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A characterization of bipartite distance-regular graphs

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ABSTRACT

It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular. Examples are given to show that the converse does not hold. Thus, a natural question is to find out when the converse is true. In this paper we give a quasi-spectral characterization of a connected bipartite weighted 2-punctually distance-regular graph whose halved graphs are distance-regular. In the case the spectral diameter is even we show that the graph characterized above is distance-regular.

1. Introduction

The study of characterizing the graphs whose eigenvalues and/or multiplicities satisfy a prescribed identity has a long history. For example, a well-known and real-world applicable theory asserts that a connected graph is bipartite if and only if its largest eigenvalue and smallest eigenvalue have the same absolute value. Recently, the eigenvectors, especially the one associated with the largest eigenvalue, are also taking into consider-

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ation, for instances, in mathematical theory: [18,19,15,16,13,22]; in applications: [7,4]. See [6, pp. 65–69] for more applications. In this paper, we will give a (quasi-spectral) characterization of graphs when an identity involving eigenvalues, multiplicities, the eigenvector corresponding to the largest eigenvalue, and partial graph structure is satisfied. The details are as follows.

Throughout this paper, let G be a connected graph with vertex set V, order $n = |V|$, diameter D, and distance function ∂. The *adjacency matrix* A of G is the binary matrix indexed by V, where the entry $(A)_{uv} = 1$ if $\partial(u, v) = 1$, and $(A)_{uv} = 0$ otherwise. Assume that A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ with corresponding multiplicities $m_0 = 1, m_1, \ldots, m_d$. The *spectrum* of G is denoted by $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, and the parameter d is called the *spectral diameter* of G. Note that $D \leqslant d$ [3]. As is known, there is a sequence of orthogonal polynomials $p_0, p_1, \ldots, p_d$ with respect to the inner product $\langle\, , \rangle_G$ (formally defined in the beginning of the next section), where $\deg p_i = i$ and $\langle p_i, p_i \rangle_G = p_i(\lambda_0)$ for $0 \leqslant i \leqslant d$ [15]. Let α be the eigenvector of A associated with λ_0 such that $\alpha^t \alpha = n$ and all entries of α are positive. Note that α is usually called the *Perron vector*, and $\alpha = (1, 1, \ldots, 1)^t$ if and only if G is regular. For $u \in V$, let α_u be the entry corresponding to u in the eigenvector α. For $0 \leqslant i \leqslant d$, define the *weighted distance-i matrix* $\widetilde{A}_i$ of G to be the matrix indexed by V such that the entry $(\widetilde{A}_i)_{uv} = \alpha_u \alpha_v$ if $\partial(u, v) = i$, and $(\widetilde{A}_i)_{uv} = 0$ otherwise. In particular, for the case G is regular, $\widetilde{A}_i$ is binary and is the so-called *distance-i matrix* A_i of G. For an integer $h \leqslant d$, we say that G is *weighted h-punctually distance-regular* if $\widetilde{A}_h = p_h(A)$. Define $\widetilde{\delta}_i = \sum_{u,v} (\widetilde{A}_i \circ \widetilde{A}_i)_{uv}/n$, where "$\circ$" is the entrywise product of matrices. A bipartite graph with bipartition (X, Y) is called (k_1, k_2)-*biregular* if every vertex in X has degree k_1 and every vertex in Y has degree k_2. The *distance-i graph* of G is the graph whose adjacency matrix is the distance-i matrix of G. For a connected bipartite graph G with bipartition (X, Y), the *halved graphs* G^X and G^Y are the two connected components of the distance-2 graph of G. It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular [5, Proposition 4.2.2]. Examples 5.1–5.3 show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Thus, a natural question is to find out when the converse is true. Our main result is the following.

Theorem 1.1. *Let G be a connected bipartite graph with bipartition (X, Y). Suppose that G is weighted 2-punctually distance-regular with even spectral diameter, and both halved graphs G^X and G^Y are distance-regular. Then G is distance-regular.*

In addition to the main result, we believe that Proposition 3.3, Theorem 3.4, Proposition 4.5 and Theorem 5.6 are of independent interest.

This paper is organized as follows. In the next section we provide some simple but useful lemmas for bipartite graphs. In Section 3, we present some results related to the spectral excess theorem [15], and characterize the graphs with $\widetilde{\delta}_i = p_i(\lambda_0)$ for $i \in \{0, 1\}$ (Lemma 3.7). In particular, this lemma is very useful for checking the reg-

ularity or biregularity of a graph. In Section 4, we study the concepts of weighted punctual distance-regularity and weighted partial distance-regularity, which can be regarded as generalizations of the concepts of punctual distance-regularity and partial distance-regularity [11,10]. In Section 5, we prove Theorem 1.1.

2. Some results for bipartite graphs

In this section we provide some simple but useful lemmas to be used later on. These results are related to the concept of orthogonal polynomials. The basic idea is to generalize the study of distance-regular graphs (see [3,5,24]).

2.1. Three-term recurrence

From the spectrum $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ we consider the $(d+1)$-dimensional vector space $\mathbb{R}_d[x]$ of real polynomials of degree at most d with inner product

$$\langle p, q\rangle_G := \sum_{i=0}^{d} \frac{m_i}{n} p(\lambda_i) q(\lambda_i) = \operatorname{tr}\bigl(p(A)q(A)\bigr)/n.$$

The *predistance polynomials* $p_0, p_1, \dots, p_d$ of G are orthogonal polynomials satisfying $\deg p_i = i$ and $\langle p_i, p_i\rangle_G = p_i(\lambda_0)$ for $0 \leqslant i \leqslant d$ [15]. Moreover, they satisfy a three-term recurrence of the form

$$xp_i = c_{i+1}p_{i+1} + a_i p_i + b_{i-1}p_{i-1} \tag{1}$$

for $0 \leqslant i \leqslant d$, where c_{i+1}, a_i, b_{i-1} are scalars in $\mathbb{R}$, called the *preintersection numbers* of G, with $b_{-1} = c_{d+1} := 0$ [12]. Note that $a_i + b_i + c_i = \lambda_0$ for $0 \leqslant i \leqslant d$, where $c_0 := 0$ and $b_d := 0$ [8]. If G is bipartite, then $a_i = 0$ for $0 \leqslant i \leqslant d$ [11], and thus $xp_i = c_{i+1}p_{i+1} + b_{i-1}p_{i-1}$. By this observation, the predistance polynomials of bipartite graphs satisfy a three-term recurrence of the form

$$x^2 p_i = X_{i+2}p_{i+2} + Y_i p_i + Z_{i-2}p_{i-2} \tag{2}$$

for $0 \leqslant i \leqslant d$, where $X_{i+2} := c_{i+1}c_{i+2}$, $Y_i := b_i c_{i+1} + b_{i-1}c_i$ and $Z_{i-2} := b_{i-2}b_{i-1}$. By directly computing, it follows that $X_i + Y_i + Z_i = \lambda_0^2$ for $0 \leqslant i \leqslant d$.

2.2. The 'odd' or 'even' part

The sum of all predistance polynomials gives the *Hoffman polynomial* H [20]:

$$H(x) := n \prod_{i=1}^{d} \frac{x - \lambda_i}{\lambda_0 - \lambda_i} = p_0 + p_1 + \cdots + p_d, \tag{3}$$

no matter whether the graph is regular or not. For a proof, see for instance [9]. Hoffman [20] proved that a connected graph G is regular if and only if $H(A) = J$, the all-ones matrix. The following result, given first in [14, p. 117] (see also [22, Lemma 2.1]) gives a generalization to nonregular graphs.

Lemma 2.1. *Let G be a connected graph with adjacency matrix A and Perron vector α. Then, $H(A) = \alpha\alpha^t$. Moreover, G is regular if and only if $H(A) = J$, the all-ones matrix.* □

By the construction of $\widetilde{A}_i$, Lemma 2.1 and (3), any connected graph G has the property that

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_D = H(A) = p_0(A) + p_1(A) + \cdots + p_d(A). \tag{4}$$

If G is bipartite, then we can rewrite (4) (in Lemma 2.2) more precisely by only taking the 'odd' or 'even' part, which was also considered in [11]. Define $\widetilde{A}^{\text{odd}} = \sum_{\text{odd } i} \widetilde{A}_i$, $p^{\text{odd}} = \sum_{\text{odd } i} p_i$ and $\widetilde{\delta}^{\text{odd}} = \sum_{\text{odd } i} \widetilde{\delta}_i$. Similarly for $\widetilde{A}^{\text{even}}$, p^{even} and $\widetilde{\delta}^{\text{even}}$. For two $n \times n$ real symmetric matrices M and N, define the inner product

$$\langle M, N\rangle := \frac{1}{n}\operatorname{tr}(MN) = \frac{1}{n}\sum_{u,v}(M \circ N)_{uv}.$$

Then $\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_D$ are orthogonal, and $\widetilde{\delta}_i = \langle \widetilde{A}_i, \widetilde{A}_i\rangle$. Hence $\langle \widetilde{A}^*, \widetilde{A}^*\rangle = \widetilde{\delta}^*$ and $\langle p^*(A), p^*(A)\rangle = p^*(\lambda_0)$ for $* \in \{\text{odd}, \text{even}\}$. If G is bipartite, then p_i is odd or even only depending on its degree i being odd or even [11]. The following lemma is proved by (4) and the fact that $(p_i(A))_{uv} = 0$ if $\partial(u, v)$ and i have distinct parity (since bipartite graphs contain no odd cycle).

Lemma 2.2. *If G is bipartite, then $\widetilde{A}^{\text{odd}} = p^{\text{odd}}(A)$ and $\widetilde{A}^{\text{even}} = p^{\text{even}}(A)$. Moreover, by taking norms, $\widetilde{\delta}^{\text{odd}} = p^{\text{odd}}(\lambda_0)$ and $\widetilde{\delta}^{\text{even}} = p^{\text{even}}(\lambda_0)$.* □

Remark 2.3. Observe that $p^{\text{odd}} = (H(x) - H(-x))/2$, $H(\lambda_0) = n$ and $H(\lambda_d) = 0$. Thus for bipartite graphs, we deduce that $\widetilde{\delta}^{\text{odd}} = \widetilde{\delta}^{\text{even}} = p^{\text{odd}}(\lambda_0) = p^{\text{even}}(\lambda_0) = n/2$.

3. The spectral excess theorem

The spectral excess theorem [15] asserts that $\widetilde{\delta}_d \leqslant p_d(\lambda_0)$ if G is regular, and equality is attained if and only if G is distance-regular. See [9,17] for short proofs, and [11,10] for some generalizations. The parameter $p_d(\lambda_0)$ is called the *spectral excess* of G, which can be expressed in terms of the spectrum, which is

$$p_d(\lambda_0) = \frac{n}{\pi_0^2}\left(\sum_{i=0}^{d}\frac{1}{m_i\pi_i^2}\right)^{-1},$$

where $\pi_i = \prod_{j\neq i} |\lambda_i - \lambda_j|$ for $0 \leqslant i \leqslant d$ [15]. The following lemma gives an expression of $p_{d-1}(\lambda_0)$ for bipartite graphs in terms of the spectrum. The proof is essentially identical to [9, p. 8–9], except the setting of the polynomials h_i.

Lemma 3.1. *Let G be a connected bipartite graph. Then*

$$p_{d-1}(\lambda_0) = n\left(2 + \sum_{i=1}^{d-1} \frac{(h_i(\lambda_0) + (-1)^{d-1}h_i(-\lambda_0))^2}{m_i h_i(\lambda_i)^2}\right)^{-1},$$

where $h_i = \prod_{j\neq 0,i,d}(x - \lambda_j)$ for $1 \leqslant i \leqslant d-1$. □

For $0 \leqslant i \leqslant d$, define $\widetilde{A}_{\geqslant i} = \sum_{j\geqslant i} \widetilde{A}_j$, $p_{\geqslant i} = \sum_{j\geqslant i} p_j$ and $\widetilde{\delta}_{\geqslant i} = \sum_{j\geqslant i} \widetilde{\delta}_j$. Similarly for $\widetilde{A}_{\leqslant i}$, $p_{\leqslant i}$ and $\widetilde{\delta}_{\leqslant i}$. The parameter $\widetilde{\delta}_D$ is referred to as the *average weighted excess* and $p_{\geqslant D}(\lambda_0)$ as the *generalized spectral excess* of G. Recently, the authors [22] proved the following 'weighted' version of the spectral excess theorem for nonregular graphs. In fact, the approach of giving weights, the entries of the Perron vector, to the vertices of a nonregular graph has been recently used many times in the literature (see, for instance, [18,19,15,16,13]).

Theorem 3.2. *(See [22].) Let G be a connected graph with diameter D. Then $\widetilde{\delta}_D \leqslant p_{\geqslant D}(\lambda_0)$ with equality if and only if $\widetilde{A}_D = p_{\geqslant D}(A)$. Moreover, suppose further that $D = d$. Then equality holds if and only if G is distance-regular.* □

Define $\widetilde{A}^{\mathrm{odd}}_{\geqslant i} = \sum_{\mathrm{odd}\, j\geqslant i} \widetilde{A}_j$, $\widetilde{\delta}^{\mathrm{odd}}_{\geqslant i} = \sum_{\mathrm{odd}\, j\geqslant i} \widetilde{\delta}_j$ and $p^{\mathrm{odd}}_{\geqslant i} = \sum_{\mathrm{odd}\, j\geqslant i} p_j$. Similarly for $\widetilde{A}^*_{\Omega}$, $\widetilde{\delta}^*_{\Omega}$ and p^*_{Ω}, where $(\Omega, *) \in \{(\geqslant i, \mathrm{even}), (\leqslant i, \mathrm{odd}), (\leqslant i, \mathrm{even})\}$. Summing the recurrence relation (1) from the terms with index $i+1$ to d, it follows that

$$xp_{\geqslant i+1} = b_i p_i + \lambda_0 p_{\geqslant i+1} - c_{i+1}p_{i+1}$$

[10, Proposition 2.5]. Note that if $\widetilde{A}_{\geqslant i+1} = p_{\geqslant i+1}(A)$ and $\partial(u,v) < i$ for $u, v \in V$, then $(Ap_{\geqslant i+1}(A))_{uv} = 0 = (\lambda_0 p_{\geqslant i+1}(A))_{uv}$, and thus $b_i(p_i(A))_{uv} = c_{i+1}(p_{i+1}(A))_{uv}$. Using this fact, we have the following result.

Proposition 3.3. *Let G be a connected bipartite graph and $i \leqslant d-1$. Then $\widetilde{A}_{\leqslant i} = p_{\leqslant i}(A)$ if and only if $\widetilde{A}_j = p_j(A)$ for $0 \leqslant j \leqslant i$.*

Proof. The sufficiency is clear. To prove necessity, we only need to show that $\widetilde{A}_i = p_i(A)$ (the remaining follows by similar argument). If $\partial(u,v) \geqslant i$, then $(\widetilde{A}_i)_{uv} = (p_i(A))_{uv}$ by assumption. Suppose $\partial(u,v) < i$. If $\partial(u,v)$ and i have different parity, then $(\widetilde{A}_i)_{uv} = 0 = (p_i(A))_{uv}$. Suppose that $\partial(u,v)$ and i have the same parity. Then $b_i(p_i(A))_{uv} = c_{i+1}(p_{i+1}(A))_{uv} = 0$. Since $b_i \neq 0$ for $i \leqslant d-1$, it follows that $(\widetilde{A}_i)_{uv} = 0 = (p_i(A))_{uv}$. □

In [22], the authors posed the problem of characterizing the graphs which satisfy equality in Theorem 3.2 (or equivalently, $\widetilde{A}_D = p_{\geqslant D}(A)$), and gave a simple solution: regular graphs with diameter 2 (in fact, these graphs are the so-called *distance-polynomial graphs* [25]). Under the condition $D = d$, such graphs are distance-regular (Theorem 3.2). Here we complete this characterization for bipartite graphs.

Theorem 3.4. *A connected bipartite graph with $\widetilde{A}_D = p_{\geqslant D}(A)$ is distance-regular.*

Proof. Note that the assumption is equivalent to $\widetilde{A}_{\leqslant D-1} = p_{\leqslant D-1}(A)$. By Proposition 3.3, $\widetilde{A}_i = p_i(A)$ for $0 \leqslant i \leqslant D-1$. By Lemma 2.2, it follows that $p^*_{\geqslant D+1}(A)$ is the zero matrix, where $* \in \{\text{odd}, \text{even}\}$ has the same parity as $D+1$. This happens only for the case $D = d$, since otherwise $p^*_{\geqslant D+1}(\lambda_0) = 0$, contradicting the fact that $p_i(\lambda_0) > 0$ for $0 \leqslant i \leqslant d$. The remaining follows from Theorem 3.2. □

Let

$$\mathrm{Proj}_N M := \frac{\langle N, M\rangle}{\langle N, N\rangle} N$$

denote the projection of M onto $\mathrm{Span}\{N\}$. Lemmas 3.5–3.6 present some inequalities related to the spectral excess theorem. The proofs are essentially the same as in [17, Lemma 1].

Lemma 3.5. *Let G be a connected graph. For $0 \leqslant i \leqslant d$,*

(i) $\widetilde{\delta}_{\geqslant i} \leqslant p_{\geqslant i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}_{\geqslant i} = p_{\geqslant i}(A)$, *and*
(ii) $\widetilde{\delta}_{\leqslant i} \geqslant p_{\leqslant i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}_{\leqslant i} = p_{\leqslant i}(A)$. □

Lemma 3.6. *Let G be a connected bipartite graph. For $0 \leqslant i \leqslant d$ and $* \in \{\text{odd}, \text{even}\}$,*

(i) $\widetilde{\delta}^*_{\geqslant i} \leqslant p^*_{\geqslant i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}^*_{\geqslant i} = p^*_{\geqslant i}(A)$, *and*
(ii) $\widetilde{\delta}^*_{\leqslant i} \geqslant p^*_{\leqslant i}(\lambda_0)$ *with equality if and only if* $\widetilde{A}^*_{\leqslant i} = p^*_{\leqslant i}(A)$. □

A natural question motivated by Lemmas 3.5–3.6 is to study the relation between the parameters $\widetilde{\delta}_i$ and $p_i(\lambda_0)$ for $0 \leqslant i \leqslant d-1$ (the case $i = d$ is given in Theorem 3.2). We give some results in the following. Note that $p_0 = 1$ and $p_1 = \lambda_0 x/\bar{k}$ (by the Gram–Schmidt procedure), where $\bar{k}$ is the average degree of G. Moreover,

$$\langle \widetilde{A}_1, A\rangle = \frac{1}{n}\sum_{u,v}(\widetilde{A}_1)_{uv} = \frac{1}{n}\mathbf{1}^t\widetilde{A}_1\mathbf{1} = \frac{1}{n}\mathbf{1}^t DAD\mathbf{1} = \lambda_0, \tag{5}$$

where $\mathbf{1}$ is the all-ones vector, and D is the diagonal matrix with entries $D_{uu} = \alpha_u$ for $u \in V$. The first part of Lemma 3.7 is simple, but plays a crucial role in proving

the regularity of a graph, which follows from the inequality $\widetilde{\delta}_{\leqslant 0} \geqslant p_{\leqslant 0}(\lambda_0)$ mentioned in Lemma 3.5. In fact, it can also be derived by the Cauchy–Schwarz inequality: $\sum_{u\in V} \alpha_u^4 \geqslant (\sum_{u\in V} \alpha_u^2)^2/n = n$. The second part of Lemma 3.7 characterizes the graphs satisfying $\widetilde{\delta}_1 = p_1(\lambda_0)$, which is useful for checking the regularity or biregularity of a graph.

Lemma 3.7. *Let G be a connected graph. Then*

1. $\widetilde{\delta}_0 \geqslant 1$ $(= p_0(\lambda_0))$ *(which is equivalent to $\sum_{u\in V} \alpha_u^4 \geqslant n$), with equality if and only if any of the following conditions holds*:
   (i) $\widetilde{A}_0 = I$ $(= p_0(A))$,
   (ii) *G is regular.*
2. $\widetilde{\delta}_1 \geqslant \lambda_0^2/\bar{k}$ $(= p_1(\lambda_0))$, *with equality if and only if any of the following conditions holds*:
   (i) $\widetilde{A}_1 = p_1(A)$,
   (ii) *G is regular or biregular.*

Proof. We only need to prove the second part. Computing $\mathrm{Proj}_{\widetilde{A}_1} p_1(A)$ by the same argument as in [17, Lemma 1] and (5), it follows that $\widetilde{\delta}_1 \geqslant p_1(\lambda_0)$, with equality if and only if $\widetilde{A}_1 = p_1(A)$. Now it remains to show that (i) $\Leftrightarrow$ (ii). To prove necessity, we give the weight α_u to the vertex $u \in V$, and the weight $\alpha_u\alpha_v$ to the edge connecting u and v. Since $\widetilde{A}_1 = p_1(A) = \lambda_0 A/\bar{k}$, all edges receive the same weight, $\lambda_0/\bar{k}$. If G is not bipartite, then it contains an odd cycle, and all vertices on this cycle must have the same weight. The assumption 'G is connected' deduces that all vertices are of the same weight. Thus G is regular. For the case G is bipartite, the condition 'all edges receive the same weight' implies that vertices in the same partite set have the same weight. Thus G is biregular. Now we prove sufficiency. If G is regular, then clearly $p_1(A) = \lambda_0 A/\bar{k} = A = \widetilde{A}_1$. Suppose that G is (k_1, k_2)-biregular with bipartition (X, Y), where $|X| = n_1$, $|Y| = n_2$. Note that $\lambda_0 = \sqrt{k_1k_2}$, $n_1k_1 = n_2k_2$ and the Perron vector

$$\alpha = (\underbrace{\alpha', \ldots, \alpha'}_{n_1} \underbrace{\alpha'', \ldots, \alpha''}_{n_2})^t,$$

where $\alpha' = \sqrt{\frac{n_1+n_2}{2n_1}}$ and $\alpha'' = \sqrt{\frac{n_1+n_2}{2n_2}}$. Thus

$$p_1(A) = \frac{\lambda_0}{\bar{k}} A = \frac{\sqrt{k_1k_2}(n_1+n_2)}{n_1k_1+n_2k_2} A = \frac{n_1+n_2}{2\sqrt{n_1n_2}} A = \alpha'\alpha'' A = \widetilde{A}_1. \qquad \square$$

The next question is to discuss the relation between $\widetilde{\delta}_2$ and $p_2(\lambda_0)$. We give the answer under the assumption G is regular, and provide an example to show that the regularity condition is necessary. Thus, there is no hope to determine the order of $\widetilde{\delta}_2$ and $p_2(\lambda_0)$ uniformly. Lemma 3.8 is proved by the inequality $\widetilde{\delta}_{\leqslant 2} \geqslant p_{\leqslant 2}(\lambda_0)$ mentioned in Lemmas 3.5 and 3.7.

Lemma 3.8. *Let G be a connected regular graph. Then $\widetilde{\delta}_2 \geqslant p_2(\lambda_0)$, with equality if and only if $\widetilde{A}_2 = p_2(A)$.* □

Example 3.9. (See [22].) Let P_3 be a path of three vertices, with spectrum $\{\sqrt{2}, 0, -\sqrt{2}\}$. Then $D = d = 2$, $p_0 = 1$, $p_1 = 3\sqrt{2}x/4$, $p_2 = 3(x^2-4/3)/4$, and $\widetilde{\delta}_2 = 3/8 < 1/2 = p_2(\lambda_0)$.

4. Weighted punctual and partial distance-regularity

The concepts of punctual distance-regularity and partial distance-regularity have been recently studied [11,10]. In this paper, we study two concepts, which are basically the same as that in [11,10], except that here we drop the regularity assumption, and the use of weighted distance matrices is taking into account. A connected graph is called *weighted h-punctually distance-regular* if $\widetilde{A}_h = p_h(A)$; and is called *weighted m-partially distance-regular* if $\widetilde{A}_i = p_i(A)$ for $i \leqslant m$. Note by Lemma 3.7 that the regularity condition is actually not necessary in the concept of partial distance-regularity. Clearly, the concepts of weighted 0-punctual distance-regularity and weighted 0-partial distance-regularity are identical. However, the weighted 1-punctual distance-regularity and the weighted 1-partial distance-regularity are not equivalent. For example, by Lemma 3.7, the path graph P_3 of three vertices is weighted 1-punctually distance-regular, but not weighted 1-partially distance-regular. The following result indicates that the concepts of weighted 2-punctual distance-regularity and weighted 2-partial distance-regularity coincide.

Proposition 4.1. *Let G be a connected graph. Then $\widetilde{A}_2 = p_2(A)$ if and only if G is weighted 2-partially distance-regular.*

Proof. We only need to prove necessity. Since $\widetilde{A}_2 = p_2(A) = aA^2 + bA + cI$ for some real numbers a, b, c with $a \neq 0$, we conclude that A^2 has a constant diagonal, which implies that G is regular. The remaining follows from Lemma 3.7. □

Proposition 4.2 states an equivalent condition of the weighted 2-punctual distance-regularity for bipartite graphs with spectral diameter $d \geqslant 3$. Note that the assumption $d \geqslant 3$ is necessary, since otherwise the path graph P_3 of three vertices gives a counterexample. The proof follows from Proposition 3.3, Lemma 3.5 and Proposition 4.1.

Proposition 4.2. *Let G be a connected bipartite graph with spectral diameter $d \geqslant 3$. Then $\widetilde{\delta}_{\leqslant 2} = p_{\leqslant 2}(\lambda_0)$ if and only if G is weighted 2-punctually distance-regular.* □

Lemma 4.3 demonstrates that for a connected bipartite weighted 2-punctually distance-regular graph, its two halved graphs have the same spectrum (with appropriate spectral diameter), and, under further assumption, it gives a lower bound or exact value of the diameter, depending on the parity of its spectral diameter.

Lemma 4.3. *Let G be a connected bipartite graph with bipartition (X, Y), diameter D, spectral diameter d and $\widetilde{A}_2 = p_2(A)$. Then the halved graphs G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Suppose further that at least one of G^X and G^Y has spectral diameter which is equal to its diameter. Then $D \geqslant d-1$ for odd d, and $D = d$ otherwise.*

Proof. Since G is bipartite, p_2 is even, that is, $p_2 = ax^2 + b$ for some real numbers a, b with $a \neq 0$. Let X_1 and Y_1 be adjacency matrices of G^X and G^Y, respectively. Note that

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

for some square matrix B (since G is regular by Proposition 4.1). Hence

$$\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} = \widetilde{A}_2 = p_2(A) = aA^2 + bI = \begin{pmatrix} aBB^T + bI & 0 \\ 0 & aB^TB + bI \end{pmatrix}.$$

Since BB^T and B^TB have the same characteristic polynomial (see for instance [26, Theorem 2.8]), G^X and G^Y have the same spectrum. Note that if λ is an eigenvalue of A with eigenvector u then $a\lambda^2 + b$ is an eigenvalue of $\widetilde{A}_2$ with the same eigenvector. Thus $\widetilde{A}_2$ has $\lceil (d+1)/2 \rceil = \lfloor d/2 \rfloor + 1$ distinct eigenvalues, and so do G^X and G^Y. Hence G^X and G^Y are of spectral diameter $\lfloor d/2 \rfloor$. If at least one of G^X and G^Y has spectral diameter which is equal to its diameter, we derived that $d \geqslant D \geqslant 2\lfloor d/2 \rfloor$, as claimed. □

At the end of this section, we give some results for connected bipartite graphs with $\widetilde{\delta}_{d-1} = p_{d-1}(\lambda_0)$. Lemma 4.4 follows from Lemma 3.6. The proof of Proposition 4.5 is basically identical to [17, Proposition 2] (it is not difficult to prove this characterization by backward induction on i, using the recurrence relation (2), Lemmas 2.2, 4.4 and 3.7).

Lemma 4.4. *Let G be a connected bipartite graph. Then $\widetilde{\delta}_{d-1} \leqslant p_{d-1}(\lambda_0)$, with equality if and only if $\widetilde{A}_{d-1} = p_{d-1}(A)$.* □

Proposition 4.5. *Let G be a connected bipartite graph with $\widetilde{\delta}_{d-1} = p_{d-1}(\lambda_0)$. Then $\widetilde{A}_i = p_i(A)$ for all i with the opposite parity of d. In particular, G is regular if d is odd, and biregular otherwise.* □

5. Proof of the main result

It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular [5, Proposition 4.2.2]. We first provide three examples to show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Here we omit the computation details which are straightforward by definitions.

Example 5.1 *(Weighted 2-punctually distance-regular and odd spectral diameter).* Consider the Möbius–Kantor graph, i.e., the generalized Petersen graph $G(8,3)$ [23], with spectrum $\{3^1, \sqrt{3}^4, 1^3, (-1)^3, (-\sqrt{3})^4, (-3)^1\}$. Then $D = 4 < 5 = d$, $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 3$, $p_3 = 2(x^3 - 5x)/5$, $p_4 = (x^4 - 10x^2 + 15)/6$, $p_5 = (x^5 - 56x^3/5 + 21x)/18$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$, and both halved graphs are distance-regular with spectrum $\{6^1, 0^4, (-2)^3\}$.

Example 5.2 *(Not weighted 2-punctually distance-regular and even spectral diameter).* Consider the Hoffman graph with spectrum $\{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, which is cospectral to the Hamming 4-cube but not distance-regular [20,5]. Then $D = d = 4$, $p_0 = 1$, $p_1 = x$, $p_2 = (x^2 - 4)/2$, $p_3 = (x^3 - 10x)/6$, $p_4 = (x^4 - 16x^2 + 24)/24$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 3\}$, and its two halved graphs are the complete graph K_8 and the complete multipartite graph $K_{2,2,2,2}$, which are both distance-regular.

Example 5.3 *(Not weighted 2-punctually distance-regular and odd spectral diameter).* Consider the graph obtained by deleting a 10-cycle from the complete bipartite graph $K_{5,5}$, with spectrum $\{3^1, ((\sqrt{5}+1)/2)^2, ((\sqrt{5}-1)/2)^2, ((-\sqrt{5}+1)/2)^2, ((-\sqrt{5}-1)/2)^2, (-3)^1\}$. Then $D = 3 < 5 = d$, $p_0 = 1$, $p_1 = x$, $p_2 = 3(x^2 - 3)/5$, $p_3 = 12(x^3 - 19x/3)/49$, $p_4 = (x^4 - 48x^2/5 + 49/5)/11$, $p_5 = (x^5 - 543x^3/49 + 2820x/147)/33$, $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1\}$, and both halves graphs are the complete graphs K_5, which are distance-regular.

The following result is related to [5, Proposition 4.2.2] (in the case that d is even).

Theorem 5.4. *Let G be a connected bipartite graph with bipartiton (X, Y) and spectral diameter d. Suppose that $\widetilde{A}_i = p_i(A)$ for even i, where $0 \leqslant i \leqslant d$. Then G is weighted 2-punctually distance-regular and both halved graphs G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$.*

Proof. By assumption, $\widetilde{A}_0 = p_0(A) = I$ and $\widetilde{A}_2 = p_2(A) = aA^2 + bI$ for some real numbers a, b with $a \neq 0$. Then G is regular and weighted 2-punctually distance-regular. By Lemma 4.3, G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Since p_{2i} is even, we can assume $p_{2i} = f_i(ax^2 + b)$ for some $f_i \in \mathbb{R}[x]$ of degree i. Thus, for $0 \leqslant i \leqslant \lfloor d/2 \rfloor$,

$$\begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \widetilde{A}_{2i} = p_{2i}(A) = f_i\big(aA^2 + bI\big) = f_i(\widetilde{A}_2) = \begin{pmatrix} f_i(X_1) & 0 \\ 0 & f_i(Y_1) \end{pmatrix},$$

where X_i and Y_i are distance-i matrices of G^X and G^Y, respectively. Therefore, G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$. $\square$

Now we are ready to prove the main result, Theorem 1.1.

Theorem 5.5. *Let G be a connected bipartite graph with bipartition (X, Y) and spectral diameter d. Suppose that G is weighted 2-punctually distance-regular and both halved graphs G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$. Then $\widetilde{\delta}_\ell = p_\ell(\lambda_0)$, where $\ell = d-1$ if d is odd, and $\ell = d$ otherwise. In particular, if d is even, then Theorem 1.1 holds.*

Proof. First note by Proposition 4.1 that G is regular. By Lemma 4.3, G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Thus G^X and G^Y have the same (pre)distance-polynomials f'_i, $0 \leqslant i \leqslant \lfloor d/2 \rfloor$. Since G^X and G^Y are distance-regular,

$$\widetilde{A}_{2i} = \begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \begin{pmatrix} f'_i(X_1) & 0 \\ 0 & f'_i(Y_1) \end{pmatrix} = f'_i(\widetilde{A}_2) = f'_i\big(p_2(A)\big) = g_{2i}(A)$$

for $0 \leqslant i \leqslant \lfloor d/2 \rfloor$, where X_i and Y_i are distance-i matrices of G^X and G^Y, respectively, and $g_{2i} \in \mathbb{R}[x]$ is even of degree $2i$. Since G is regular, $\widetilde{A}_\ell J = g_\ell(A)J = g_\ell(\lambda_0)J$. Then each row of $\widetilde{A}_\ell$ has exactly $g_\ell(\lambda_0)$ ones, and thus $\widetilde{\delta}_\ell = g_\ell(\lambda_0)$. Now it remains to show that $g_\ell = p_\ell$. Note that $\langle g_\ell, g_\ell \rangle_G = \langle g_\ell(A), g_\ell(A) \rangle = \langle \widetilde{A}_\ell, \widetilde{A}_\ell \rangle = \widetilde{\delta}_\ell = g_\ell(\lambda_0)$. For every polynomial $p \in \mathbb{R}_{\ell-1}[x]$, $\langle g_\ell, p \rangle_G = \langle \widetilde{A}_\ell, p(A) \rangle = 0$. By the uniqueness of the predistance polynomials, it follows that $g_\ell = p_\ell$. Moreover, if d is even, then by Theorem 3.2, G is distance-regular. $\square$

Putting Proposition 4.5, Theorems 3.2, 5.4 and 5.5 together, we can conclude the following theorem.

Theorem 5.6. *Let G be a connected bipartite graph with bipartition (X, Y) and spectral diameter d. Then the following conditions are equivalent.*

(i) *$\widetilde{A}_i = p_i(A)$ for even i, where $0 \leqslant i \leqslant d$;*
(ii) *$\widetilde{\delta}_\ell = p_\ell(\lambda_0)$, where $\ell = d-1$ if d is odd, and $\ell = d$ otherwise;*
(iii) *G is weighted 2-punctually distance-regular and both halved graphs G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$.* $\square$

Applying a result in [1, Theorem 4.2], Theorem 5.6(i) seems to be improved to the condition that only $i \in \{0, d-2\}$ is necessary when d is even. Unfortunately, there is a flaw in the proof of this result [2].

Note that the Möbius–Kantor graph (Example 5.1) with odd spectral diameter satisfies Theorem 5.6(i)–(iii) with $D = d-1$. The following example shows that a bipartite graph with odd spectral diameter satisfying Theorem 5.6(i)–(iii) and $D = d$ needs not to be distance-regular.

Example 5.7. Consider the regular bipartite graphs on 20 vertices obtained from the Desargues graph by the Godsil–McKay switching, which is not distance-regular with spectrum $\{3^1, 2^4, 1^5, (-1)^5, (-2)^4, (-3)^1\}$ [21]. Then $D = d = 5$, $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 3$, $p_3 = (x^3 - 5x)/2$, $p_4 = (x^4 - 9x^2 + 12)/4$, $p_5 = (x^5 - 11x^3 + 22x)/12$, $\widetilde{A}_i =$

$p_i(A)$ for $i \in \{0, 1, 2, 4\}$, and both halved graphs are distance-regular with spectrum $\{6^1, 1^4, (-2)^5\}$.

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