Multi-server machine repair problems under a (V,R) synchronous single vacation policy

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A R T I C L E   I N F O

A B S T R A C T

This paper considers a machine repair problem with M operating machines and S standbys, in which R repairmen are responsible for supervising these machines and operate a (V,R) vacation policy. With such policy, if the number of the failed machines is reduced to $R - V (R > V)$ (there exists V idle repairmen) at a service completion, these V idle servers will together take a synchronous vacation (or leave for other secondary job). Upon returning from the vacation, they do not take a vacation again and remain idle until the first arriving failed machine arrives. The steady-state probabilities are solved in terms of matrix forms and the system performance measures are obtained. Algorithmic procedures are provided to deal with the optimization problem of discrete/continuous decision variables while maintaining a minimum specified level of system availability.

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1. Introduction

We consider a machine repair problem where a group of M operating machines is under the supervision of one or more repairmen (servers) in the repair facility. These operating units are assumed unreliable and may fail at any time. This incidence may lead to loss of production because the failed machine must stay in the repair facility for some time. To avoid any loss of production, the plant always keeps some standby machines, say $S (S < M)$, so that a standby machine can immediately act as a substitute when an operating machine fails. When a machine fails, it is immediately sent to the repair facility for repair and backed up by a standby, if available. Meanwhile, the repairmen may together leave for a synchronous single vacation of random length whenever there exists $V (V < R)$ idle repairmen. The so-called single vacation means that at the end of the vacation the repairmen remain idle until the first arriving failed machine arrives. A real-world example of this vacation model can be realized in the manufacturing/production-assembly system where the servers in their idle time may be assigned to perform some extra operations such as additional work, preventive maintenance. On the other hand, machine repair related problems represent a group of very important problems used to analyze timesharing computer systems, multi-programmed computer systems and multi-access communication channels (see [1]).

In a classical machine repair model, it is assumed that the servers remain idle until the failed machines present (i.e., each server is always available for the waiting failed ones). Such type of machine repair problems also received considerable attention in the literature, as shown by the literature surveys of Ke and Wang [2], Wang et al. [3,4] and Haque and Armstrong [5]. So far, however, only a few works is taken into consideration the server vacations in machine repair problems (see [6–8]). Gupta [6] first analyzed a machine repair problem with warm standbys and server vacations, in which the single server

leaves a vacation when the repair facility is empty. Gupta's work [6] gave an algorithm to compute the steady-state probability distribution of the number of failed machines in the system. Gupta's models were extended to server-breakdown case by Ke [7], who derived system performance measures and performed a cost sensitivity analysis. Ke and Lin [8] dealt with the reliability measures of a multi-server machine repair model with standby and multiple vacations. Ke and Wang [9] examined the steady-state results for a machine repair problem with two types of standby and multi-server vacations under two vacation policies. They performed a sensitivity analysis to investigate the effect on the joint optimum number of standbys and servers if the system parameters take on other specific values. Later, Jain et al. [10] studied the machine repair problem with mixed standbys (warm and cold) where the failed unit may balk or renege in case of heavy load of failed units. Ke and Lin [11] performed a sensitivity analysis of two machine repair models including various repair rates in each phase and two-phase with differing numbers of technicians. Ke et al. [12] considered three vacation policies of machine repair problem in production systems with spares and server vacations. Furthermore, Garg et al. [13] investigated the availability of crank-case manufacturing system in an automobile industry. They showed that the availability of the system can be improved using proper maintenance planning and scheduling. More recently, Yue et al. [14] considered a machine repair problem with warm standbys and two heterogeneous repairmen. They investigated the problem from the viewpoints of both queueing and reliability. Wang et al. [15] provided an optimization analysis of the machine repair problem with balking and variable number of servers Recently, Ke et al. [16] proposed a multi-repairmen problem with warm standbys, pressure coefficient, imperfect coverage and server breakdown. They performed a comparative analysis among two optimal approaches for searching discrete and continuous parameters.

Existing research works on machine repair problems with multi-server vacations, including those above, mainly focused on *server individual vacation at system empty* (*i.e.*, at each repair completion instant, the server individually takes a vacation each time system empty). From practical viewpoint, however, some servers may together take vacations when the number of failed machines reduced a predetermined threshold (see [17–19]). Ke and Wu [20] considered a machine repair problem operating a (R, V, K) synchronous vacation policy, where the vacation policy is a multiple (infinite) vacation. It should be noted that multiple vacation policy is different from single vacation policy, which the former cannot be reduced to the latter (referred to [21,9]). Comparable work on machine repair problems with synchronous single vacation policy is rarely found in the literature. Thus, we develop a multi-server machine repair problem with standby where the servers apply a (V, R) synchronous single vacation policy when the number of failed machines is reduced to $R - V$. Besides the lack of research work on this problem, our study is also motivated by some practical systems as follows.

Consider a firm has many departments including the research and development (R&D) department, the equipment department, the productive department, the quality control department, etc. The employees in the equipment department are responsible for and improving the machine reliability, evaluating the availability and performance of the machines, managing the components for the repair of machines, and creating and updating the relative log file. The main tasks of the employees are maintaining and keeping the operations of equipment in productive department. To provide enough production capacity to satisfy the orders placed by the customers. The number of operating machines should be greater than or equal to a threshold value called as M. It is assumed there are S standby machines as spares for the operating machines. In the equipment department, there are R employees who provide maintenance service for these machines. Suppose these employees are configurable. When the workload in the equipment department become lower, partial manpower, V of R employees, will be dispatched to organize the relative log file and loop up the maintain records to monitor the statues of each machines. That is, the employees may leave the system a random time. It can be regarded as the synchronous vacation of employees. For the machine maintain service in the equipment department, it is a multi-server queueing system with synchronous vacation policy where partial server will take a synchronous vacation together. We can investigate this queueing system to evaluate the employee's performance.

Our model can be implemented to another practical problem based on the work of Chelst et al. [22]. Consider a coal transportation system with $(M + S)$ identical trains which are response for transiting the coal from the mines to the unloader system to dump the coal. The unloader system involves R employees to unload the train. Partial (V) employees may be assigned to execute some secondary tasks such as maintenance or clean when they are idle. Once the secondary tasks are completed, the employees will return to the unloader system. The employees and the trains correspond to servers and machines, respectively. The unloaded times and the times of executing secondary tasks can be regarded as service times and vacation times, respectively. In this study, we wish to develop a computational model that helps managers for the following important questions: (1) Under a certain cost, what is the optimum spare machines and optimal (V, R) policy that minimize the expected cost of this system; That is how many standbys are needed and how many repairmen utilize their idle time during the operation. (2) After standbys and (V, R) policy are decided, how to adjust the service rate and vacation rate such that the cost is possibly reduced.

The paper is organized as follows; In Section 2, the system assumptions are described. In Section 3, the steady-state equations are obtained and the computable forms of the steady-state probabilities are derived using the matrix-analytic method. Some system performance measures are derived in Section 4. In Section 5, a cost model is developed to determine the optimal values of servers, standbys, vacation servers, service rate and vacation rate in order to minimize the total expected cost per unit time while maintaining a specified level of system availability. Some numerical examples and sensitivity analyses are provided. Section 6 concludes.

2. System description

In this research, a machine repairable system with M identical and independent machines operating simultaneously in parallel, S standby machines, and R repairman who are responsible for maintaining these machines is considered. Our analysis is based on the following assumptions.

2.1. Assumptions

1. M operating machines are required for the functioning of the system. In other words, the system is short if only if $S + 1$ or more machines fail.
2. Operating machines are subject to breakdowns according to an independent Poisson distribution with rate λ. When an operating machine breaks down, it will immediately be backed up by an available standby.
3. Each of the standby machines fails independently of the others with Poisson rate α, where $(0 \leqslant \alpha \leqslant \lambda)$. When a standby machine moves into an operating state, its characteristics will be that of an operating machine.
4. Every failure machines are repaired by R repairmen in the order of failures, that is, the FCFS discipline. The repair time is assumed to be independent and identically exponentially distribution with parameter μ.
5. When a failed standby machine is repaired, it is as good as a new one and goes into standby state unless the system is short. At this time, the repaired machine will be sent back to an operating state immediately.
6. Each repairman can repair only one failed unit at a time. The failed unit that on arriving at the repair facility finds all repairmen busy or on a vacation must wait in the queue until a repairman is available.
7. When the failure machines queueing up for repair is less than $(R–V)$, that is, the number of idle repairman is more than V, these V idle repairmen will take a single vacation together.
8. The system allows only V repairmen on vacation at any time. The vacation time is distributed as an exponential with rate θ. The various stochastic processes involved in this system are independent of each other.

It should be noted that the repairmen adopt a synchronous single vacation policy and they wait idly for the first failed machine to arrive as the vacation period terminated.

3. Steady-state results

For the M/M/R machine repair model with standbys under a (V, R) synchronous single vacation policy, we describe the state of the system by the pairs $\{(i, n): i = V, B, \text{ and } n = 0, 1, ..., M + S\}$, where $i = V(B)$ indicates that there are V repairmen on vacation state (or not), and n represents the number of failed machines in the system. The mean failure rate λ_n and mean repair rate μ_n for this system are given by

$$\lambda_n = \begin{cases} M\lambda + (S - n)\alpha, & 0 \leqslant n \leqslant S; \\ [M - (n - S)]\lambda, & S \leqslant n \leqslant M + S; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu_n = \begin{cases} n\mu, & 1 \leqslant n \leqslant R; \\ 0, & \text{otherwise}. \end{cases}$$

In steady-state, the following notations are used
$P_{V,n} \equiv$ probability that there are n failed machines in the system when there are V repairmen on vacation.
$P_{B,n} \equiv$ probability that there are n failed machines in the system when there are no repairmen on vacation, where $0 \leqslant n \leqslant M + S$.

3.1. Steady-state equations

Using birth and death process and referring to the steady-transition-rate diagram shown in Fig. 1, the steady-state equations for the M/M/R machine repair problem with standbys under a (V,R) synchronous single vacation policy are obtained as follows.

(1) $i = V$

$$(\lambda_0 + \theta)P_{V,0} = \mu_1 P_{V,1}, \tag{1}$$

$$(\lambda_i + \mu_i + \theta)P_{V,i} = \lambda_{i-1}P_{V,i-1} + \mu_{i+1}P_{V,i+1}, \quad 1 \leqslant i \leqslant R - V - 1, \tag{2}$$

$$(\lambda_{R-V} + \mu_{R-V} + \theta)P_{V,R-V} = \lambda_{R-V-1}P_{V,R-V-1} + \mu_{R-V}P_{V,R-V+1} + \mu_{R-V+1}P_{B,R-V+1}, \tag{3}$$

$$(\lambda_i + \mu_{R-V} + \theta)P_{V,i} = \lambda_{i-1}P_{V,i-1} + \mu_{R-V}P_{V,i+1}, \quad R - V + 1 \leqslant i \leqslant M + S - 1, \tag{4}$$

$$(\mu_{R-V} + \theta)P_{V,M+S} = \lambda_{M+S-1}P_{V,M+S-1}. \tag{5}$$

(2) $i = B$

$$\lambda_0 P_{B,0} = \mu_1 P_{B,1} + \theta P_{V,0}, \tag{6}$$

$$(\lambda_i + \mu_i)P_{B,i} = \lambda_{i-1}P_{B,i-1} + \mu_{i+1}P_{B,i+1} + \theta P_{V,i}, \quad 1 \leqslant i \leqslant R - V - 1, \tag{7}$$

$$(\lambda_{R-V} + \mu_{R-V})P_{B,R-V} = \lambda_{R-V-1}P_{B,R-V-1} + \theta P_{V,R-V}, \tag{8}$$

$$(\lambda_i + \mu_i)P_{B,i} = \lambda_{i-1}P_{B,i-1} + \mu_{i+1}P_{B,i+1} + \theta P_{V,i}, \quad R - V + 1 \leqslant i \leqslant R - 1, \tag{9}$$

$$(\lambda_i + \mu_R)P_{B,i} = \lambda_{i-1}P_{B,i-1} + \mu_R P_{B,i+1} + \theta P_{V,i}, \quad R \leqslant i \leqslant M + S - 1, \tag{10}$$

$$\mu_R P_{B,M+S} = \lambda_{M+S-1}P_{B,M+S-1} + \theta P_{V,M+S}. \tag{11}$$

There is no way of solving (1)–(11) in a recursive manner to develop the explicit expressions for the steady-state probabilities. In the next section, we provide a matrix-analytic method to deal with this problem.

3.2. Matrix-analytic solutions

To analyze the resulting system of linear equations (1)–(11), a matrix-form property is used. Following the concepts by Neuts [23], one finds that the transition rate matrix **Q** of this Markov chains can be partitioned as the following form:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{\Gamma}_{(M+S+1)\times(M+S+1)} & \mathbf{\Pi}_{(M+S+1)\times(M+S+1)} \\ \mathbf{\Phi}_{(M+S+1)\times(M+S+1)} & \mathbf{\Omega}_{(M+S+1)\times(M+S+1)} \end{bmatrix}. \tag{12}$$

The matrix **Q** is a square matrix of order $2(M + S + 1)$ and each entry of the matrix **Q** is listed in the following:

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_0 & \mu_1 \\ \lambda_0 & \gamma_1 & \mu_2 \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \lambda_{R-V-2} & \gamma_{R-V-1} & \mu_{R-V} \\ & & & & \lambda_{R-V-1} & \gamma_{R-V} & \mu_{R-V} \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & \lambda_{M+S-2} & \gamma_{M+S-1} & \mu_{R-V} \\ & & & & & & & & \lambda_{M+S-1} & \gamma_{M+S} \end{bmatrix}. \tag{13}$$

All elements in **Π** are equal to zero except $\mathbf{\Pi}_{R-V+1,R-V+2} = \mu_{R-V+1}$ and $\mathbf{\Phi} = \theta I_{M+S+1}$. I_{M+S+1} denotes the identity matrix of order $M + S + 1$.

$$\mathbf{\Omega}_{(M+S+1)\times(M+S+1)} = \begin{bmatrix} \sigma_0 & \mu_1 \\ \lambda_o & \sigma_1 & \mu_2 \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \lambda_{R-V-2} & \sigma_{R-V-1} & \mu_{R-V} \\ & & & & \lambda_{R-V-1} & \sigma_{R-V} & 0 \\ & & & & & \lambda_{R-V} & \sigma_{R-V+1} & \mu_{R-V+2} \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & \lambda_{M+S-2} & \sigma_{M+S-1} & \mu_R \\ & & & & & & & & \lambda_{M+S-1} & \sigma_{M+S} \end{bmatrix}. \tag{14}$$

The diagonal elements of $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ indicated by γ_i and σ_i, $0 \leqslant i \leqslant M + S$, are such that the sum of each column of $\mathbf{Q}$ is zero.

Let $\mathbf{P}$ denote steady-state probability vector of $\mathbf{Q}$. By partitioning the vector $\mathbf{P} = [\mathbf{P}_V \mathbf{P}_B]^T$ with $\mathbf{P}_V$ and $\mathbf{P}_B$ are both $(M + S + 1) \times 1$ vector, one finds that the steady-state equations $\mathbf{QP} = \mathbf{0}$ are given by

$$\mathbf{\Gamma P}_V + \mathbf{\Pi P}_B = \mathbf{0},$$
$$\mathbf{\Phi P}_V + \mathbf{\Omega P}_B = \mathbf{0}. \tag{15}$$

Using the following normalizing equation:

$$\sum_{i \in \{V,B\}} \sum_{n=0}^{M+S} P_{i,n} = e^T \mathbf{P} = 1, \tag{16}$$

where e represents a column vector with suitable size and each component equal to one. Eq. (16) is substituted into the first (redundant) row in Eq. (15) to yield

$$\mathbf{Q}^* \mathbf{P} = \begin{bmatrix} \mathbf{\Gamma}^* & \mathbf{\Pi}^* \\ \mathbf{\Phi} & \mathbf{\Omega} \end{bmatrix} \quad \mathbf{P} = [1, 0, \ldots, 0]^T, \tag{17}$$

$\mathbf{\Gamma}^*$ and $\mathbf{\Pi}^*$ are the matrices which are obtained by replacing each elements in the first row of $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ with one (for Eq. (16), the normalization condition). The solution of Eq. (17) provides the steady-state probabilities as

$$\mathbf{P} = (\mathbf{Q}^*)^{-1} [1, 0, \ldots, 0]^T = \begin{bmatrix} \mathbf{Q}_{11.2}^{-1} & -\mathbf{Q}_{11.2}^{-1} ll^* \mathbf{\Omega}^{-1} \\ -\mathbf{\Omega}^{-1} \mathbf{\Phi} \mathbf{Q}_{11.2}^{-1} & \mathbf{\Omega}^{-1} \mathbf{\Phi} \mathbf{Q}_{11.2}^{-1} \mathbf{\Pi}^* \mathbf{\Omega}^{-1} + \mathbf{\Omega}^{-1} \end{bmatrix} [1, 0, \ldots, 0]^T. \tag{18}$$

where $\mathbf{Q}_{11.2} = \mathbf{\Gamma}^* - \mathbf{\Pi}^* \mathbf{\Omega}^{-1} \mathbf{\Phi}$. Finally, it is observed that the steady-state probabilities $\mathbf{P}_V$ and $\mathbf{P}_B$ are equal to the first column of matrix $\mathbf{Q}_{11.2}^{-1}$ and $-\mathbf{\Omega}^{-1} \mathbf{\Phi} \mathbf{Q}_{11.2}^{-1}$, respectively.

4. Performance analysis

In this section, we deal with the steady-state availability and the mean time to system failure analysis. Also, the explicit expressions of some performance measures for the machine repair problem are included.

4.1. Availability and reliability analysis

It is assumed that the system breaks down if and only if $(S + 1)$ or more machines fail. The steady-state availability can be calculated as

$$A.V. = P(0 \leqslant n \leqslant s) = \sum_{n=0}^{S} P_{V,n} + \sum_{n=0}^{S} P_{B,n} = V_{s+1} (\mathbf{P}_V + \mathbf{P}_B)$$
$$= v_{s+1} (I_{M+S+1} - \mathbf{\Omega}^{-1} \mathbf{\Phi}) \mathbf{Q}_{11.2}^{-1} [1, 0, \ldots, 0]^T, \tag{19}$$

where v_k represents a row vector with suitable size which the first k elements are equal to 1 and zero otherwise.

To calculate the MTTF, we reduce the original transition rate matrix and delete the rows and columns for the absorbing state(s). The new matrix is called $\mathbf{B}$ as
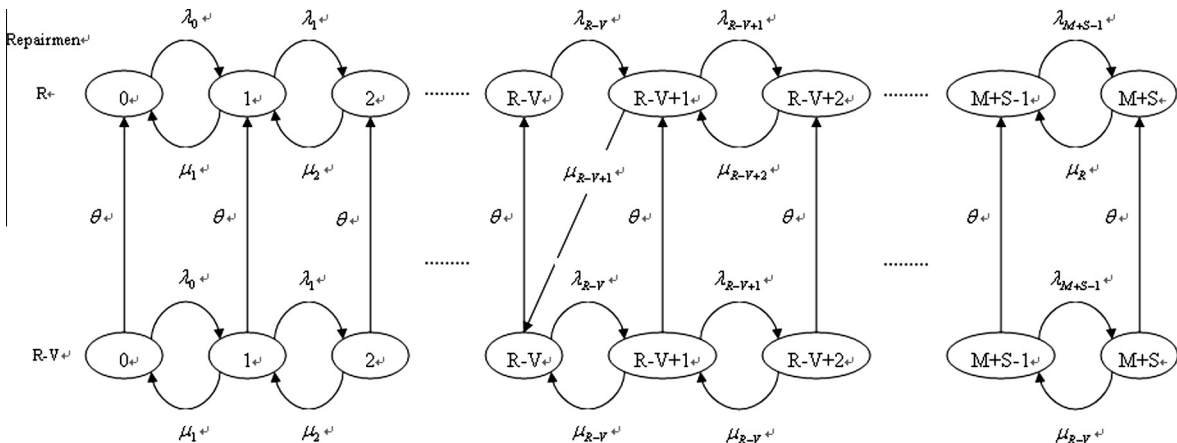

Fig. 1. The steady-transition-rate diagram for the multiple-server machine repair problems with standbys under a (V, R) synchronous single vacation policy.

$$\mathbf{B} = \begin{bmatrix} \mathbf{\Gamma}_r & \mathbf{\Pi}_r \\ \mathbf{\Phi}_r & \mathbf{\Omega}_r \end{bmatrix}^{\mathrm{T}}, \tag{20}$$

where $\mathbf{\Gamma}_r$, $\mathbf{\Pi}_r$, $\mathbf{\Phi}_r$ and $\mathbf{\Omega}_r$ denote the square sub-matrix of $\mathbf{\Gamma}$, $\mathbf{\Pi}$, $\mathbf{\Phi}$ and $\mathbf{\Omega}$, respectively. The subscribe "r" means reducing the matrix by deleting the $(S+2)$th $-(M+S+1)$th rows and $(S+2)$th $-(M+S+1)$th columns. Then, the expected time to reach an absorbing state is calculated from

$$E[T_{\mathbf{P}(\mathbf{0})\rightarrow\mathbf{P}_{(\text{absorbing})}}] = \mathbf{P}(0)^{\mathrm{T}} \int_0^\infty e^{\mathbf{B}t} dt = \mathbf{P}(0)^{\mathrm{T}}(-\mathbf{B}^{-1})e. \tag{21}$$

where $\mathbf{P}(0) = [1, 0, ..., 0]^{\mathrm{T}}$ denotes the initial conditions for this problem.

4.2. Other system performance measures

Our analysis is based on the following system performance measures. Let

$E[F] \equiv$ the expected number of failed machines in the system,
$E[F_q] \equiv$ the expected number of failed machines in the queue,
$E[O] \equiv$ the expected number of operating machines in the system,
$E[S] \equiv$ the expected number of acting standby machines in the system,
$E[B] \equiv$ the expected number of busy repairmen in the system,
$E[V] \equiv$ the expected number of vacation repairmen in the system,
$E[I] \equiv$ the expected number of idle repairmen in the system,
$M.A. \equiv$ machine availability (the fraction of the total time that the machines are working),
$O.U. \equiv$ operative utilization (the fraction of busy servers).

The expressions for $E[F]$, $E[F_q]$, $E[O]$, $E[S]$, $E[B]$, $E[V]$ and $E[I]$ are obtained as follows:

$$E[F] = \sum_{n=0}^{M+S} n(P_{V,n} + P_{B,n}) = [0, 1, 2, \ldots, M+S](\mathbf{P}_V + \mathbf{P}_B) = [0, 1, 2, \ldots, M+S](I_{M+S+1} - \mathbf{\Omega}^{-1}\mathbf{\Phi})\mathbf{Q}_{11.2}^{-1}[1, 0, \ldots, 0]^{\mathrm{T}}, \tag{22}$$

$$E[F_q] = \sum_{n=0}^{M+S} max\{0, n - (R - V)\}P_{V,n} + \sum_{n=0}^{M+S} max\{0, n - R\}P_{B,n} = \sum_{n=R-V+1}^{M+S} [n - (R - V)]P_{V,n} + \sum_{n=R+1}^{M+S} (n - R)P_{B,n}, \tag{23}$$

$$E[O] = \sum_{n=0}^{M+S} min\{M, M+S-n\}(P_{V,n} + P_{B,n}), \tag{24}$$

$$E[S] = \sum_{n=0}^{M+S} max\{0, S-n\}(P_{V,n} + P_{B,n}) = \sum_{n=0}^{S-1} (S-n)(P_{V,n} + P_{B,n}), \tag{25}$$

$$E[B] = \sum_{n=0}^{M+S} min\{n, R-V\}P_{V,n} + \sum_{n=0}^{M+S} min\{n, R\}P_{B,n}, \tag{26}$$

$$E[V] = \sum_{n=0}^{M+S} V P_{V,n} = V e^T \mathbf{P}_V = V e^T \mathbf{Q}_{112}^{-1}[1, 0, \ldots, 0]^{\mathrm{T}}, \tag{27}$$

$$E[I] = R - E[B] - E[V]. \tag{28}$$

Following Benson and Cox [24], the machine availability and the operative utilization are defined by

$$M.A. = 1 - \frac{E[F]}{M+S} \quad \text{and} \quad O.U. = \frac{E[B]}{R}. \tag{29}$$

Furthermore, using the Little's formula we obtain the expected waiting time in the system, $E[W]$, and in the queue $E[W_q]$ as

$$E[W] = E[F]/\lambda_e \quad \text{and} \quad E[W_q] = E[F_q]/\lambda_e, \tag{30}$$

where $\lambda_e = \sum_{i \in \{V,B\}} \sum_{n=0}^{M+S} \lambda_n P_{i,n}$ is the effective arrival rate into the system.

5. Cost analysis

In this section, we construct a total expected cost function per unit time based on the system performance measures, and impose a constraint on the system availability in which S, R and V are discrete decision variables.

First let

$C_h \equiv$ cost per unit time when one failed machine joins the system,
$C_e \equiv$ cost per unit time of a failed machine after all spares are exhaused (downtime cost),
$C_s \equiv$ cost per unit time when one machine is functioning as a spare (inventory cost),
$C_b \equiv$ cost per unit time when one repairmen is busy,
$C_i \equiv$ cost per unit time when one repairmen is idle,
$C_f \equiv$ cost per unit time of each repairmen,
$\gamma \equiv$ reward per unit time when one repair is on vacation.

5.1. Direct search method for optimal S, R and V

Using the definitions of these cost elements listed above, the total expected cost function per unit time is given by

$$T_{\text{cost}}(S, R, V) = C_h E[F] + C_s E[S] + C_e(M - E[O]) + C_b E[B] + C_i E[I] + RC_f - \gamma E[V]. \tag{31}$$

Some production system always requires minimum of M machines in operation or a certain level of system availability. Our object is to determine the optimum number of standbys S, say S^*, the optimum number of repairmen R, say R^*, and the optimum vacation policy level V, say V^*, simultaneous which minimize the cost function $T_{\text{cost}}(S, R, V)$ and the system availability is maintained at a certain level.

Following the concept of Hilliard [25], the cost minimization problem can be illustrated mathematically as

$$\underset{S,R,V}{\text{Minimize}}\, T_{\text{cost}}(S, R, V). \tag{32}$$

$$\text{Subject to} \quad A.V. = \sum_{n=0}^{S}(P_{V,n} + P_{B,n}) \geqslant A, \tag{33}$$

where $A.V.$ is the steady-state probability that at least M machines are in operation and function properly (system availability) and A is the availability level required.

A direct search method may be used to obtain potentially useful results. The optimization algorithm is a direct search approach over a grid whose boundaries for decision variables are selected in order to guarantee that the global optimum is obtained in the interior region (see [25]). The direct search algorithm is applied in the set {$M \geqslant S$ and $M \geqslant R \geqslant V$; S, R, V are positive integers}.

The specific steps in the direct search algorithm for obtaining the optimal value (S^*, R^*, V^*) are as follows:

Step 1. Find the optimal number of repairmen, and the optimal vacation policy level, for S standbys, *i.e.*,

$$\underset{R,V}{\text{Min}}\, T_{\text{cost}}(S, R, V) = T_{\text{cost}}(S, R^*, V^*)$$

subject to the availability constraint is satisfied.
Step 2. Step 2. Find the set of all minimum cost solutions for $S = 1, 2, ..., M$, *i.e.*, $\Theta = \{T_{\text{cost}}(S, R^*, V^*):S = 1, 2, ..., M\}$.
Step 3. Find the minimum cost solutions in this set, *i.e.*,

$$\underset{S}{\text{min}}\Theta = T_{\text{cost}}(S^*, R^*, V^*).$$

We provide an example to illustrate the direct search algorithm.

Example. Consider $M = 15$ and the system parameters $\lambda = 0.6$, μ = 2.5, θ = 0.2, α = 0.3, the cost elements and availability as follows

C_h = 10, C_e = 125, C_s = 50, C_b = 75, C_i = 40, C_f = 80, γ = 60, and A = 0.9.

Step 1. Find R^* and V^* for S standbys necessary to satisfy the required availability, where $S = 1, 2, ..., 15$. (see Table 1).
*Step 2.* From Table 1, Θ={$1209.55,$1108.82,$1048.50,$1050.06,$1085.63,...}.
Step 3. From step 2, the optimal solution $T_{\text{cost}}(S^*, R^*, V^*)$ = $1048.50 is achieved at $S^* = 8$, $R^* = 7$, $V^* = 2$ and the corresponding availability is 0.90311.

Table 1
The expected cost $T_{\text{cost}}(S^*, R^*, V^*)$ and the system availability $A.V.$ ($\lambda = 0.6$, $\mu = 2.5$, $\theta = 0.2$, $\alpha = 0.3$).

S	6	7	8	9	10
(S^*, R^*, V^*)	(6,8,1)	(7,7,1)	(8,7,2)	(9,6,1)	(10,6,1)
$T_{\text{cost}}(S^*, R^*, V^*)$	1209.55	1108.82	1048.50	1050.06	1085.63
$A.V.$	0.91014	0.92727	0.90311	0.92069	0.93692
S	11	12	13	14	15
(S^*, R^*, V^*)	(11,6,1)	(12,6,1)	(13,6,1)	(14,6,1)	(15,6,1)
$T_{\text{cost}}(S^*, R^*, V^*)$	1122.34	1159.65	1197.19	1234.70	1271.99
$A.V.$	0.94896	0.95805	0.96502	0.97044	0.97472
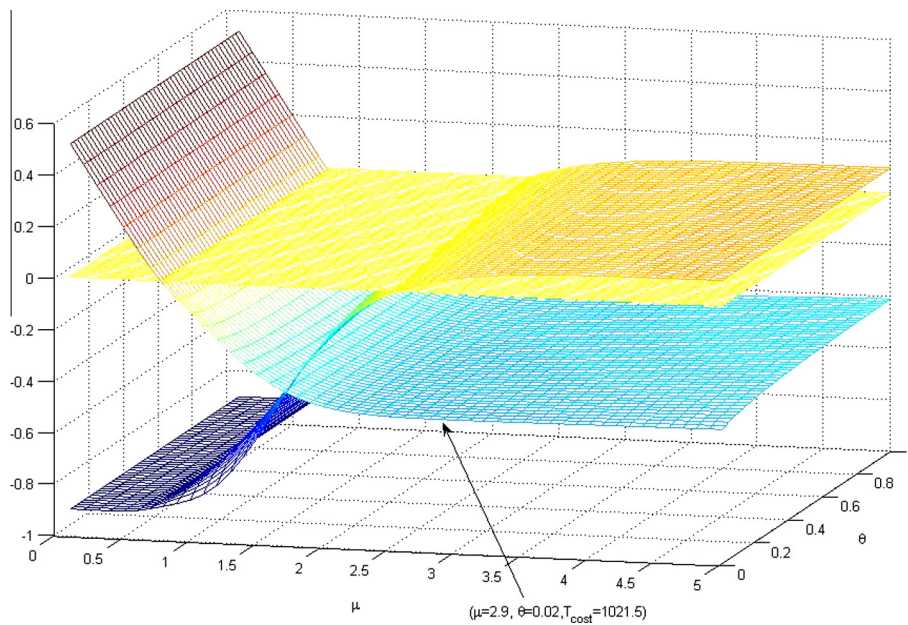
Fig. 2. Cost and availability surfaces for $(M, S^*, R^*, V^*) = (15, 8, 7, 2)$ as $\lambda = 0.6$ and $\alpha = 0.3$.

5.2. Optimal (μ, θ)

In practice, the service rate and vacation rate could be adjusted to minimize the total cost as the number of machines, standby, repairmen, and vacation policy level are known. It is assumed that there exists a maximum service rate of each repairman μ_U, a maximum vacation rate θ_U of these V repairmen and a given budget C. Then, this cost minimization problem can be illustrated mathematically as

$$\underset{0 \leqslant \mu \leqslant \mu_U, 0 \leqslant \theta \leqslant \theta_U}{\text{Minimize}} \ T_{\text{cost}}(\mu, \theta)/C - 1. \tag{34}$$

$$\text{Subject to} \ \ A.V. = \sum_{n=0}^{S} (P_{V,n} + P_{B,n}) \geqslant A. \tag{35}$$

The object function can be considered as an alternative form of the original cost function. After fixing the discrete variables, we deal with the optimization of the continuous variables. Under the same cost elements listed above and given $\lambda = 0.6$, $\alpha = 0.05$, $\mu_U = 5.0$, $C = 2000$, three surfaces $T_{\text{cost}}(\mu, \theta)/C - 1$, $A.V. - A$, and $z = 0$ for $(M, S^*, R^*, V^*) = (15, 8, 7, 2)$ are represented graphically in Fig. 2. The optimal service rate μ^* and the optimal vacation rate θ^* are the point achieving the lowest (minimum) cost in the area (feasible region) of $A.V. - A \geqslant 0$ (availability constraint). From Fig. 2, one sees that the optimal solution is $(\mu^*, \theta^*) = (2.9, 0.02)$ and the corresponding minimum object function is -0.48925 ($T_{\text{cost}}(\mu^*, \theta^*) = (1 - 0.48925) * 2000 = 1021.5$).

Table 2

System performance measures of the machine-repair problem with standbys under a (V,R) synchronous single vacation policy under optimal operating conditions ($\mu = 2.5, \theta = 0.2$).

(λ, α)	(0.3, 0.3)	(0.9, 0.3)	(1.5, 0.3)	(0.6, 0)	(0.6, 0.3)	(0.6, 0.6)
(S^*, R^*, V^*)	(5, 4, 1)	(11, 9, 2)	(14, 13, 1)	(8, 6, 1)	(8, 7, 2)	(10, 6, 1)
T_{cost}	643.358	1416.81	2145.64	1022.11	1048.50	1073.32
A.V.	0.91941	0.90417	0.90509	0.92854	0.90311	0.90528
$E[F]$	2.61779	7.00162	9.93347	4.22717	4.84068	6.30274
$E[F_q]$	0.53192	1.17675	0.55327	0.66541	0.88737	1.81540
$E[O]$	14.8474	14.7711	14.7755	14.8407	14.7850	14.7891
$E[S]$	2.53484	4.22728	4.29107	3.93212	3.37432	3.90813
$E[B]$	2.08587	5.82487	9.38020	3.56177	3.95332	4.48734
$E[V]$	0.81645	1.77055	0.84624	0.81343	1.73861	0.88529
$E[I]$	1.09768	1.40459	2.77355	1.62481	1.30807	0.62737
M.A.	0.86911	0.73071	0.65747	0.81621	0.78954	0.74789
O.U	0.52147	0.64721	0.72155	0.59363	0.56476	0.74789

Table 3

System performance measures of the machine-repair problem with standbys under a (V,R) synchronous single vacation policy under optimal operating conditions ($\lambda = 0.6, \alpha = 0.3$).

(μ, θ)	(1.2, 0.5)	(2.4, 0.5)	(3.6, 0.5)	(2.5, 0.2)	(2.5, 0.5)	(2.5, 1.0)
(S^*, R^*, V^*)	(14, 10, 10)	(9, 6, 1)	(6, 5, 1)	(8, 7, 2)	(8, 6, 1)	(10, 6, 2)
T_{cost}	1847.76	1065.20	822.23	1048.50	1030.20	1042.37
A.V.	0.90330	0.91424	0.93190	0.90311	0.91024	0.92091
$E[F]$	9.75034	5.30466	3.15025	4.84069	4.77319	5.76921
$E[F_q]$	1.24683	1.11666	0.42376	0.88737	0.80929	1.68337
$E[O]$	14.7644	14.8087	14.8681	14.7850	14.8057	14.8179
$E[S]$	4.48532	3.88670	2.98165	3.37432	3.42110	4.41291
$E[B]$	8.50350	4.18800	2.72649	3.95332	3.96390	4.08584
$E[V]$	0.01088	0.72677	0.70269	1.73861	0.71615	1.21740
$E[I]$	1.48562	1.08524	1.57082	1.30807	1.31995	0.69676
M.A.	6.63782	0.77897	0.84999	0.78954	0.79247	0.76923
O.U.	0.85035	0.69800	0.54530	0.56476	0.66065	0.68097

Table 4

System performance measures of the machine-repair problem with standbys under a (V,R) synchronous single vacation policy under optimal operating conditions ($M = 15, S = 8, R = 7, V = 2$).

$(\lambda, \alpha))$	(0.3, 0.3)	(0.6, 0.3)	(0.9, 0.3)	(0.6, 0)	(0.6, 0.3)	(0.6, 0.6)
(μ^*, θ^*)	(1.5, 0.02)	(2.9, 0.02)	(4.3, 0.02)	(2.7, 0.02)	(2.9, 0.02)	(3, 0.02)
T_{cost}	1020.33	1021.50	1021.60	1026.46	1021.50	1017.43
A.V.	0.94548	0.95160	0.95404	0.95014	0.95160	0.94472
$E[F]$	4.36272	4.04968	3.92574	3.82927	4.04968	4.37077
$E[F_q]$	0.63529	0.54765	0.51498	0.51958	0.54765	0.64493
$E[O]$	14.8935	14.90267	14.9066	14.8936	14.90267	14.8917
$E[S]$	3.74381	4.04764	4.16762	4.27711	4.04764	3.73750
$E[B]$	3.72746	3.50203	3.41076	3.30969	3.50203	3.72585
$E[V]$	1.94268	1.96119	1.97082	1.94829	1.96119	1.97089
$E[I]$	1.32986	1.53677	1.61842	1.74202	1.53677	1.30326
M.A.	0.81032	0.82393	0.82932	0.83351	0.82393	0.80997
O.U.	0.53249	0.50029	0.48725	0.47281	0.50029	0.53226

5.3. Sensitivity analysis

Considering $M = 15$, the same cost elements and system parameters listed in above example, we perform a sensitivity analysis for changes in the joint optimum value (S^*, R^*, V^*) along with changes in specific values of the system parameters λ, μ, θ and α. The minimum expected cost $T_{cost}(S^*, R^*, V^*)$ and values of various system performance measures A.V. E [F], $E[F_q]$, $E[O]$, $E[S]$, $E[B]$, $E[V]$, $E[I]$, M.A., and O.U. at the optimum values (S^*, R^*, V^*) are shown in Tables 2 and 3 for different values of (λ, α) and (μ, θ).

From Table 2, we observe that (i) $T_{cost}(S^*, R^*, V^*)$ increases as λ or α increases; (ii) S^* and R^* increase as λ increases; and (iii) S^* increases as α increases. One sees from Table 3 that (i) $T_{cost}(S^*, R^*, V^*)$ decreases as μ increases; (ii) S^* and R^* decrease as μ increases; and (iii) S^* increases as θ increases.

Similarly, we are also interesting in the effect of (μ^*, θ^*) by changing the values of other two continuous system parameters λ and α when $M = 15$ and $(S^*, R^*, V^*) = (8, 7, 2)$ are determined. In Table 4, the optimum value (μ^*, θ^*) and several system

performance measures for specific values of λ and α are given. It is observed that μ^* increases as λ or α increases (roughly insensitive to α) and θ^* is changeless as λ or α changes. Apparently, increasing the service rate is more effective in reducing cost than adjusting the vacation rate. Consequently, the adjustment (increase) of the service rate μ will be considered firstly until $\mu^* = \mu_U$ then adjusts (increases) the vacation rate θ next.

6. Conclusions

The systematic methodology provided in this paper works efficiently for a machine repair model with standbys under a synchronous single vacation policy. The stationary probability vectors were obtained in terms of matrix forms using the technique of matrix partition. Firstly, we developed the steady-state solutions in matrix forms for the machine repair model by using the Markov process. These solutions were used to obtain the various system performance measures, such as the steady-state availability, MTTF, the expected number of failed machines in the queue / system, the expected number of idle, busy and vacation servers, machine availability, operative utilization, etc. Next, we developed a cost model for the machine repair model to determine the joint optimum number of standbys, servers and vacation servers in order to minimize the steady-state expected cost per unit time, while maintaining a specified level of system availability. After the determination of the three discrete decision variables, the optimal adjustments of service rate and vacation rate were also considered. Two procedures were provided to handle this optimization problem. Finally, a sensitivity analysis was performed to investigate the effect on the joint optimum values if the system parameters take on other specific values. We extended the traditional vacation policy to more generalization one. The investigated model can be used to evaluate the performance of some practical queueing systems similar to mentioned earlier in Introduction. Specifically, it may be employed to fit the system with configurable servers which have multiple tasks.

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