2nk}{2k} = n.$$

The same argument gives $L_A(\gamma^{-1}) = \frac{n}{2}$.

When $n = 2m$, we have $L_A(\gamma) = l_A(\kappa_\gamma(gKZ)) = 2m$ and $L_A(\gamma^{-1}) = l_A(\kappa_{\gamma^{-1}}(gKZ)) = m$. By Corollary 5.5.2, $[\gamma]$ and $\kappa_\gamma(gKZ)$ have the same type, which is $(0, m)$.

When $n = 2m+1$, we have $L_A(\gamma^{-1}) = \frac{2m+1}{2}$, which implies that $r_\gamma^{-1}$ is ramified rank-one split (and so is $r_\gamma$). Now suppose $[\gamma]$ is of type $(i, j)$. Applying Eq. (5.2) to $g^{-1}\gamma g$ and $g^{-1}\gamma^{-1}g$, we obtain

$$i + 2j = 2m + 1 \quad \text{and} \quad m + \frac{1}{2} \leqslant 2i + j \leqslant m + 2.$$

It is easy to see that $(i, j) = (1, m)$ is the only non-negative integral solution. □

We now show that the conditions in the above proposition characterize closed pointed galleries.

**Proposition 9.4.2.** *Suppose $\gamma \in \Gamma$ satisfies either (1) $[\gamma]$ is of type $(0, n)$, or (2) $\gamma$ is ramified rank-one split and $[\gamma]$ is of type $(1, n)$. If the three vertices $g\sigma^i KZ$ of the chamber $g\tilde{B}$ all lie in $\Delta_G([\gamma])$, then there is a unique $i \in \{0, 1, 2\}$ such that $\kappa_\gamma(g\sigma^i BZ)$ is a closed pointed gallery.*

**Proof.** The uniqueness of $i$ follows from the third statement in Proposition 9.3.1; we shall show it exists. Denote by $g\mathcal{A}$ the apartment containing the two chambers $g\tilde{B}$ and $\gamma g\tilde{B}$. Replacing $g$ by $gb$ for some $b \in \tilde{B}$ if necessary, we may assume that $\mathcal{A}$ is the standard apartment whose pointed chambers are represented by $DS_3\tilde{B}$, where $D$ is the group of diagonal matrices in $G$ and $S_3$ is the subgroup of permutation matrices in $G$. Write $g^{-1}\gamma g = Msb$ for some $M \in D$, $s \in S_3$ and $b \in BZ$. Since the vertices of $gBZ$ are in $\Delta_G([\gamma])$, by Proposition 5.5.1, $\kappa_\gamma(g\sigma^i KZ)$ has the same type as $[\gamma]$ for all $i$.

Case (I). $[\gamma]$ has type $(0, n)$. Then $g^{-1}\gamma g$, $\sigma^{-1}g^{-1}\gamma g\sigma$ and $\sigma g^{-1}\gamma g\sigma^{-1}$ all lie in $T_{0,n}$. In this case,

$$M \in \left\{ \begin{pmatrix} \pi^n & & \\ & \pi^n & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \pi^n & & \\ & 1 & \\ & & \pi^n \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \pi^n & \\ & & \pi^n \end{pmatrix} \right\}$$
$$= \left\{ \sigma^i \begin{pmatrix} 1 & & \\ & \pi^n & \\ & & \pi^n \end{pmatrix} \sigma^{-i} \colon i = 0, 1, 2 \right\}.$$

In other words, $M = \sigma^i \left(\begin{smallmatrix} 1 & & \\ & \pi^n & \\ & & \pi^n \end{smallmatrix}\right) \sigma^{-i} = \sigma^i (t_1\sigma)^{2n} \sigma^{-i}$ for some $i$. We shall show that $s$ is the identity matrix. If so, then, since $B\sigma = \sigma B$, we have $b\sigma^i = \sigma^i b'$ with $b' \in B$ and

$$\left(g\sigma^i\right)^{-1} \gamma \left(g\sigma^i\right) = \sigma^{-i} M b \sigma^i = \sigma^{-i} M \sigma^i b' \in (L_B)^{2n}.$$

Thus $\kappa_\gamma(g\sigma^i BZ)$ is a closed pointed gallery.

It suffices to consider the case $M = \left(\begin{smallmatrix} 1 & & \\ & \pi^n & \\ & & \pi^n \end{smallmatrix}\right)$ as the other cases are similar. To determine $s$, write $\sigma = \left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & \pi \end{smallmatrix}\right) s_3$ with $s_3 \in S_3$. Observe that

$$\begin{aligned} \sigma^{-1} g^{-1} \gamma g \sigma &= \sigma^{-1} M s b \sigma = \sigma^{-1} M s \sigma b'' \\ &= s_3^{-1} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi^{-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \pi^n & \\ & & \pi^n \end{pmatrix} s \begin{pmatrix} 1 & & \\ & 1 & \\ & & \pi \end{pmatrix} s_3 b'' \end{aligned}$$

and $s\begin{pmatrix}1&&\\&1&\\&&\pi\end{pmatrix}$ is $\begin{pmatrix}\pi&&\\&1&\\&&1\end{pmatrix}s$, $\begin{pmatrix}1&&\\&\pi&\\&&1\end{pmatrix}s$, or $\begin{pmatrix}1&&\\&1&\\&&\pi\end{pmatrix}s$ according as the first, second, or third row of $s$ is $(0\ 0\ 1)$. In order that $\sigma^{-1}g^{-1}\gamma g\sigma \in T_{0,n}$, the third row of $s$ must be $(0\ 0\ 1)$. Similarly, $\sigma g^{-1}\gamma g\sigma^{-1} \in T_{0,n}$ implies the first row of $s$ should be $(1\ 0\ 0)$. Therefore $s$ is the identity matrix.

Case (II). $\gamma$ is ramified rank-one split and $[\gamma]$ has type $(1,n)$. Then $g^{-1}\gamma g$, $\sigma^{-1}g^{-1}\gamma g\sigma$ and $\sigma g^{-1}\gamma g\sigma^{-1}$ all lie in $T_{1,n}$. In this case,

$$M \in \left\{\sigma^i\begin{pmatrix}1&&\\&\pi^n&\\&&\pi^{n+1}\end{pmatrix}\sigma^{-i}, \sigma^i\begin{pmatrix}1&&\\&\pi^{n+1}&\\&&\pi^n\end{pmatrix}\sigma^{-i}\colon\ i=0,1,2\right\}.$$

As before, it suffices to consider the cases $M=\begin{pmatrix}1&&\\&\pi^n&\\&&\pi^{n+1}\end{pmatrix}$ or $\begin{pmatrix}1&&\\&\pi^{n+1}&\\&&\pi^n\end{pmatrix}$.

A similar argument as in Case (I) yields

$$Ms=\begin{pmatrix}1&&\\&\pi^n&\\&&\pi^{n+1}\end{pmatrix},\qquad \begin{pmatrix}1&&\\&&\pi^n\\&\pi^{n+1}&\end{pmatrix},\quad \text{or}\quad \begin{pmatrix}1&&\\&\pi^{n+1}&\\&&\pi^n\end{pmatrix}.$$

Observe that $[\gamma^2]$ has type $(0,2n+1)$. Since $\Delta_G([\gamma])\subset\Delta_G([\gamma^2])$ by Proposition 8.4.5, we have $(g^{-1}\gamma g)^2=(Msb)^2\in T_{0,2n+1}$. On the other hand, if $Ms=\begin{pmatrix}1&&\\&\pi^n&\\&&\pi^{n+1}\end{pmatrix}$ or $\begin{pmatrix}1&&\\&\pi^{n+1}&\\&&\pi^n\end{pmatrix}$, then a direct computation shows $(Msb)^2\in T_{2,2n}$, which is a contradiction. Therefore $g^{-1}\gamma g=Msb=\begin{pmatrix}1&&\\&&\pi^n\\&\pi^{n+1}&\end{pmatrix}b\in L_B^{2n+1}$. Thus $\kappa_\gamma(g\sigma^iBZ)$ is a closed pointed gallery for some $i$. $\square$

The above two propositions and Proposition 9.3.1 together imply

**Theorem 9.4.3.** *Given $\gamma\in\Gamma$, the number of closed pointed galleries in $X_\Gamma$ of the form $\kappa_\gamma(gBZ)$ is equal to the number of chambers with vertices $P_\gamma gKZ$, where $gKZ\in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_G([\gamma])$.*

## 10. Chamber zeta function of $X_\Gamma$

### *10.1. Type 1 chamber zeta function of $X_\Gamma$*

Two closed galleries in $X_\Gamma$ are called equivalent if one is obtained from the other by changing the starting chamber. A closed gallery is called *primitive* if it is not a repetition of another closed gallery of shorter length. For a primitive tailless closed gallery $C$ of length $n$, denote by $[C]$ the collection of the $n$ closed galleries equivalent to $C$.

The type 1 chamber zeta function of $X_\Gamma$ is defined as an Euler product:

$$Z_{2,1}(X_\Gamma,u)=\prod_{[C]}\left(1-u^{l(C)}\right)^{-1},\tag{10.1}$$

where $[C]$ runs through the equivalence classes of primitive, tailless, type 1 closed galleries in $X_\Gamma$. Let $M_n(X_\Gamma)$ denote the number of tailless, type 1 closed galleries in $X_\Gamma$ of length $n$.

**Proposition 10.1.1.** *The type* 1 *chamber zeta function of* $X_\Gamma$ *is a rational function with the following expressions*:

$$Z_{2,1}(X_\Gamma,u)=\exp\left(\sum_{n\geqslant 1}\frac{M_n(X_\Gamma)}{n}u^n\right)=\frac{1}{\det(I-L_B u)}. \tag{10.2}$$

**Proof.** For $n\geqslant 1$, $\operatorname{Tr} L_B^n$ on $X_\Gamma$ counts the number of closed pointed galleries of length $n$, which is equal to $M_n(X_\Gamma)$ by Proposition 9.3.1. The equalities follow from the same argument as the proof of Proposition 6.1.1. □

*10.2. Comparing chamber zeta function and edge zeta function*

In this subsection we give an explicit formula for $M_n(X_\Gamma)$, the number of closed pointed galleries in $X_\Gamma$ of length $n$, similar to Theorem 8.5.1. This is achieved by computing the difference between logarithmic derivatives of edge zeta function and chamber zeta function and applying Theorem 8.5.1.

**Theorem 10.2.1.**

$$\begin{aligned}
&u\frac{d}{du}\log Z_{1,2}(X_\Gamma,u)-u\frac{d}{du}\log Z_{2,1}(X_\Gamma,-u)\\
&\quad=\sum_{n\geqslant 1}\Bigg(\sum_{\substack{\gamma\in[\Gamma]\text{ irregular,}\\ [\gamma]\text{ of type }(0,n)}}-(q-1)\operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\\
&\qquad+\sum_{\substack{\gamma\in[\Gamma]\text{ unramified rank-one split,}\\ [\gamma]\text{ of type }(0,n)}}2\operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\\
&\qquad+\sum_{\substack{\gamma\in[\Gamma]\text{ ramified rank-one split,}\\ [\gamma]\text{ of type }(0,n)\text{ or }(1,n)}}\operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\Bigg),
\end{aligned}$$

*where* $\operatorname{vol}([\gamma])$ *is defined by* (5.4).

**Proof.** Combining Theorem 9.4.3 and Proposition 10.1.1, we have

$$\begin{aligned}
&u\frac{d}{du}\log Z_{2,1}(X_\Gamma,-u)\\
&\quad=\sum_{n\geqslant 1}\Bigg(\sum_{\gamma\in[\Gamma],\ [\gamma]\text{ of type }(0,n)}N_B(\gamma)u^{2n}-\sum_{\substack{\gamma\in[\Gamma]\text{ ramified rank-one split,}\\ [\gamma]\text{ of type }(1,n)}}N_B(\gamma)u^{2n+1}\Bigg),
\end{aligned}$$

where $N_B(\gamma)$ is the number of chambers with vertices $P_\gamma gKZ$, where $gKZ \in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_G([\gamma])$. From the definition of $\Delta_G([\gamma])$, it is clear that $N_B(\gamma) = N_B(\gamma^{-1})$ so that

$$u\frac{d}{du}\log Z_{2,1}(X_\Gamma, -u) = \sum_{n\geqslant 1}\Bigg(\sum_{\gamma\in[\Gamma],\,[\gamma]\text{ of type }(n,0)} N_B(\gamma)u^{2n} - \sum_{\substack{\gamma\in[\Gamma]\text{ ramified rank-one split},\\ [\gamma]\text{ of type}(n,1)}} N_B(\gamma)u^{2n+1}\Bigg).$$

On the other hand, for type 2 edge zeta function we have

$$\begin{aligned} u\frac{d}{du}\log Z_{1,2}(X_\Gamma, u) &= u\frac{d}{du}\log Z_{1,1}\big(X_\Gamma, u^2\big) \\ &= \sum_{\gamma\in[\Gamma]}\ \sum_{\kappa_\gamma(gKZ)\text{ tailless, type 1}} 2u^{2l_A(\kappa_\gamma(gKZ))} \\ &= \sum_{n\geqslant 1}\ \sum_{\gamma\in[\Gamma],\,[\gamma]\text{ of type }(n,0)} 2N_K(\gamma)u^{2l_A([\gamma])}, \end{aligned}$$

where $N_K(\gamma) = \mathrm{vol}([\gamma])\omega_{[\gamma]}$ is the number of tailless type 1 cycles in $[\gamma]$ (cf. Theorem 8.5.1). We shall compare this with the number $N_B(\gamma)$. Recall that for $[\gamma]$ of type 1, we have $\Delta_G([\gamma]) = \Delta_A([\gamma])$.

Case I. $\gamma$ split with $[\gamma]$ of type $(n,0)$. Then $r_\gamma = \mathrm{diag}(1,a,b)$, where $1,a,b$ are distinct with $\mathrm{ord}_\pi(a) = 0$ and $\mathrm{ord}_\pi b = n$. Let $\delta = \mathrm{ord}_\pi(1-a)$. The centralizer $C_G(r_\gamma)$ consists of diagonal elements in $G$. By Corollary 7.3.2, $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ has cardinality $N_K(\gamma) = \mathrm{vol}([\gamma])q^\delta$, represented by vertices $h_{i,j}v_xKZ$, where $h_{i,j} = \mathrm{diag}(1,\pi^i,\pi^j) \in C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/(C_G(r_\gamma)\cap KZ)$ and $v_x = \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1\end{pmatrix}$ with $x \in \pi^{-\delta}\mathcal{O}_F/\mathcal{O}_F$.

There are $q+1$ chambers sharing the type 1 edge $E_0 := (KZ, \mathrm{diag}(1,1,\pi)KZ)$ with the third vertex $u_cKZ := \begin{pmatrix} \pi & c & \\ & 1 & \\ & & \pi\end{pmatrix}KZ$, $c \in \mathcal{O}_F/\pi\mathcal{O}_F$, and $u_\infty KZ := \begin{pmatrix} 1 & & \\ & \pi & \\ & & \pi\end{pmatrix}KZ$. Left multiplication by $h_{i,j}v_x$ sends $E_0$ to $(h_{i,j}v_xKZ, h_{i,j+1}v_xKZ)$ and the third vertex to $h_{i,j}v_xu_cKZ = \begin{pmatrix} 1 & (c+x)/\pi & \\ & \pi^{i-1} & \\ & & \pi^j\end{pmatrix}KZ$ and $h_{i,j}v_xu_\infty KZ = \begin{pmatrix} 1 & x\pi & \\ & \pi^{i+1} & \\ & & \pi^{j+1}\end{pmatrix}KZ$, respectively. We count the number of such vertices belonging to $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$.

There is only one integral $x$, namely, $x = 0$. When $\delta = 0$, each type 1 edge $(h_{i,j}v_0KZ, h_{i,j+1}v_0KZ)$ can be extended to a pointed chamber by adding only one of the two vertices $h_{i+1,j+1}v_0KZ$ and $h_{i-1,j}v_0KZ$ in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$. Once the starting pointed chamber $gBZ$ is chosen, the closed pointed gallery $\kappa_\gamma(gBZ)$ is determined. Hence $N_B(\gamma) = 2\#(C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])) = 2N_K(\gamma)$.

Next assume $\delta \geqslant 1$. In this case, each type 1 edge $(h_{i,j}v_0KZ, h_{i,j+1}v_0KZ)$ can be extended to a pointed chamber by adding one of the $q+1$ vertices $h_{i,j}v_0u_cKZ$ and $h_{i,j}v_0u_\infty KZ$ in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$. The same holds when $h_{i,j}v_0$ is replaced by $h_{i,j}v_x$ for $-1 \geqslant \mathrm{ord}_\pi x \geqslant -\delta+1$. This gives rise to $(q+1)(q^{\delta-1}-1)$ pointed chambers. Finally, when $\operatorname{ord}_\pi x = -\delta$, each type 1 edge $(h_{i,j}v_xKZ, h_{i,j+1}v_xKZ)$ can be extended to a pointed chamber by adding only one vertex $h_{i,j}v_xu_\infty KZ$ in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$, so there are $(q-1)q^{\delta-1}$ pointed chambers. Put together, we get $N_B(\gamma) = \operatorname{vol}([\gamma])(q+1+(q+1)(q^{\delta-1}-1)+(q-1)q^{\delta-1}) = \operatorname{vol}([\gamma])2q^\delta = 2N_K(\gamma)$.

Hence there is no contribution to $u\frac{d}{du}\log Z_{1,2}(X_\Gamma,u) - u\frac{d}{du}\log Z_{2,1}(X_\Gamma,-u)$ from $\gamma$ split and $[\gamma]$ of type $(n,0)$.

Case II. $\gamma$ irregular with $[\gamma]$ of type $(n,0)$. Then $r_\gamma = \operatorname{diag}(1,1,b)$, where $\operatorname{ord}_\pi b = n$. By Corollary 7.3.2, $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ has cardinality $N_K(\gamma) = \operatorname{vol}([\gamma])$. By the same method as in Case I, one checks that all $q+1$ chambers sharing an edge with two vertices in $\Delta_A([\gamma])$ have the third vertex also lie in $\Delta_A([\gamma])$. Hence $N_B(\gamma) = (q+1)N_K(\gamma)$ and the contribution of an irregular $\gamma$ with $[\gamma]$ of type $(n,0)$ to $u\frac{d}{du}\log Z_{1,2}(X_\Gamma,u) - u\frac{d}{du}\log Z_{2,1}(X_\Gamma,-u)$ is $-(q-1)\operatorname{vol}([\gamma])u^{2n}$.

Case III. $\gamma$ unramified rank-one split with $[\gamma]$ of type $(n,0)$. In this case $r_\gamma = \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$, and the eigenvalues $a$, $e+d\lambda$ and $e+d\bar{\lambda}$ of $r_\gamma$ generate an unramified quadratic extension $L$ over $F$. The type assumption on $\gamma$ implies that $\operatorname{ord}_\pi a = n$ and $\min(\operatorname{ord}_\pi e, \operatorname{ord}_\pi d) = 0$ so that $e+d\lambda$ and $e+d\bar{\lambda}$ are units in $L$. Let $\delta = \operatorname{ord}_\pi d$.

As discussed in Section 8.1, the double cosets $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/C_G(r_\gamma)\cap KZ$ are represented by $h_m = \operatorname{diag}(\pi^m,1,1)$, $m \bmod \operatorname{vol}([\gamma])$. By Proposition 8.4.4, $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ has cardinality $N_K(\gamma) = \operatorname{vol}([\gamma])\frac{q^\delta+q^{\delta-1}-2}{q-1}$ and is represented by $h_mg_{i,j,u}KZ$ and $h_mg_{i,z}KZ$, where $m \bmod \operatorname{vol}([\gamma])$, $g_{i,j,u} = \begin{pmatrix} 1 & & \\ & \pi^{i-j} & u \\ & & \pi^j \end{pmatrix}$ with $0 \leqslant j \leqslant i \leqslant \delta$, $u \in \mathcal{O}_F^\times/\pi^{i-j}\mathcal{O}_F$ for $j < i$ and $u = 0$ for $j = i$, and $g_{i,z} = \begin{pmatrix} 1 & & \\ & \pi^i & z \\ & & 1 \end{pmatrix}$ with $1 \leqslant i \leqslant \delta$ and $z \in \pi\mathcal{O}_F/\pi^i\mathcal{O}_F$.

It remains to count the number of pointed chambers with vertices in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ containing a type 1 edge $(gKZ, g\operatorname{diag}(\pi,1,1)KZ)$ for $g = h_mg_{i,j,u}$ or $h_mg_{i,z}$. When $\delta = 0$, there are no $g_{i,z}$ and only one $g_{i,j,u}$, equal to the identity matrix, hence the vertices in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ are $h_mKZ$, $m \bmod \operatorname{vol}([\gamma])$. It is clear that there are no pointed chambers formed by these vertices. Hence $N_K(\gamma) = \operatorname{vol}([\gamma])$ and $N_B(\gamma) = 0$ when $\delta = 0$.

Next assume $\delta \geqslant 1$. There are $q+1$ chambers sharing the type 1 edge $E_1 := (KZ, \operatorname{diag}(\pi,1,1)KZ)$ with the third vertex being $w_xKZ := \begin{pmatrix} \pi & & \\ & \pi & x \\ & & 1 \end{pmatrix}KZ$ with $x \in \mathcal{O}_F/\pi\mathcal{O}_F$ and $w_\infty KZ := \operatorname{diag}(1,\pi^{-1},1)KZ$, respectively. Left multiplication by $g = h_mg_{i,j,u}$ or $h_mg_{i,z}$ sends the edge $E_1$ to the type 1 edge $(gKZ, g\operatorname{diag}(\pi,1,1)KZ)$, so we need to count the number of distinct vertices among $gw_xKZ$ and $gw_\infty KZ$ which fall in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$. Observe that

$$h_mg_{i,j,u}w_xKZ = \begin{pmatrix} \pi^{m+1} & & \\ & \pi^{i-j+1} & x\pi^{i-j}+u \\ & & \pi^j \end{pmatrix}KZ,$$

$$h_mg_{i,j,u}w_\infty KZ = \begin{pmatrix} \pi^m & & \\ & \pi^{i-j-1} & u \\ & & \pi^j \end{pmatrix}KZ,$$

$$h_m g_{i,z} w_x KZ = \begin{pmatrix} \pi^{m+1} & & \\ & \pi^{i+1} & x\pi^i + z \\ & & 1 \end{pmatrix} KZ, \quad \text{and}$$

$$h_m g_{i,z} w_\infty KZ = \begin{pmatrix} \pi^{m} & & \\ & \pi^{i-1} & z \\ & & 1 \end{pmatrix} KZ.$$

It is straightforward to check that, for $0 \leqslant i \leqslant \delta-1$, all $gw_x KZ$ and $gw_\infty KZ$ are distinct vertices in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$, thus they give rise to are $\mathrm{vol}([\gamma])(q+1)\frac{q^\delta+q^{\delta-1}-2}{q-1}$ pointed chambers. When $i=\delta$, for each $g$ above, only $gw_\infty KZ$ lies in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash \Delta_A([\gamma])$, hence they yield $\mathrm{vol}([\gamma])(q^\delta + q^{\delta-1})$ pointed chambers. Altogether, $N_B(\gamma)$ is equal to $2N_K(\gamma) - 2\,\mathrm{vol}([\gamma])$ for $\delta \geqslant 0$.

In conclusion, the contribution to $u\frac{d}{du}\log Z_{1,2}(X_\Gamma, u) - u\frac{d}{du}\log Z_{2,1}(X_\Gamma, -u)$ from $\gamma$ unramified rank-one split with $[\gamma]$ of type $(n,0)$ is $2\,\mathrm{vol}([\gamma])u^{2n}$.

Case IV. $\gamma$ ramified rank-one split with $[\gamma]$ of type $(n,0)$. Then $r_\gamma = \begin{pmatrix} a & & \\ & e & dc \\ & d & e+db \end{pmatrix}$ and the eigenvalues $a$, $e+d\lambda$ and $e+d\bar{\lambda}$ of $\gamma$ generate a ramified quadratic extension $L$ over $F$. In this case, $\mathrm{ord}_\pi a = n$ and $\mathrm{ord}_\pi e = 0$ so that $e+d\lambda$ and $e+d\bar{\lambda}$ are units in $L$. Let $\delta = \mathrm{ord}_\pi d$.

As discussed in Section 8.1, $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash C_G(r_\gamma)/C_G(r_\gamma) \cap KZ$ has cardinality $\mathrm{vol}([\gamma])$, and it is represented by $h = \mathrm{diag}(\pi^m, 1, 1)$ with $0 \leqslant m \leqslant (\mathrm{vol}([\gamma])-1)/2$ and $\mathrm{diag}(\pi^m,1,1)\pi_L$ with $0 \leqslant m \leqslant (\mathrm{vol}([\gamma])-3)/2$ if $\mathrm{vol}([\gamma])$ is odd, and by $h = \mathrm{diag}(\pi^m, 1, 1)$ and $\mathrm{diag}(\pi^m,1,1)\pi_L$ with $m \bmod \mathrm{vol}([\gamma])/2$ if $\mathrm{vol}([\gamma])$ is even. Here $\pi_L = \begin{pmatrix} 1 & & \\ & & c \\ & 1 & b \end{pmatrix}$ is imbedded in $G$.

It follows from Proposition 8.4.4 that $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A(\gamma)$ is represented by $hg_{i,j,u}KZ$ for $g_{i,j,u}$ as in Case III and $h$ as above, so the total number of vertices is $\mathrm{vol}([\gamma])(q^{\delta+1}-1)/(q-1) = N_K(\gamma)$. To count the number of pointed chambers we proceed as in Case III by counting, for each $g = hg_{i,j,u}$, the number of $gw_x KZ$ and $gw_\infty KZ$ which lie in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A(\gamma)$.

We first discuss the case $\delta = 0$. Then there is only one $g_{0,0,u}$, equal to the identity matrix. All representatives are given by $hKZ$. Observe that $\mathrm{diag}(\pi^m,1,1)\pi_L KZ = \begin{pmatrix} \pi^m & & \\ & \pi & 0 \\ & & 1 \end{pmatrix} KZ$. So there is only one vertex $gw_0KZ$ which will form a chamber containing the type 1 edge $(gKZ, g\,\mathrm{diag}(\pi,1,1)KZ)$. Hence the number of pointed chambers is $N_B(\gamma) = \mathrm{vol}([\gamma]) = 2N_K(\gamma) - \mathrm{vol}([\gamma])$ for $\delta = 0$.

Now assume $\delta \geqslant 1$. One sees from the explicit computation in Case III that for $g = hg_{i,j,u}$, all $q+1$ vertices $gw_xKZ$ and $gw_\infty KZ$ are distinct vertices in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A([\gamma])$ provided that $0 \leqslant i \leqslant \delta-1$; when $i=\delta$, only one vertex, $gw_\infty KZ$, lies in $C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A(\gamma)$. They give rise to $\mathrm{vol}([\gamma])((q^\delta-1)(q+1)/(q-1)+q^\delta) = \mathrm{vol}([\gamma])(2(q^{\delta+1}-1)/(q-1)-1)$ pointed chambers. Therefore $N_B(\gamma) = 2N_K(\gamma) - \mathrm{vol}([\gamma])$ for $\delta \geqslant 1$.

This shows that the contribution to $u\frac{d}{du}\log Z_{1,2}(X_\Gamma, u) - u\frac{d}{du}\log Z_{2,1}(X_\Gamma, -u)$ from a ramified rank-one split $\gamma$ with $[\gamma]$ of type $(n,0)$ is $\mathrm{vol}([\gamma])u^{2n}$.

Case V. $[\gamma]$ of type $(n,1)$. Then it has no contribution to the type 2 edge zeta function, and it has contribution to the type 1 chamber zeta function only when $\gamma$ is ramified rank-one split. Then $r_\gamma$ has eigenvalues $a, e+d\lambda, e+d\bar{\lambda}$, where $a,e,d \in F$, $\mathrm{ord}_\pi a = 2n-1$, $\mathrm{ord}_\pi e \geqslant 1$ and $\delta = \mathrm{ord}_\pi d = 0$ by the analysis above Theorem 8.3.1. Its contribution to $u\frac{d}{du}\log Z_{2,1}(X_\Gamma, -u)$ is $-N_B(\gamma)u^{2n+1}$ with $N_B(\gamma) = \#C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_G([\gamma])$. Since $\delta = 0$ and $\mu = 0$ by the remark following Theorem 8.4.3, we have $\Delta_G([\gamma]) = \Delta_A([\gamma])$ such that $N_B(\gamma) = \mathrm{vol}([\gamma])$ by Corollary 8.4.2.

This completes the proof of the theorem. □

An immediate consequence of the theorem above is a description of the number $M_n(X_\Gamma)$ of closed, type 1, tailless geodesic galleries of length $n$ in $X_\Gamma$, given below:

**Corollary 10.2.2.** (1) *If $n = 2m+1$ is odd, then*

$$M_n(X_\Gamma) = \sum_{\substack{\gamma\in[\Gamma] \text{ ramified rank-one split,} \\ [\gamma] \text{ of type } (1,m)}} \mathrm{vol}\bigl([\gamma]\bigr);$$

(2) *If $n = 2m$ is even, then*

$$\begin{aligned} M_n(X_\Gamma) = {} & \sum_{\substack{\gamma\in[\Gamma] \text{ split,} \\ [\gamma] \text{ of type } (0,m)}} \mathrm{vol}\bigl([\gamma]\bigr)\omega_{[\gamma]} + \sum_{\substack{\gamma\in[\Gamma] \text{ irregular,} \\ [\gamma] \text{ of type } (0,m)}} \mathrm{vol}\bigl([\gamma]\bigr)q \\ & + \sum_{\substack{\gamma\in[\Gamma] \text{ unramified rank-one split,} \\ [\gamma] \text{ of type } (0,m)}} \mathrm{vol}\bigl([\gamma]\bigr)(\omega_{[\gamma]} - 2) \\ & + \sum_{\substack{\gamma\in[\Gamma] \text{ ramified rank-one split,} \\ [\gamma] \text{ of type } (0,m)}} \mathrm{vol}\bigl([\gamma]\bigr)(\omega_{[\gamma]} - 1). \end{aligned}$$

*Here* $\mathrm{vol}([\gamma])$ *is defined by* (5.4) *and* $\omega_{[\gamma]}$ *is as in Theorem* 7.2.1 *and Proposition* 8.4.4.

## 11. A proof of Theorem C

### *11.1. Hecke operators on $X_\Gamma$ and cycle counting*

The action of the Hecke operator $T_{n,m}$ on $L^2(\Gamma\backslash G/KZ)$ is represented by the matrix $B_{n,m}$, whose rows and columns are indexed by vertices of $X_\Gamma$ such that the entry at the row indexed by $\Gamma gKZ$ and column indexed by $\Gamma g'KZ$ records the number of homotopy classes of 1-geodesic paths of type $(n,m)$ from $\Gamma gKZ$ to $\Gamma g'KZ$ in $X_\Gamma$. Alternatively, this is the number of $\gamma \in \Gamma$ such that the homotopy classes of the 1-geodesics in $\mathcal{B}$ from $gKZ$ to $\gamma g'KZ$ have type $(n,m)$. The trace of $B_{n,m}$ then gives the number of 1-geodesic cycles of type $(n,m)$ up to homotopy. In other words,

$$\mathrm{Tr}(B_{n,m}) = \#\bigl\{\kappa_\gamma(gKZ) \bigm| \gamma \in [\Gamma],\ \kappa_\gamma(gKZ) \in [\gamma] \text{ has type } (n,m)\bigr\}.$$

To facilitate our computations, form two kinds of formal power series:

$$\sum_{\substack{n,m\geqslant 0\\(n,m)\neq(0,0)}} \mathrm{Tr}(B_{n,m})u^{n+2m} = \sum_{\gamma\in[\Gamma],\,\gamma\neq id}\ \sum_{\kappa_\gamma(gKZ)\in[\gamma]} u^{l_A(\kappa_\gamma(gKZ))}, \tag{11.1}$$

and

$$\sum_{n>0}\mathrm{Tr}(B_{n,0})u^n = \sum_{\gamma\in[\Gamma],\,\gamma\neq id}\ \sum_{\kappa_\gamma(gKZ)\in[\gamma]\text{ has type 1}} u^{l_A(\kappa_\gamma(gKZ))}. \tag{11.2}$$

We can relate the left hand side of the zeta identity (1.6) to cycle counting:

**Proposition 11.1.1.**

$$\begin{aligned} &u\frac{d}{du}\log\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I-A_1u+A_2qu^2-q^3u^3I)}\\ &\quad= q\left(\sum_{n>0}\mathrm{Tr}(B_{n,0})u^n\right)-(q-1)\left(\sum_{\substack{n,m\geqslant 0\\(n,m)\neq(0,0)}}\mathrm{Tr}(B_{n,m})u^{n+2m}\right)\frac{1-q^2u^3}{1-u^3}, \end{aligned} \tag{11.3}$$

*where the operators act on $L^2(\Gamma\backslash G/KZ)$, $\chi(X_\Gamma)=\frac{(q+1)(q-1)^2}{3}V$ is the Euler characteristic of $X_\Gamma$, and $V$ is the number of vertices in $X_\Gamma$.*

**Proof.** As $B_{n,m}$ is $T_{n,m}$ acting on the space $L^2(\Gamma\backslash G/KZ)$, so (2.1) also holds with $T_{n,m}$ replaced by $B_{n,m}$. In other words,

$$\begin{aligned} &u\frac{d}{du}\mathrm{Tr}\log\frac{(1-u^3)^rI}{(I-A_1u+A_2qu^2-q^3u^3I)}\\ &\quad= q\left(\sum_{n>0}\mathrm{Tr}(B_{n,0})u^n\right)-(q-1)\left(\sum_{\substack{n,m\geqslant 0\\(n,m)\neq(0,0)}}\mathrm{Tr}(B_{n,m})u^{n+2m}\right)\frac{1-q^2u^3}{1-u^3}, \end{aligned}$$

where $r=\frac{(q+1)(q-1)^2}{3}$. Recall that each vertex is incident to $q^2+q+1$ type 1 edges and $q^2+q+1$ type 2 edges so that the total number of undirected edges in $X_\Gamma$ is $\frac{2(q^2+q+1)}{2}V$. Since each edge is contained in $(q+1)$ chambers, the number of chambers in $X_\Gamma$ is $\frac{(q+1)}{3}(q^2+q+1)V$. Therefore the Euler characteristic of $X_\Gamma$ is

$$\chi(X_\Gamma)=V-\left(q^2+q+1\right)V+\frac{(q+1)}{3}\left(q^2+q+1\right)V=\frac{(q-1)^2(q+1)}{3}V=rV.$$

Using the identity

$$\log(\det A)=\mathrm{Tr}(\log A)$$

for a $V \times V$ matrix $A$, we have

$$u\frac{d}{du}\operatorname{Tr}\log\frac{(1-u^3)^r I}{(I-A_1u+A_2qu^2-q^3u^3I)} = u\frac{d}{du}\log\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I-A_1u+A_2qu^2-q^3u^3I)},$$

which proves the proposition. □

*11.2. Type 1 edge zeta function revisited*

Although the type 1 edge zeta function only concerns type 1 tailless cycles, to prove the theorem we shall involve all homotopy cycles. Denote by $P_{n,m,s}$, $P_{n,m,i}$, $Q_{n,m}$, and $R_{n,m}$ the number of algebraically minimal homotopy cycles of type $(n,m)$ contained in the conjugacy classes of split, irregular, unramified rank-one split, and ramified rank-one split $\gamma$'s, respectively. More precisely,

$$\begin{aligned} P_{n,m,s} &= \sum_{\substack{\gamma\in[\Gamma] \text{ split} \\ [\gamma] \text{ of type } (n,m)}} \#\big(C_{P_\gamma^{-1}\Gamma P_\gamma}(r_\gamma)\backslash\Delta_A\big([\gamma]\big)\big) \\ &= \sum_{\substack{\gamma\in[\Gamma] \text{ split} \\ [\gamma] \text{ of type } (n,m)}} \operatorname{vol}\big([\gamma]\big)\omega_{[\gamma]}, \end{aligned} \tag{11.4}$$

and $P_{n,m,i}$, $Q_{n,m}$, and $R_{n,m}$ are similarly defined by changing the type of $\gamma$ accordingly. Recall that an irregular $\gamma$ has type 1 or 2 so that $P_{n,m,i} = 0$ if $nm \neq 0$. Further since $\gamma$ has type $(n,0)$ if and only if $\gamma^{-1}$ has type $(0,n)$, we have $P_{n,0,i} = P_{0,n,,i}$. By Theorem 8.5.1, the type 1 edge zeta function can be restated as

**Proposition 11.2.1.**

$$u\frac{d}{du}\log Z_{1,1}(X_\Gamma,u) = \sum_{n>0}(P_{n,0,s}+P_{n,0,i}+Q_{n,0}+R_{n,0})u^n.$$

*11.3. The number of homotopy cycles of type $(n,m)$*

In order to gain information on $P_{n,0,s}$, $P_{n,0,i}$, $Q_{n,0}$ and $R_{n,0}$, we extend the summation to include homotopy cycles of type $(n,m)$. Recall that the number of such cycles is $\operatorname{Tr}(B_{n,m})$, and cycles with tails are also included. Their relation with the number of algebraically tailless cycles is given below.

**Proposition 11.3.1.** *With the same notation as in Theorem 8.4.3, we have*

$$\sum_{\substack{n,m\geqslant 0 \\ (n,m)\neq(0,0)}} \operatorname{Tr}(B_{n,m})u^{n+2m}$$

$$= \left( \sum_{\substack{n,m \geqslant 0 \\ (n,m) \neq (0,0)}} P_{n,m,s} u^{n+2m} \right) \frac{1-u^3}{1-q^3u^3} + \left( \sum_{\substack{n,m \geqslant 0 \\ (n,m) \neq (0,0)}} P_{n,m,i} u^{n+2m} \right) \frac{1-u^3}{1-q^2u^3}$$

$$+ \sum_{\substack{\gamma \in [\Gamma] \\ [\gamma] \text{ unram. rank-one split}}} \operatorname{vol}([\gamma]) u^{l_A([\gamma])}$$

$$\times \left( \frac{q^{\delta([\gamma])+1} + q^{\delta([\gamma])} - 2}{q-1} + \frac{(q+1)q^{\delta([\gamma])+2}u^3}{1-q^3u^3} \right) \left( \frac{1-u^3}{1-q^2u^3} \right)$$

$$+ \sum_{\substack{\gamma \in [\Gamma] \\ [\gamma] \text{ ram. rank-one split}}} \operatorname{vol}([\gamma]) q^{\mu([\gamma])} u^{l_A([\gamma])} \left( \frac{q^{\delta([\gamma])+1} - 1}{q-1} + \frac{q^{\delta([\gamma])+3}u^3}{1-q^3u^3} \right) \frac{1-u^3}{1-q^2u^3}.$$

**Proof.** Break the right side of (11.1) into four parts, over split, irregular, unramified rank-one split, and ramified rank-one split $\gamma$'s, respectively. Applying Theorem 7.2.1 to the split and irregular part, Theorem 8.4.3 to the unramified and ramified rank-one split parts, and using the definitions of $P_{n,m,s}$ and $P_{n,m,i}$, we get the desired formula. □

Next we compute the number of type 1 homotopy cycles on $X_\Gamma$.

**Proposition 11.3.2.** *With the same notation as in Theorem 8.4.3, we have*

$$\sum_{n>0} \operatorname{Tr}(B_{n,0}) u^n$$

$$= (1-q^{-1}) \left( \sum_{(n,m) \neq (0,0)} P_{n,m,s} u^{n+2m} \right) \frac{1-q^2u^3}{1-q^3u^3}$$

$$+ \sum_{\substack{\gamma \in [\Gamma] \\ [\gamma] \text{ unram. rank-one split}}} \operatorname{vol}([\gamma]) u^{l_A([\gamma])} \left( q^{\delta([\gamma])} + q^{\delta([\gamma])-1} + \frac{(q^2-1)q^{\delta([\gamma])+1}u^3}{1-q^3u^3} \right)$$

$$+ \sum_{\substack{\gamma \in [\Gamma] \\ [\gamma] \text{ ram. rank-one split}}} \operatorname{vol}([\gamma]) u^{l_A([\gamma])} \left( q^{\delta} (q^{\mu} - \mu) + \frac{(q-1)q^{\delta+\mu+2}u^3}{1-q^3u^3} \right)$$

$$+ q^{-1} \sum_{n>0} (P_{n,0,s} + qP_{n,0,i} + Q_{n,0} + R_{n,0}) u^n - 2q^{-1}$$

$$\times \sum_{\substack{\gamma \in [\Gamma], \text{ type } 1 \\ [\gamma] \text{ unram. rank-one split}}} \operatorname{vol}([\gamma]) u^{l_A([\gamma])}$$

$$+ \sum_{\substack{\gamma \in [\Gamma], \text{ type } 1 \\ [\gamma] \text{ ram. rank-one split}}} \operatorname{vol}([\gamma]) u^{l_A([\gamma])} \left( -q^{\mu([\gamma])-1} + \mu([\gamma]) q^{\delta([\gamma])} \right).$$

**Proof.** By definition,

$$\sum_{n>0}\operatorname{Tr}(B_{n,0})u^n=\sum_{\gamma\in[\Gamma]}\sum_{\kappa_\gamma(gK)\in[\gamma]\text{ type }1}u^{l_A(\kappa_\gamma(gK))}.$$

We split the sum over $\gamma$ into four parts according to $\gamma$ split, irregular, unramified rank-one split, or ramified rank-one split. For the split part, we add (A) and (B) of Theorem 7.3.1 and use the definition of $P_{n,m,s}$ to arrive at the sum

$$\left(1-q^{-1}\right)\left(\sum_{(n,m)\neq(0,0)}P_{n,m,s}u^{n+2m}\right)\frac{1-q^2u^3}{1-q^3u^3}+q^{-1}\left(\sum_{n>0}P_{n,0,s}u^n\right).$$

For the irregular part, Theorem 7.3.1, (C) gives the contribution $\sum_{n>0}P_{n,0,i}u^n$. For the unramified (resp. ramified) rank-one split part, we add (A2) and (A3) (resp. (B2) and (B3)) of Theorem 8.4.3 to get

$$\begin{aligned}
&\sum_{\substack{\gamma\in[\Gamma]\\ [\gamma]\text{ unram. rank-one split}}}\operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\left(q^{\delta([\gamma])}+q^{\delta([\gamma])-1}+\frac{(q^2-1)q^{\delta([\gamma])+1}u^3}{1-q^3u^3}\right)\\
&+\sum_{\substack{\gamma\in[\Gamma],\text{ type }1\\ [\gamma]\text{ unram. rank-one split}}}\operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\frac{q^{\delta([\gamma])}+q^{\delta([\gamma])-1}-2}{q-1}\\
&+\sum_{\substack{\gamma\in[\Gamma]\\ [\gamma]\text{ ram. rank-one split}}}\operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\\
&\times\left(q^{\delta([\gamma])}\bigl(q^{\mu([\gamma])}-\mu\bigl([\gamma]\bigr)\bigr)+\frac{(q-1)q^{\delta([\gamma])+\mu([\gamma])+2}u^3}{1-q^3u^3}\right)\\
&+\sum_{\substack{\gamma\in[\Gamma],\text{ type }1\\ [\gamma]\text{ ram. rank-one split}}}\operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\left(q^{\mu([\gamma])}\frac{q^{\delta([\gamma])}-1}{q-1}+\mu\bigl([\gamma]\bigr)q^{\delta([\gamma])}\right).
\end{aligned}\tag{11.5}$$

It follows from Proposition 8.4.4 and the definitions of $Q_{n,0}$ and $R_{n,0}$ that

$$\begin{aligned}
&\sum_{\substack{\gamma\in[\Gamma],\text{ type }1\\ [\gamma]\text{ unram. rank-one split}}}\operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}\frac{q^{\delta([\gamma])}+q^{\delta([\gamma])-1}-2}{q-1}\\
&\quad=q^{-1}\sum_{n>0}Q_{n,0}u^n-2q^{-1}\sum_{\substack{\gamma\in[\Gamma],\text{ type }1\\ [\gamma]\text{ unram. rank-one split}}}\operatorname{vol}\bigl([\gamma]\bigr)u^{l_A([\gamma])}
\end{aligned}\tag{11.6}$$

and

$$\sum_{\substack{\gamma\in[\Gamma],\ \text{type 1}\\ [\gamma]\ \text{ram. rank-one split}}} \operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\left(q^{\mu([\gamma])}\frac{q^{\delta([\gamma])}-1}{q-1}+\mu\big([\gamma]\big)q^{\delta([\gamma])}\right)$$
$$= q^{-1}\sum_{n>0} R_{n,0}u^n$$
$$+ \sum_{\substack{\gamma\in[\Gamma],\ \text{type 1}\\ [\gamma]\ \text{ram. rank-one split}}} \operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\big(-q^{\mu([\gamma])-1}+\mu\big([\gamma]\big)q^{\delta([\gamma])}\big). \tag{11.7}$$

Finally, plug (11.6) and (11.7) into (11.5) to complete the proof. □

### 11.4. *Proof of Theorem C*

Combining Propositions 11.3.2 and 11.3.1, we obtain

$$q\left(\sum_{n>0}\operatorname{Tr}(B_{n,0})u^n\right)-(q-1)\left(\sum_{\substack{n,m\geqslant 0\\ (n,m)\neq(0,0)}}\operatorname{Tr}(B_{n,m})u^{n+2m}\right)\left(\frac{1-q^2u^3}{1-u^3}\right)$$
$$=\sum_{n>0}(P_{n,0,s}+P_{n,0,i}+Q_{n,0}+R_{n,0})u^n-(q-1)\sum_{\gamma\in[\Gamma],\ \text{irregular, type 2}}\operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}$$
$$+\sum_{\substack{\gamma\in[\Gamma],\ \text{not type 1}\\ [\gamma]\ \text{unram. rank-one split}}} 2\operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}$$
$$+\sum_{\substack{\gamma\in[\Gamma],\ \text{not type 1}\\ [\gamma]\ \text{ram. rank-one split}}} \operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\big(q^{\mu([\gamma])}-\mu\big([\gamma]\big)q^{\delta([\gamma])+1}\big)$$

since all irregular elements have type 1 or 2. As before, to a rank-one split $\gamma$, we associate $r_\gamma=\begin{pmatrix} a & & \\ & e & dc \\ & d & e+db\end{pmatrix}$. First assume $\gamma$ is unramified rank-one split. By Theorem 8.3.1, $[\gamma]$ has type $(n,m)=(\operatorname{ord}_\pi a,\min(\operatorname{ord}_\pi e,\operatorname{ord}_\pi d))$, hence $[\gamma]$ is not of type 1 if and only if $a$ is a unit, which is equivalent to its inverse $[\gamma^{-1}]$ having type $(m,0)$. Note that $l_A([\gamma])=2m=2l_A([\gamma^{-1}])$ by Theorem 8.3.1. Next assume that $[\gamma]$ is ramified rank-one split. Since $\mu([\gamma])=1$ implies $\delta([\gamma])=0$, we have $q^{\mu([\gamma])}-\mu([\gamma])q^{\delta([\gamma])+1}=0$ in this case. Thus we need only consider the case $\mu([\gamma])=0$ so that $q^{\mu([\gamma])}-\mu([\gamma])q^{\delta([\gamma])+1}=1$. Then $[\gamma]$ is not of type 1 if and only if $a$ is a unit, in which case it has type $(0,\operatorname{ord}_\pi e)$ if $\operatorname{ord}_\pi e\leqslant\operatorname{ord}_\pi d$, and type $(1,\operatorname{ord}_\pi d)$ if $\operatorname{ord}_\pi d<\operatorname{ord}_\pi e$ by Theorem 8.3.1. Further, we see that $[\gamma^{-1}]$ has type $(\operatorname{ord}_\pi e,0)$ so that $l_A([\gamma])=2l_A([\gamma^{-1}])=2\operatorname{ord}_\pi e$ in the former case, and in the latter case, $[\gamma^{-1}]$ has type $(\operatorname{ord}_\pi d,1)$, $[\gamma^{-2}]$ has type $(2\operatorname{ord}_\pi d+1,0)$ and $l_A([\gamma])=1+2\operatorname{ord}_\pi d=l_A([\gamma^{-2}])$. Further, we have $\operatorname{vol}([\gamma])=\operatorname{vol}([\gamma^{-1}])=\operatorname{vol}([\gamma^{-2}])$ for $\gamma$ rank-one split. Consequently, we may replace $\gamma$ by $\gamma^{-1}$ and rewrite

$$\sum_{\substack{\gamma\in[\Gamma],\ \text{not type 1}\\ [\gamma]\ \text{unram. rank-one split}}} 2\operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}$$

$$
\begin{aligned}
&+ \sum_{\substack{\gamma\in[\Gamma],\ \text{not type 1}\\ [\gamma]\ \text{ram. rank-one split}}} \operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma])}\big(q^{\mu([\gamma])} - \mu\big([\gamma]\big)q^{\delta([\gamma])+1}\big)\\
&= \sum_{\gamma\in[\Gamma],\ \text{type 1 unram. rank-one split}} 2\operatorname{vol}\big([\gamma]\big)u^{2l_A([\gamma])}\\
&+ \sum_{\gamma\in[\Gamma],\ \text{type 1 ram. rank-one split}} \operatorname{vol}\big([\gamma]\big)u^{2l_A([\gamma])}\\
&+ \sum_{\gamma\in[\Gamma],\ [\gamma]\ \text{of type}\ (m,1),\ \text{ram. rank-one split}} \operatorname{vol}\big([\gamma]\big)u^{l_A([\gamma^2])}.
\end{aligned}
$$

Together with the term $-(q-1)\sum_{\gamma\in[\Gamma],\ \text{irregular, type 2}}\operatorname{vol}([\gamma])u^{l_A([\gamma])}$, it gives the difference of the logarithmic derivatives of $Z_1(X_\Gamma, u^2)$ and $Z_B(X_\Gamma, -u)$ by Theorem 10.2.1. Here we used the fact that the inverse of a type 2 irregular element $\gamma$ is type 1 irregular, and $\operatorname{vol}([\gamma]) = \operatorname{vol}([\gamma^{-1}])$.

Combined with Propositions 11.1.1 and 11.2.1, this proves

**Proposition 11.4.1.**

$$
\begin{aligned}
&u\frac{d}{du}\log\left(\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I - A_1u + qA_2u^2 - q^3Iu^3)}\right)\\
&\quad = q\left(\sum_{n>0}\operatorname{Tr}(B_{n,0})u^n\right) - (q-1)\left(\sum_{\substack{n,m\geqslant 0\\ (n,m)\neq(0,0)}}\operatorname{Tr}(B_{n,m})u^{n+2m}\right)\left(\frac{1-q^2u^3}{1-u^3}\right)\\
&\quad = u\frac{d}{du}\log Z_{1,1}(X_\Gamma, u) + u\frac{d}{du}\log Z_{1,2}(X_\Gamma, u) - u\frac{d}{du}\log Z_{2,1}(X_\Gamma, -u).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\frac{(1-u^3)^{\chi(X_\Gamma)}}{\det(I - A_1u + qA_2u^2 - q^3Iu^3)} &= c\frac{Z_{1,1}(X_\Gamma, u)Z_{1,2}(X_\Gamma, u)}{Z_{2,1}(X_\Gamma, -u)}\\
&= c\frac{\det(1 + L_Bu)}{\det(I - L_Eu)\det(I - (L_E)^t u^2)}
\end{aligned}
$$

for some constant $c$. Here the last equality comes from Propositions 10.1.1 and 6.1.1. Since both sides are formal power series with constant term 1, we find $c = 1$. This concludes the proof of Theorem C.

## References


[1] H. Bass, The Ihara–Selberg zeta function of a tree lattice, Internat. J. Math. 3 (6) (1992) 717–797.

[2] A. Deitmar, J.W. Hoffman, The Ihara–Selberg zeta function for PGL(3) and Hecke operators, Internat. J. Math. 17 (2006) 143–156.

[3] Y. Fang, W.-C.W. Li, C.-J. Wang, The zeta functions of complexes from $Sp(4)$, Int. Math. Res. Not. IMRN (2012), http://dx.doi.org/10.1093/imrn/rns007.

[4] Y.Z. Flicker, Matching of Orbital Integrals on GL(4) and GSp(2), Amer. Math. Soc., 1999.
[5] B. Garrett, Buildings and Classical Groups, Chapman & Hall/CRC, 1997.
[6] K. Hashimoto, Zeta functions of finite graphs and representations of $p$-adic groups, in: Automorphic Forms and Geometry of Arithmetic Varieties, in: Adv. Stud. Pure Math., vol. 15, Academic Press, Boston, MA, 1989, pp. 211–280.
[7] K. Hashimoto, Artin type $L$-functions and the density theorem for prime cycles on finite graphs, Internat. J. Math. 3 (6) (1992) 809–826.
[8] J.W. Hoffman, Remarks on the zeta function of a graph, in: Dynamical Systems and Differential Equations, Wilmington, NC, 2002, Discrete Contin. Dyn. Syst. (suppl.) (2003) 413–422.
[9] Y. Ihara, On discrete subgroups of the two by two projective linear group over $p$-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[10] M.-H. Kang, W.-C.W. Li, C.-J. Wang, The Zeta functions of complexes from PGL(3): a representation-theoretic approach, Israel J. Math. 177 (2010) 335–347.
[11] L. Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, Invent. Math. 147 (2002) 1–241.
[12] G. Laumon, M. Rapoport, U. Stuhler, $\mathcal{D}$-elliptic sheaves and the Langlands correspondence, Invent. Math. 113 (1993) 217–238.
[13] W.-C.W. Li, Ramanujan hypergraphs, Geom. Funct. Anal. 14 (2004) 380–399.
[14] A. Lubotzky, R. Phillips, P. Sarnak, Ramanujan graphs, Combinatorica 8 (1988) 261–277.
[15] A. Lubotzky, B. Samuels, U. Vishne, Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$, European J. Combin. 26 (6) (2005) 965–993.
[16] A. Lubotzky, B. Samuels, U. Vishne, Ramanujan complexes of type $\tilde{A}_d$, Israel J. Math. 149 (2005) 267–299.
[17] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, 1972.
[18] A. Sarveniazi, Explicit construction of a Ramanujan $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph, Duke Math. J. 139 (1) (2007) 141–171.
[19] I. Satake, Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields, Inst. Hautes Études Sci. Publ. Math. 18 (1963) 5–69.
[20] H. Stark, A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996) 124–165.
[21] T. Tamagawa, On the $\zeta$-functions of a division algebra, Ann. of Math. (2) 77 (1963) 387–405.