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# Spider web networks: a family of optimal, fault tolerant, hamiltonian bipartite graphs

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**Abstract**

In this paper, we propose a honeycomb mesh variation, called a spider web network. Assume that $m$ and $n$ are positive even integers with $m \geqslant 4$. A spider web network $\mathrm{SW}(m,n)$ is a 3-regular bipartite planar graph with bipartition $C$ and $D$. We prove that the honeycomb rectangular mesh $\mathrm{HREM}(m,n)$ is a spanning subgraph of $\mathrm{SW}(m,n)$. We also prove that $\mathrm{SW}(m,n) - e$ is hamiltonian for any $e \in E$ and $\mathrm{SW}(m,n) - \{c,d\}$ remains hamiltonian for any $c \in C$ and $d \in D$. These hamiltonian properties are optimal.


*Keywords:* Bipartite; 1-edge hamiltonian; $1_p$-hamiltonian; Optimal

## 1. Introduction

Throughout this paper, we assume that $m, n$ are positive even integers with $m \geqslant 4$. We use $[r]_s$ to denote $r(\mathbf{mod}\, s)$.

Network topology is a crucial factor for an interconnection network since it determines the performance of the network. Many interconnection network topologies have been proposed in the literature for the purpose of connecting a large number of processing elements. Network topology is always represented by a graph where the nodes represent processors and the edges represent the links between processors. One of the most popular architectures is mesh-connected computers [1]. Each processor is placed into a square or rectangular grid

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and connected by a communication link to its neighbors in up to four directions.

It is well known that there are three possible tessellations of a plane with regular polygons of the same kind: square, triangular and hexagonal, corresponding to dividing a plane into regular squares, triangles and hexagons, respectively. Some computer and communication networks have been built based on this observation. The square tessellation is the basis for mesh-connected computers. The triangular tessellation is the basis for defining hexagonal meshed multiprocessors [2,3]. The hexagonal tessellation is the basis for defining honeycombed meshes [4,5].

Stojmenovic [5] introduced three different honeycomb meshes, the honeycomb rectangular mesh, honeycomb rhombic mesh and honeycomb hexagonal mesh. Most of these meshes are not regular. Moreover, any honeycomb mesh is not hamiltonian unless it is small in size [6]. To remedy these drawbacks, the honeycomb rectangular torus, honeycomb rhombic torus and honeycomb hexagonal torus are proposed [5]. Any such torus is 3-regular. However, all honeycomb tori are not planar. In this paper, we propose a variation of honeycomb meshes, called a spider web network.

In the following section, we give some graph terms that are used in this paper and a formal definition of spider web networks. The spider graph $\mathrm{SW}(m,n)$ is a bipartite graph with bipartition $C$ and $D$. Moreover, the honeycomb mesh $\mathrm{HREM}(m,n)$ forms a spanning subgraph of $\mathrm{SW}(m,n)$. In Section 3, we prove that $\mathrm{SW}(m,n)-e$ is hamiltonian for any $e \in E$. In Section 4, we prove that $\mathrm{SW}(m,n)-\{c,d\}$ remains hamiltonian for any $c \in C$ and $d \in D$. These hamiltonian properties are optimal. A conclusion is given in the final section.

## 2. Spider web networks

Usually, computer networks are represented by graphs where nodes represent processors and edges represent the links between processors. In this paper, a network is represented as an undirected graph. For the graph definition and notation, we follow [7]. $G=(V,E)$ is a *graph* if $V$ is a finite set and $E$ is a subset of $\{(a,b)|(a,b)$ is an unordered pair of $V\}$. We say that $V$ is the *node set* and $E$ is the *edge set* of $G$. Two nodes $a$ and $b$ are *adjacent* if $(a,b)\in E$.

The *honeycomb rectangular mesh* $\mathrm{HREM}(m,n)$ is the graph with the node set $\{(i,j)|0\leqslant i<m, 0\leqslant j<n\}$ such that $(i,j)$ and $(k,l)$ are adjacent if they satisfy one of the following conditions:

1. $i=k$ and $j=l\pm 1$;
2. $j=l$ and $k=i+1$ if $i+j$ is odd; and
3. $j=l$ and $k=i-1$ if $i+j$ is even.

For example, a honeycomb rectangular mesh HREM$(8,6)$ is shown in Fig. 1.

A *spider web network* SW$(m,n)$, where $m$, $n$ are even integers with $m \geqslant 4$, $n \geqslant 2$, is the graph with the vertex set $\{(i,j)|0 \leqslant i < m, 0 \leqslant j < n\}$ such that $(i,j)$ and $(k,l)$ are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1$;
2. $j = l$ and $k = [i+1]_m$ if $i+j$ is odd or $j = n-1$; and
3. $j = l$ and $k = [i-1]_m$ if $i+j$ is even or $j = 0$.

For example, a spider graph SW$(8,6)$ is shown in Fig. 2(a). Another layout of SW$(8,6)$ is shown in Fig. 2(b) with the dashed lines indicating the edges of SW$(m,n)$ that are not in HREM$(m,n)$. Obviously, HREM$(m,n)$ is a spanning subgraph of SW$(m,n)$. The *inner cycle* of SW$(m,n)$ is $\langle(0,0),(1,0),\ldots,(m-1,0),(0,0)\rangle$ whereas the *outer cycle* of SW$(m,n)$ is $\langle(0,n-1),(1,n-1),\ldots,(m-1,n-1),(0,n-1)\rangle$. It is obvious that any spider web network is a planar 3-regular bipartite graph. A vertex $(i,j)$ is labeled black when $i+j$ is even and white if otherwise.

One of the major requirements of designing the network topology is a network's hamiltonian properties. For example, the "token ring" approach is

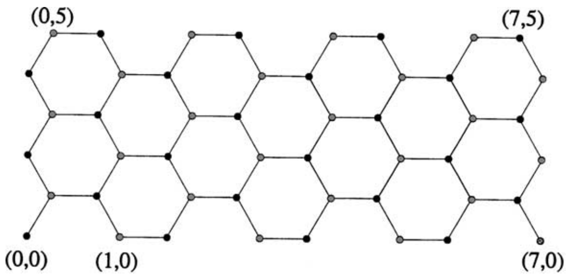


Fig. 1. HREM(8,6).

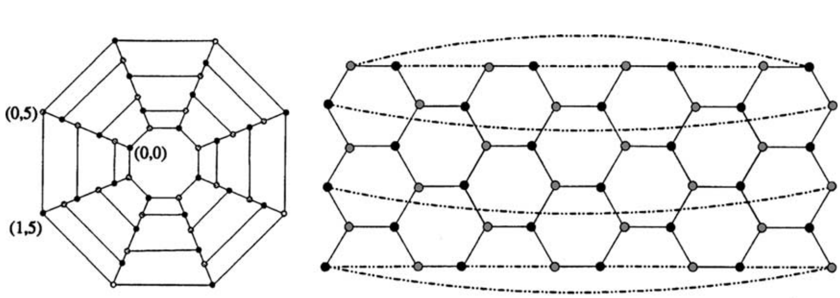


Fig. 2. SW(8,6).

used in distributed operating systems. Fault tolerance is also desirable in massive parallel systems that have a relatively high probability of failure.

A *path* is a sequence of consecutive adjacent nodes. A path is usually delimited by $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. We use $P^{-1}$ to denote the path $\langle x_{n-1}, x_{n-2}, \ldots, x_1, x_0\rangle$ if $P$ is the path $\langle x_0, x_1, x_2, \ldots, x_{n-1}\rangle$. A path is called a *hamiltonian path* if its nodes are distinct and span $V$. A *cycle* is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a *hamiltonian cycle* if its nodes are distinct except for the first node and the last node and if they span $V$. A *hamiltonian graph* is a graph with a hamiltonian cycle. The honeycomb rectangular mesh $\mathrm{HREM}(8,6)$ is not hamiltonian because $\deg_{\mathrm{HREM}(8,6)}(0,0) = 1$.

A graph $G = (V, E)$ is *1-edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$. Obviously, any 1-edge hamiltonian graph is hamiltonian. A 1-edge hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all 1-edge hamiltonian graphs with the same number of vertices as $G$. A graph $G = (V, E)$ is *1-node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. A 1-node hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all 1-node hamiltonian graphs with the same number of vertices as $G$. A graph $G = (V, E)$ is *1-hamiltonian* if it is 1-edge hamiltonian and 1-node hamiltonian. A 1-hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all 1-hamiltonian graphs with the same number of vertices as $G$. The study of optimal 1-hamiltonian graphs is motivated by the design of optimal fault-tolerant token rings in computer networks. Numbers of optimal 1-hamiltonian graphs have been proposed [8–10]. Obviously, $\deg_G(x) \geqslant 3$ for any vertex $x$ in a 1-edge hamiltonian, 1-node hamiltonian, or 1-hamiltonian graph $G$.

However, any bipartite graph is not 1-hamiltonian. Any cycle of a bipartite graph contains the same number of vertices in each partite set. Thus, the deletion of a vertex from a hamiltonian bipartite graph results in a non-hamiltonian graph. Let $G$ be a bipartite graph with bipartition $C$ and $D$. We use $\mathscr{F}(G)$ to denote $\{\{c, d\} | c \in C, d \in D\}$. A hamiltonian bipartite graph is $1_p$-*hamiltonian* if $G - F$ remains hamiltonian for any $F \in \mathscr{F}(G)$. Obviously, $\deg_G(x) \geqslant 3$ for any vertex $x$ in a $1_p$-hamiltonian graph $G$. A $1_p$-hamiltonian graph $G$ is *optimal* if it contains the least number of edges among all $1_p$-hamiltonian graphs with the same number of vertices as $G$.

## 3. A recursive property of SW($m, n$)

Using the definition of a spider web network, $\mathrm{SW}(m, n+2)$ can be constructed from $\mathrm{SW}(m, n)$ as follows: Let $S$ denote the edge subset $\{((i, n-1), ([i-1]_m, n-1)) | i = 0, 2, 4, \ldots, m-2\}$ of $\mathrm{SW}(m, n)$. Let $\mathrm{SW}^*(m, n)$ denote the spanning subgraph of $\mathrm{SW}(m, n)$ with edge set $E(\mathrm{SW}(m, n)) - S$. Let

$V^n = \{(i,k)|0 \leqslant i < m; k = n, n+1\}$, and $E^n = \{((i,k),(i,k+1))|0 \leqslant i < m; k = n-1, n\} \cup \{((i,n),([i-1]_m, n))|i = 0,2,4,\dots,m-2\} \cup \{((i,n+1),([i+1]_m, n+1))|0 \leqslant i < m\}$. Then $V(\mathrm{SW}(m,n+2)) = V(\mathrm{SW}(m,n)) \cup V^n$, $E(\mathrm{SW}(m,n+2)) = (E(\mathrm{SW}(m,n)) - S) \cup E^n$. For this reason, we can view $\mathrm{SW}(m,n)$ as a substructure of $\mathrm{SW}(m,n+2)$ if there is no confusion.

Let $F' \subset V(\mathrm{SW}^*(m,n)) \cup E(\mathrm{SW}^*(m,n))$ be a faulty set with $|F'| \leqslant 2$, such that $F'$ contains an edge in $E(\mathrm{SW}^*(m,n))$ if $|F'| = 1$ and $F' \subset \mathscr{F}(\mathrm{SW}^*(m,n))$ if $|F'| = 2$. Suppose that $\mathscr{C}$ is a hamiltonian cycle of $\mathrm{SW}(m,n) - F'$, in which $(i, n-1)$ is fault free for some $0 \leqslant i < m$. Now, we are going to construct a hamiltonian cycle of $\mathrm{SW}(m,n+2)$ as follows:

*Case 1*: **there is some edge in** $S \cap E(\mathscr{C})$. We can pick an edge $((r, n-1),([r-1]_m, n-1)) \in \mathscr{C}$ for some even integer $0 \leqslant r < m-1$. For $0 \leqslant i \leqslant m-2$, we define $e^* = (([r+i]_m, n-1),([r+i+1]_m, n-1))$, and $Q_i$ as

$$Q_i = \langle ([r+i]_m, n+1), ([r+i+1]_m, n+1) \rangle \quad \text{if } [r+i]_2 = 0;$$

$$Q_i = \langle ([r+i]_m, n+1), ([r+i+1]_m, n+1) \rangle \quad \text{if } [r+i]_2 = 1 \quad \text{and} \quad e^* \in \mathscr{C};$$

$$Q_i = \langle ([r+i]_m, n+1), ([r+i]_m, n), ([r+i+1]_m, n), ([r+i+1]_m, n+1) \rangle$$
$$\text{if otherwise.}$$

Then set the path $Q$ as $\langle (r, n+1), Q_0, ([r+1]_m, n+1), Q_1, ([r+2]_m, n+1) \dots ([r-2]_m, n+1), Q_{m-2}, ([r-1]_m, n+1) \rangle$.

Now we perform the following algorithm on $\mathscr{C}$:

**Algorithm 1** (*Extend* ($\mathscr{C}$))

1. Replace those edges $((i,n-1),([i-1]_m, n-1)) \in \mathscr{C}$, where $i \neq r$ and $i$ is even, with the path $\langle (i,n-1),(i,n),([i-1]_m,n),([i-1]_m,n-1) \rangle$.
2. Replace the edge $((r,n-1),(r-1,n-1))$ with the path $\langle (r,n-1), (r,n),(r,n+1),Q,(r-1,n+1),(r-1,n),(r-1,n-1) \rangle$.

Obviously, the resultant of Algorithm 1 is a hamiltonian cycle of $\mathrm{SW}(m,n+2) - F'$.

*Case* 2: **there is no edge in** $S \cap E(\mathscr{C})$. Obviously, $((i,n-1),(i-1,n-1)) \in \mathscr{C}$ for every odd $i$ with $1 \leqslant i < m$. The hamiltonian cycle of $\mathrm{SW}(m,n+2) - F'$ can be easily constructed by replacing every $((i,n-1),(i-1,n-1))$, where $i$ is odd and $1 \leqslant i < m$, with the path $\langle (i,n-1),(i,n),(i,n+1),(i-1,n+1), (i-1,n),(i-1,n-1) \rangle$.

Thus, we have the following theorem.

**Theorem 3.1.** *Let $F' \subset V(\mathrm{SW}^*(m,n)) \cup E(\mathrm{SW}^*(m,n))$ be a faulty set with $|F'| \leqslant 2$, such that $F'$ contains an edge in $E(\mathrm{SW}^*(m,n))$ if $|F'| = 1$ and $F' \subset$*

*$\mathscr{F}(\mathrm{SW}^*(m,n))$ if $|F'| = 2$. Suppose that $(i, n-1)$ with $0 \leqslant i < m$ is faulty free, then $\mathrm{SW}(m, n+2) - F'$ is hamiltonian if $\mathrm{SW}(m,n) - F'$ is hamiltonian.*

## 4. SW$(m, n)$ is 1-edge hamiltonian

For $j = 0$ or $n-1$, $I_j(i,k)$ denotes $\langle (i,j), ([i+1]_m, j), ([i+2]_m, j), \ldots, (k,j) \rangle$, and $I_j^{-1}(i,k)$ denotes $\langle (k,j), ([k-1]_m, j), ([k-2]_m, j), \ldots, (i,j) \rangle$. In addition, let $H_i(j,k)$ denote the path $\langle (i,j), (i,j+1), (i,j+2), \ldots, (i,k) \rangle$, and $H_i^{-1}(j,k) = \langle (i,k), (i,k-1), \ldots, (i,j) \rangle$ for $0 \leqslant i < m$, $0 \leqslant j,k < n$.

**Theorem 4.1.** SW$(m,n)$ *is* 1-*edge hamiltonian for any even integers* $m$, $n$ *with* $m \geqslant 4$, $n \geqslant 2$.

**Proof.** We prove this theorem by induction. We first prove SW$(m,2)$ is 1-edge hamiltonian. Let $e$ be an edge of SW$(m,2)$. By the symmetric property of SW$(m,2)$, we may assume that $e$ is either $((0,0),(m-1,0))$ or $((i,0),(i,1))$ with $i \neq 0, m-1$. Obviously, $\langle (0,0), I_0(0, m-1), (m-1,0), (m-1,1), I_1^{-1}(0, m-1), (0,1), (0,0) \rangle$ forms a hamiltonian cycle of SW$(m,2) - e$.

Consider SW$(m,4)$. Let $e \in E(\mathrm{SW}(m,4))$. There are three cases: (1) $e = ((i,j),([i+1]_m, j))$ for $0 \leqslant i < m$ if $j = 0,3$, or $i = 0,2,4,\ldots,m-2$ if $j=1$, or $i = 1,3,\ldots,m-1$ if $j = 2$; (2) $e = ((i,j),(i,j+1))$ for $0 \leqslant i < m$, $j = 0,2$; (3) $e = ((i,1),(i,2))$ for $0 \leqslant i < m$. In Case 1 and Case 2, we may assume that $e \in E(\mathrm{SW}^*(m,2))$ since the inner cycle and the outer cycle are symmetric. Because SW$(m,2)$ is 1-edge hamiltonian, there exists a hamiltonian cycle of SW$(m,4) - e$ using Theorem 3.1. For Case 3, suppose $e = ((0,1),(0,2))$ using the symmetric property of SW$(m,4)$. Let $P_i = \langle (i+1,0), (i,0), H_i(0,n-1), (i,n-1), (i-1,n-1), H_{i-1}^{-1}(0,n-1), (i-1,0) \rangle$. Obviously, $\langle (0,0), (0,1), (1,1), H_1(1,3), (1,3), (0,3), (0,2), (m-1,2), (m-1,3), (m-2,3), H_{m-2}^{-1}(1,3), (m-2,1), (m-1,1), (m-1,0), (m-2,0), P_{m-3}, (m-4,0), P_{m-5}, (m-6,0), \ldots, P_3, (2,0), (1,0), (0,0) \rangle$ forms a hamiltonian cycle of SW$(m,4) - e$. Thus, SW$(m,4)$ is 1-edge hamiltonian.

By inductive hypothesis, assume SW$(m,k)$ is 1-edge hamiltonian for some even integer $k$ with $k \geqslant 4$. Let $e$ be an edge of SW$(m, k+2)$. Since the inner

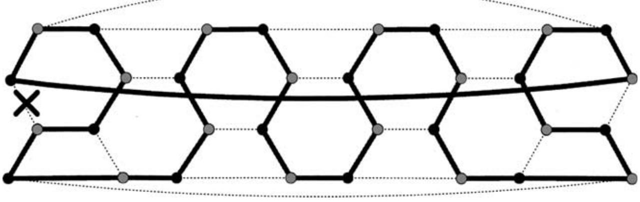

Fig. 3. Illustration of Theorem 4.1.

cycle and the outer cycle of $\mathrm{SW}(m,k+2)$ are symmetrical, we may assume that $e$ is in $\mathrm{SW}^*(m,k)$. Then there exists a hamiltonian cycle of $\mathrm{SW}(m,k)-e$. Applying Theorem 3.1, $\mathrm{SW}(m,k+2)-e$ is hamiltonian.

Hence any spider web network $\mathrm{SW}(m,n)$ is 1-edge hamiltonian. Fig. 3 gives an illustration. □

## 5. SW$(m,n)$ is $1_p$-hamiltonian

**Lemma 5.1.** $\mathrm{SW}(m,2)$ *is* $1_p$*-hamiltonian for* $m\geqslant 4$.

**Proof.** Let $F\in\mathscr{F}(\mathrm{SW}(m,2))$. By the symmetric property of $\mathrm{SW}(m,2)$, we may assume that $(0,0)\in F$. So, the other vertex in $F$ is $(x,y)$, where $x+y$ is odd. Define two paths:

$$p_i(k,k+1)=\langle (i-1,k),(i-1,k+1),(i,k+1),(i,k),(i+1,k)\rangle,$$
$$q_i(k+1,k)=\langle (i-1,k+1),(i-1,k),(i,k),(i,k+1),(i+1,k+1)\rangle.$$

To simplify the notation, $p_i=p_i(0,1)$ and $q_i=q_i(1,0)$.

Suppose that $y=1$. Then we have a hamiltonian cycle of $\mathrm{SW}(m,2)-F$:

$$\langle (1,0),(2,0),p_3,(4,0),p_5,(6,0),\ldots,(x,0),(x+1,0),p_{x+2},$$
$$(x+3,0),p_{x+4},\ldots,(m-1,0),(m-1,1),(0,1),(1,1),(1,0)\rangle.$$

Suppose that $y=0$. There exists a hamiltonian cycle of $\mathrm{SW}(m,2)-F$:

$$\langle (0,1),(1,1),q_2,(3,1),q_4,(5,1),\ldots,(x,1),(x+1,1),q_{x+2},(x+3,1),\ldots,$$
$$q_{m-3},(m-2,1),(m-2,0),(m-1,0),(m-1,1),(0,1)\rangle.$$

Hence $\mathrm{SW}(m,2)$ is $1_p$-hamiltonian. □

**Lemma 5.2.** *There exist* $\frac{m}{2}-1$ *disjoint paths,* $P_1^n,P_2^n,\ldots,P_{\frac{m}{2}-1}^n$*, that span* $\mathrm{SW}^*(m,n)-\{(0,0)\}$ *such that* $P_l^n$ *joins* $(2l,n-1)$ *to* $(2l+1,n-1)$ *for* $1\leqslant l<\frac{m}{2}-1$*, and* $P_{\frac{m}{2}-1}^n$ *joins* $(0,n-1)$ *to* $(m-2,n-1)$.

**Proof.** We prove this lemma by induction. For $n=2$, we set $P_l^2$ as $\langle (2l,1),(2l+1,1)\rangle$ for $1\leqslant l<\frac{m}{2}-1$, and set $P_{\frac{m}{2}-1}^2$ as $\langle (0,1),(1,1),(1,0),I_0(1,m-1),(m-1,0),(m-1,1),(m-2,1)$. Obviously, $P_l^2$'s satisfy the requirement of the lemma for $0\leqslant l\leqslant\frac{m}{2}-1$. Now assume that the lemma holds for $n=k$, where $k$ is even. Then, there exist $\frac{m}{2}-1$ disjoint paths, $P_1^k,P_2^k,\ldots,P_{\frac{m}{2}-1}^k$, that span $\mathrm{SW}^*(m,k)-\{(0,0)\}$ such that $P_l^k$ joins $(2l,k-1)$ to $(2l+1,k-1)$ for $1\leqslant l<\frac{m}{2}-1$, and $P_{\frac{m}{2}-1}^k$ joins $(0,k-1)$ to $(m-2,k-1)$.

Now, we set $P_l^{k+2}$ as $\langle (2l,k+1),(2l+1,k+1)\rangle$ for $1\leqslant l<\frac{m}{2}-1$. Define $f_i=\langle (i,k-1),(i,k),(i+1,k),(i+1,k-1),P_{(i+1)/2}^k,(i+2,k-1)\rangle$ and set $P_{\frac{m}{2}-1}^{k+2}$ as:

$$\langle (0,k+1),(1,k+1),(1,k),(2,k),(2,k-1),P_1^k,(3,k-1),f_3,(5,k-1),\\ f_5,(7,k-1),\ldots,f_{m-5},(m-3,k-1),(m-3,k),(m-2,k),(m-2,k-2),\\ (P^k_{\frac{m}{2}-1})^{-1},(0,k-1),(0,k),(m-1,k),(m-1,k+1),(m-2,k+1)\rangle.$$

$P_l^{k+2}$, $1 \leqslant l \leqslant \frac{m}{2}-1$, satisfies the requirement of lemma. Hence the lemma is proved. See Fig. 4(a) for an illustration. □

**Lemma 5.3.** *Assume that r is an even integer*, $0 < r \leqslant m-2$. *There exist* $\frac{r}{2}$ *disjoint paths*, $Q_1^n, Q_2^n, \ldots, Q_{\frac{r}{2}}^n$, *that span* $\mathrm{SW}^*(m,n) - \{(r,0)\}$, *such that* $Q_l^n$ *joins* $(2l, n-1)$ *to* $(2l+1, n-1)$ *for* $1 \leqslant l \leqslant \frac{r}{2}-1$, *and* $Q_{\frac{r}{2}}^n$ *joins* $(0, n-1)$ *to* $(r, n-1)$.

**Proof.** We prove this lemma by induction. For $n=2$, we set $Q_l^2$ as $\langle (2l,1),(2l,0),(2l+1,0),(2l+1,1)\rangle$ for $1 \leqslant l \leqslant \frac{r}{2}-1$, and set $Q_{\frac{r}{2}}^2$ as $\langle (0,1),$ $(1,1),(1,0),(0,0),(m-1,0),(m-1,1),q_{m-2}^{-1},(m-3,1),q_{m-4}^{-1},(m-5,1),\ldots,$ $q_{r+2}^{-1},(r+1,1),(r,1)\rangle$. Obviously, $Q_l^2$'s satisfy the requirement of the lemma for $1 \leqslant l \leqslant \frac{r}{2}$. We assume that the lemma holds for $n=k$ where $k$ is even. Then, there exist $\frac{r}{2}$ disjoint paths, $Q_l^k$, $1 \leqslant l \leqslant \frac{r}{2}$, that span $\mathrm{SW}^*(m,k) - \{(r,0)\}$ such that $Q_l^k$ joins $(2l,k-1)$ to $(2l+1,k-1)$ for $1 \leqslant l < \frac{r}{2}$, and $Q_{\frac{r}{2}}^k$ joins $(0,k-1)$ to $(r,k-1)$.

Now, we set $Q_l^{k+2}$ as $\langle (2l,k+1),(2l+1,k+1)\rangle$ for $1 \leqslant l < \frac{r}{2}$. Define $g_i = \langle (i,k-1),Q_{i/2}^k,(i+1,k-1),(i+1,k),(i+2,k),(i+2,k-1)\rangle$, and set $Q_{\frac{r}{2}}^{k+2}$ as:

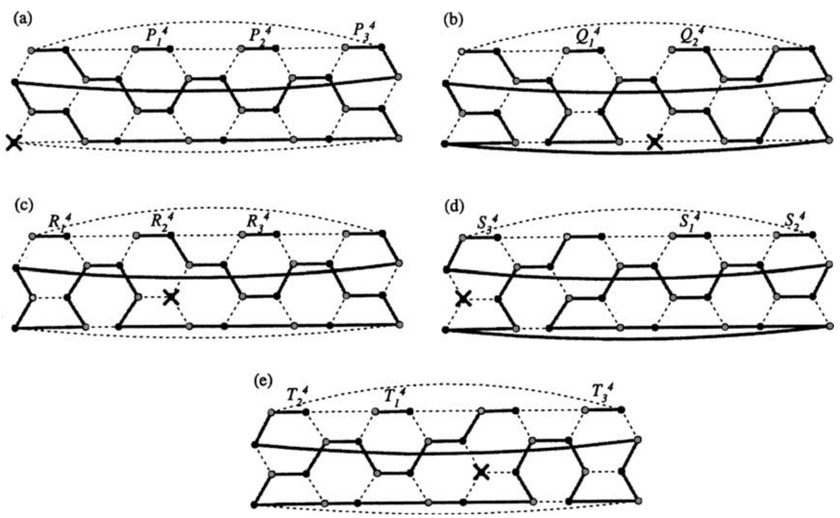


Fig. 4. An illustration for Lemmas 5.2–5.6.

$$\langle(0,k+1),(1,k+1),(1,k),(2,k),(2,k-1),g_2,(4,k-1),g_4,\\ (6,k-1),\dots,g_{r-2},(r,k-1),(Q^k_{\frac{r}{2}})^{-1},(0,k-1),(0,k),(m-1,k),\\ (m-1,k+1),q^{-1}_{m-2}(k+1,k),(m-3,k+1),\dots,q^{-1}_{r+2}(k+1,k),\\ (r+1,k+1),(r,k+1)\rangle.$$

Obviously, $Q_l^{k+2}$, for $1 \leqslant l \leqslant \frac{r}{2}$, satisfies the requirement of lemma. Hence the lemma is proved. See Fig. 4(b) for an illustration, where $r = 4$. □

**Lemma 5.4.** *Assume that s is a positive odd integer. There exist* $\frac{m}{2}-1$ *disjoint paths,* $R_l^n$, *where* $1 \leqslant l < \frac{m}{2}$ *that span* $\mathrm{SW}^*(m,n)-\{(s,1)\}$ *such that* $R_l^n$ *joins* $(2(l-1),n-1)$ *to* $(2l-1,n-1)$ *for* $l \neq \frac{s+1}{2}$, *and* $R_{\frac{s+1}{2}}$ *joins* $(s-1,n-1)$ *to* $(m-2,n-1)$.

**Proof.** We prove this lemma by induction. For $n=2$, we set

$$R_l = \langle(2(l-1),1),(2(l-1),0),(2l-1,0),(2l-1,1)\rangle \\ \text{for } 1 \leqslant l \leqslant \frac{s-1}{2},$$

$$R_l = \langle(2(l-1),1),(2l-1,1)\rangle \quad \text{for } \frac{s+3}{2} \leqslant l \leqslant \frac{m}{2}-1.$$

Besides, $R^2_{\frac{s+1}{2}}$ as $\langle(s-1,1),(s-1,0),I_0(s-1,m-1),(m-1,0),(m-1,1),(m-2,1)\rangle$. Obviously, $R_l^2$ satisfies the requirement of the lemma for $1 \leqslant l \leqslant \frac{m}{2}-1$. Now assume that the lemma holds for $n=k$ where $k$ is even. Then, there exist $\frac{m}{2}-1$ disjoint paths, $R_l^k$'s, that span $\mathrm{SW}^*(m,k)-\{(s,1)\}$ such that $R_l^k$ joins $(2(l-1),k-1)$ to $(2l-1,k-1)$ for $1 \leqslant l < \frac{m}{2}, l \neq \frac{s+1}{2}$, and $R^k_{\frac{s+1}{2}}$ joins $(s-1,k-1)$ to $(m-2,k-1)$.

Now, we set $R_l^{k+2}$ as $\langle(2(l-1),k+1),(2l-1,k+1)\rangle$ for $1 \leqslant l < \frac{m}{2}, l \neq \frac{s+1}{2}$. Define $g_i = \langle(i,k-1),R^k_{i/2},(i+1,k-1),(i+1,k),(i+2,k),(i+2,k-1)\rangle$, and set $R^{k+2}_{\frac{s+1}{2}}$ as:

$$\langle(s-1,k+1),(s,k+1),(s,k),(s+1,k),(s+1,k-1),\\ g_{s+1},(s+3,k-1),\dots,g_{m-4},(m-2,k-1),(R^k_{(s+1)/2})^{-1},(s-1,k-1),\\ g^{-1}_{s-3},(s-3,k-1),\dots,g_0^{-1},(0,k-1),(0,k),(m-1,k),(m-1,k+1),\\ (m-2,k+1)\rangle.$$

Since $R_l^{k+2}$, for $1 \leqslant l \leqslant \frac{s}{2}$, satisfies the requirement of lemma, the lemma is proved. See Fig. 4(c), where $s = 3$. □

**Lemma 5.5.** *There exist $\frac{m}{2}-1$ disjoint paths, $S_l^n$, where $1 \leqslant l < \frac{m}{2}$ that span $\mathrm{SW}^*(m,n)-\{(0,1)\}$ such that $S_l^n$ joins $(2l+2,n-1)$ to $(2l+3,n-1)$ for $1 \leqslant l \leqslant \frac{m}{2}-2$ and $S_{\frac{m}{2}-1}^n$ joins $(1,n-1)$ to $(3,n-1)$.*

**Proof.** We prove this lemma by induction. For $n=2$, we set $S_l^2 = \langle (2l+2,1),(2l+3,1)\rangle$ for $1 \leqslant l \leqslant \frac{m}{2}-2$, and $S_{\frac{m}{2}-1}^2$ as $\langle (1,1),(1,0), (0,0),(m-1,0), I_0^{-1}(2,m-1),(2,0),(2,1),(3,1)\rangle$. Obviously, $S_l^2$'s satisfy the requirement of the lemma for $1 \leqslant l \leqslant \frac{m}{2}-1$. Now assume that the lemma holds for $n=k$ where $k$ is even. Then, there exist $\frac{m}{2}-1$ disjoint paths, $S_l^k$'s, that span $\mathrm{SW}^*(m,k)-\{(0,1)\}$ such that $S_l^k$ joins $(2l+2,k-1)$ to $(2l+3,k-1)$ for $1 \leqslant l \leqslant \frac{m}{2}-2$, and $S_{\frac{m}{2}-1}^k$ joins $(1,k-1)$ to $(3,k-1)$.

Now, we set $S_l^{k+2}$ as $\langle (2l+2,k+1),(2l+3,k+1)\rangle$ for $1 \leqslant l \leqslant \frac{m}{2}-2$. Define $h_i = \langle (i,k-1),(i,k),(i+1,k),(i+1,k-1),S_{\frac{i-1}{2}}^k,(i+2,k-1)\rangle$, and set $S_{\frac{m}{2}-1}^{k+2}$ as:

$$\begin{aligned}&\langle (1,k+1),(0,k+1),(0,k),(m-1,k),(m-1,k-1),h_{m-3}^{-1},\\ &\quad (m-3,k-1),h_{m-5}^{-1},(m-5,k-1),\ldots,h_3^{-1},(3,k-1),(S_{\frac{m}{2}-1}^k)^{-1},(1,k-1),\\ &\quad (1,k),(2,k),(2,k+1),(3,k+1)\rangle.\end{aligned}$$

$S_l^{k+2}$, $1 \leqslant l \leqslant \frac{m}{2}-1$, satisfies the requirement of lemma, so the lemma is proved. See Fig. 4(d) for an illustration. □

**Lemma 5.6.** *Assume that $t$ is an even integer, $0 < t \leqslant m-2$. There exist $\frac{m}{2}-1$ disjoint paths, $T_l^n$, where $1 \leqslant l < \frac{m}{2}$ that span $\mathrm{SW}^*(m,n)-\{(t,1)\}$ such that $T_l^n$ joins $(2l,n-1)$ to $(2l+1,n-1)$ for $1 \leqslant l \leqslant \frac{m}{2}-1$ and $l \neq \frac{t}{2}$, and $T_{\frac{t}{2}}^n$ joins $(1,n-1)$ to $(t+1,n-1)$.*

**Proof.** We prove this lemma by induction. For $n=2$, we set $T_{\frac{t}{2}}^2 = \langle (1,1),(0,1),(0,0),I_0(0,t+1),(t+1,0),(t+1,1)\rangle$.

$$T_l = \langle (2l,1),(2l+1,1)\rangle \quad \text{for } 1 \leqslant l \leqslant \frac{t-2}{2},$$

$$T_l = \langle (2l,1),(2l,0),(2l+1,0),(2l+1,1)\rangle \quad \text{for } \frac{t+2}{2} \leqslant l \leqslant \frac{m-2}{2}.$$

Obviously, $T_l^2$'s satisfy the requirement of the lemma for $1 \leqslant l \leqslant \frac{m}{2}-1$. Now assume that the lemma holds for $n=k$ where $k$ is even. Then, there exist $\frac{m}{2}-1$ disjoint paths, $T_l^k$'s, that span $\mathrm{SW}^*(m,k)-\{(t,1)\}$ such that $T_l^k$ joins $(2l,k-1)$ to $(2l+1,k-1)$ for $1 \leqslant l \leqslant \frac{m}{2}-1$ and $l \neq \frac{t}{2}$, and $T_{\frac{t}{2}}^k$ joins $(1,k-1)$ to $(t+1,k-1)$.

Now, we set $T_l^{k+2}$ as $\langle (2l,k+1),(2l+1,k+1)\rangle$ for $1 \leqslant l \leqslant \frac{m}{2}-1$, and $l \neq \frac{t}{2}$. Define $h_i = \langle (i,k-1),(i,k),(i+1,k),(i+1,k-1),T_{\frac{i+1}{2}}^k,(i+2,k-1)\rangle$, and set $T_{\frac{t}{2}}^{k+2}$ as:

$$\begin{aligned}&\langle (1,k+1),(0,k+1),(0,k),(m-1,k),(m-1,k-1),h_{m-3}^{-1},\\&\quad (m-3,k-1),h_{m-5}^{-1},(m-5,k-1),\ldots,h_{t+1}^{-1},(t+1,k-1),(T_{\frac{t}{2}}^{k})^{-1},\\&\quad (1,k-1),h_1,(3,k-1),\ldots,h_{t-3},(t-1,k-1),(t-1,k),(t,k),\\&\quad (t,k+1),(t+1,k+1)\rangle.\end{aligned}$$

$T_l^{k+2}$, $1\leqslant l\leqslant \frac{m}{2}-1$, satisfies the requirement of lemma, so the lemma is proved. See Fig. 4(e), where $t=4$. $\square$

**Theorem 5.1.** $\mathrm{SW}(m,n)$ is $1_p$-hamiltonian for any even integers $m$, $n$ with $m\geqslant 4$, $n\geqslant 2$.

**Proof.** This theorem is proved by induction. Using Lemma 5.1, $\mathrm{SW}(m,2)$ is $1_p$-hamiltonian. Assume that $\mathrm{SW}(m,k)$ is $1_p$-hamiltonian for some even integer $k$ with $k\geqslant 2$.

Now, we want to prove $\mathrm{SW}(m,k+2)$ is $1_p$-hamiltonian. Let $F\in\mathscr{F}(\mathrm{SW}(m,k+2))$. Obviously, one of the following cases holds: (1) $\{(i,j)|0\leqslant i<m,j=k,k+1\}\cap F=\emptyset$, (2) $\{(i,j)|0\leqslant i<m,j=0,1\}\cap F=\emptyset$, and (3) $|\{(i,j)|0\leqslant i<m,j=k,k+1\}\cap F|=1$ and $|\{(i,j)|0\leqslant i<m,j=0,1\}\cap F|=1$.

*Case* 1: $\{(i,j)|0\leqslant i<m,j=k,k+1\}\cap F=\emptyset$. Then $F\in\mathscr{F}(\mathrm{SW}(m,k))$. By induction, $\mathrm{SW}(m,k)-F$ is hamiltonian. Applying Theorem 3.1, $\mathrm{SW}(m,k+2)-F$ is hamiltonian.

*Case* 2: $\{(i,j)|0\leqslant i<m,j=0,1\}\cap F=\emptyset$. Since the inner cycle and the outer cycle are symmetrical in any spider web network, $\mathrm{SW}(m,k+2)-F$ is hamiltonian as in Case 1.

*Case* 3: $|\{(i,j)|0\leqslant i<m,j=k,k+1\}\cap F|=1$ and $|\{(i,j)|0\leqslant i<m,j=0,1\}\cap F|=1$. By the symmetric property of the spider web networks, we have the following five cases: (3.1) $F=\{(0,0),(0,k+1)\}$, (3.2) $F=\{(r,0),(0,k+1)\}$ with $r$ an non-zero even integer, (3.3) $F=\{(s,1),(0,k+1)\}$ with $s$ an odd integer, (3.4) $F=\{(0,1),(0,k)\}$, and (3.5) $F=\{(t,1),(0,k)\}$ with $t$ an non-zero even integer.

*Case* (3.1): $F=\{(0,0),(0,k+1)\}$. Using Lemma 5.2, there exist $\frac{m}{2}-1$ disjoint paths, $P_1^k,P_2^k,\ldots,P_{\frac{m}{2}-1}^k$, that span $\mathrm{SW}^*(m,k)-\{(0,0)\}$ such that $P_l^k$ joins $(2l,k-1)$ to $(2l+1,k-1)$ for $1\leqslant l<\frac{m}{2}-1$, and $P_{\frac{m}{2}-1}^k$ joins $(0,k-1)$ to $(m-2,k-1)$.

Define $C_1(i)=\langle (i,k-1),(i,k),(i-1,k),(i-1,k-1),(P_{\frac{i-2}{2}}^k)^{-1},(i-2,k-1)\rangle$. Obviously, $\langle (0,k-1),P_{\frac{m}{2}-1}^k,(m-2,k-1),C_1(m-2),(m-4,k-1),\ldots,C_1(4),(2,k-1),(2,k),(1,k),(1,k+1),I_{k+1}(1,m-1),(m-1,k+1),(m-1,k),(0,k),(0,k-1)\rangle$ forms a hamiltonian cycle of $\mathrm{SW}(m,k+2)-F$. See Fig. 5(a).

*Case* (3.2): $F=\{(r,0),(0,k+1)\}$. By Lemma 5.3, there exist $\frac{r}{2}$ disjoint paths, $Q_1^k,Q_2^k,\ldots,Q_{\frac{r}{2}}^k$, that span $\mathrm{SW}^*(m,k)-\{(r,0)\}$ such that $Q_l^k$ joins $(2l,k-1)$ to $(2l+1,k-1)$ for $1\leqslant l<\frac{r}{2}$, and $Q_{\frac{r}{2}}^k$ joins $(0,k-1)$ to $(r,k-1)$.

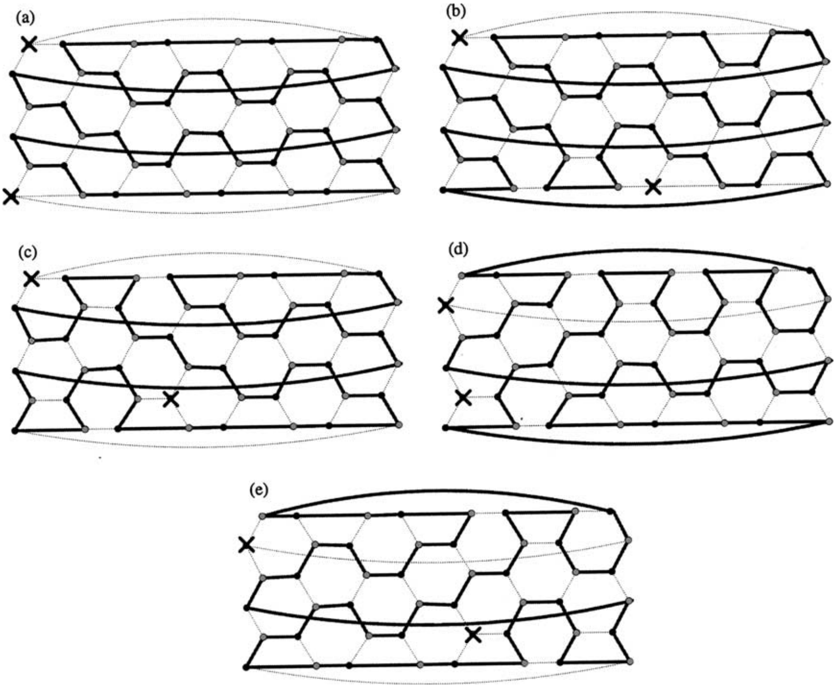


Fig. 5. Illustration for Theorem 5.1, Case (3.1)–(3.5).

Define $C_2(i) = \langle (i,k-1),(i,k),(i-1,k),(i-1,k-1),(Q^k_{\frac{i-2}{2}})^{-1},(i-2,k-1)\rangle$, $B(i) \equiv \langle (i,k+1),(i,k),(i+1,k),(i+1,k+1),(i+2,k+1)\rangle$. Obviously, $\langle (0,k-1), Q^k_{\frac{r}{2}}, (r,k-1), C_2(r), (r-2,k-1), \ldots, C_2(4), (2,k-1), (2,k), (1,k), (1,k+1), I_{k+1}(1,r+1), (r+1,k+1), B(r+1), (r+3,k+1), \ldots, B(m-3), (m-1,k+1), (m-1,k), (0,k), (0,k-1)\rangle$ forms a hamiltonian cycle of $\mathrm{SW}(m,\ k+2) - F$. See Fig. 5(b), where $r = 4$.

*Case* (3.3): $F = \{(s,1),(0,k+1)\}$. By Lemma 5.4, there exist $\frac{m}{2} - 1$ disjoint paths, $R^k_1, R^k_2, \ldots, R^k_{\frac{m}{2}-1}$, that span $\mathrm{SW}^*(m,k) - \{(s,1)\}$ such that $R^k_l$ joins $(2l-2,k-1)$ to $(2l-1,k-1)$ for $1 \leqslant l < \frac{m}{2}$ and $l \neq \frac{s+1}{2}$, and $R^k_{\frac{s+1}{2}}$ joins $(s-1, k-1)$ to $(m-2,k-1)$.

Define $C_3(i) = \langle (i,k-1),(i,k),(i-1,k),(i-1,k-1),(R^k_{\frac{i}{2}})^{-1},(i-2,k-1)\rangle$, $C'_3(i) \equiv \langle (i,k-1), R^k_{\frac{i+2}{2}}, (i+1,k-1),(i+1,k),\ (i+1,k+1),(i+2,k+1),(i+2,k),(i+2,k-1)\rangle$. Obviously, $\langle (s-1,k-1), R^k_{\frac{s+1}{2}}, (m-2,k-1), C_3(m-2), (m-4,k-1), C_3(m-4), (m-6,k-1) \ldots C_3(s+3), (s+1,k-1), (s+1,k), (s,k),\ (s,k+1), I_{k+1}(s,m-1), (m-1,k+1), (m-1,k), (0,k), (0,k-1), C'_3(0), (2,k-1), C'_3(2), (4,k-1) \ldots, C'_3(s-3), (s-1,k-1)\rangle$ forms a hamiltonian cycle of $\mathrm{SW}(m,k+2) - F$. See Fig. 5(c), where $s = 3$.

*Case* (3.4): $F = \{(0,1),(0,k)\}$. By Lemma 5.5, there exist $\frac{m}{2}-1$ disjoint paths, $S_1^k, S_2^k, \ldots, S_{\frac{m}{2}-1}^k$, that span $\mathrm{SW}^*(m,k) - \{(0,1)\}$ such that $S_l^k$ joins $(2l+2, k-1)$ to $(2l+3, k-1)$ for $1 \leqslant l \leqslant \frac{m}{2}-2$, and $S_{\frac{m}{2}-1}^k$ joins $(1, k-1)$ to $(3, k-1)$.

Define $C_4(i) = \langle (i,k-1),(i,k),(i,k+1),(i+1,k+1),(i+1,k),(i+1, k-1), S_{\frac{i-1}{2}}^k, (i+2,k-1)\rangle$. Obviously, $\langle (1,k-1), S_{\frac{m}{2}-1}^k, (3,k-1), C_4(3), (5,k-1), C_4(5), (7,k-1), \ldots, C_4(m-3), (m-1,k-1), (m-1,k), (m-1,k+1), (0,k+1), (1,k+1), (2,k+1), (2,k), (1,k), (1,k-1)\rangle$ forms a hamiltonian cycle of $\mathrm{SW}(m,k+2) - F$. See Fig. 5(d).

*Case* (3.5): $F = \{(t,1),(0,k)\}$. By Lemma 5.6, there exist $\frac{m}{2}-1$ disjoint paths, $T_1^k, T_2^k, \ldots, T_{\frac{m}{2}-1}^k$, that span $\mathrm{SW}^*(m,k) - \{(t,1)\}$ such that $T_l^k$ joins $(2l, k-1)$ to $(2l+1, k-1)$ for $1 \leqslant l \leqslant \frac{m}{2}-1$ and $l \neq \frac{t}{2}$, and $T_{\frac{t}{2}}^k$ joins $(1,k-1)$ to $(t+1,k-1)$.

Define $C_5(i) = \langle (i,k-1),(i,k),(i,k+1),(i+1,k+1),(i+1,k),(i+1,k-1), T_{\frac{i+1}{2}}^k, (i+2,k-1)\rangle$, and $C_5'(i) = \langle (i,k-1), (T_{\frac{i-1}{2}}^k)^{-1}, (i-1,k-1),(i-1,k),(i-2,k), (i-2,k-1)\rangle$. Obviously, $\langle (1,k-1), T_{\frac{t}{2}}, (t+1,k-1), C_5(t+1), (t+3,k-1), C_5(t+3), (t+5,k-1), \ldots, C_5(m-3), (m-1,k-1), (m-1,k), (m-1,k+1), (0,k+1), I_{k+1}(0,t), (t,k+1), (t,k), (t-1,k), (t-1,k-1), C_5'(t-1), (t-3,k-1), C_5'(t-3), (t-5,k-1), \ldots, C_5'(3), (1,k-1)\rangle$ forms a hamiltonian cycle of $\mathrm{SW}(m,k+2) - F$. See Fig. 5(e), where $t = 4$.

Thus we have proved the theorem. □

## 6. Concluding remarks

Since the honeycomb rectangular mesh $\mathrm{HREM}(m,n)$ is a spanning subgraph of $\mathrm{SW}(m,n)$, the spider web network can be viewed as a variation of the honeycomb mesh. The spider web networks we proposed are 3-regular planar graphs. Moreover, they are 1-edge hamiltonian and $1_p$-hamiltonian. Since the spider web network is 3-regular, it is optimal.

It is very easy to see that the diameter of the spider web network $\mathrm{SW}(m,n)$ is $\mathrm{O}(m+n)$. By choosing $m = \mathrm{O}(n)$, the diameter of $\mathrm{SW}(m,n)$ is $\mathrm{O}(\sqrt{N})$ where $N = mn$ is the number of vertices in $\mathrm{SW}(m,n)$. It would be interesting to find other planar, 3-regular, 1-edge hamiltonian, and $1_p$-hamiltonian graphs with smaller diameters.

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