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Cluster synchronization in networks of neurons with chemical synapses

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In this work, we study the cluster synchronization of chemically coupled and generally formulated networks which are allowed to be nonidentical. The sufficient condition for the existence of stably synchronous clusters is derived. Specifically, we only need to check the stability of the origins of m decoupled linear systems. Here, m is the number of subpopulations. Examples of nonidentical networks such as Hindmarsh-Rose (HR) neurons with various choices of parameters in different subpopulations, or HR neurons in one subpopulation and FitzHugh-Nagumo neurons in the other subpopulation are provided. Explicit threshold for the coupling strength that guarantees the stably cluster synchronization can be obtained. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4862484]

Cluster synchronization has attracted increasing attention due to its applications in brain science, engineering control, ecology, communication engineering, and distributed computation. Most of the existing works with application to brain science focused on networks of neurons that are electrically coupled. The first theoretical work for investigating cluster synchronization in networks of neurons with chemical synapses is due to Belykh and Hasler.[1,2] However, their work is not completely rigorous. In this paper, we study this topic and obtain essential condition on network topology so that our sufficient conditions for cluster synchronization are amount to checking if the origins of a certain decoupled linear systems are asymptotically stable. We are then able to derive explicit thresholds for the coupling strength that generates various stable dynamics in their clusters. Moreover, our model is generally formulated that is allowed to be nonidentical.

I. INTRODUCTION

Intercellular communication among brain cells is one of the most important characteristics of all animal species. Brain connectivity has a hierarchy of different levels ranging from the microscale to the mesoscale to the macroscale. The simplest macroscale rhythm in networks of neurons is the complete synchrony when all neurons fire in unison. There are many theoretical works[3–14] on this matter. Typically, irregular bursting synchronization, regular bursting synchronization, and fixed point synchronization arise, respectively, at a lower coupling strength, an intermediate coupling strength and a larger coupling strength. Moreover, its synchronization depends heavily on the intrinsic properties of the individual neurons, the coupling strength and its network topology.

Cluster synchronization,[15–19] a mesoscale phenomenon, is another example of cooperative rhythms. In this case, all neurons within each subpopulation (cluster) fire in perfect

synchrony. Most of the theoretical work in this direction consider the diffusively (electrically) coupled network. Recently, the cluster synchronization of a synaptically (chemically) and identically coupled network of HR neurons was investigated by Belykh and Hasler.[1,2]

The purpose of this paper is to study the existence and stability of synchronous clusters of generally formulated and synaptically coupled networks of possibly nonidentical neurons. Our results contain the following. First, we give a necessary and sufficient condition on the network topology so that cooperative rhythm in subpopulations is a possibility. Second, the sufficient condition for the existence of stably synchronous clusters is derived. In particular, we only need to check the stability of the origins of m decoupled linear systems. Here, m is the number of clusters in the network. Third, examples of nonidentical networks such as HR neurons with various choices of parameters in different subpopulations, or HR neurons in one subpopulation and FitzHugh-Nagumo neurons in the other subpopulation are provided. Explicit threshold for the coupling strength that guarantees the stably cluster synchronization is obtained.

We organize the paper as follow. The abstract formulation of model is introduced in Sec. II. The sufficient and necessary condition for the existence of cluster synchronization manifold (CSM) is also recorded there. In Sec. III, we derive a sufficient condition for the existence of stably synchronous clusters. The comparisons between our problem and results with some related work in the recent literatures are given in the end of the section. In Sec. IV, some examples are provided to illustrate the effectiveness of our results. These examples give the coexistence of the various stable dynamics such as spiking and fixed point on their clusters. Some concluding remarks are stated in Sec. V. The proof of our main result is recorded in the Appendix.

II. FORMULATION

The spiking or bursting behaviors of a single neuron can be captured by neuron equations such as Hodgkin-Huxley, Hindmarsh-Rose, Morris-Lecar, and FitzHugh-Nagumo models. These models take the form

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$$\dot{V} = f_1(V, \boldsymbol{w}, \boldsymbol{p}),$$
$$\dot{\boldsymbol{w}} = f_2(V, \boldsymbol{w}, \boldsymbol{p}), \tag{1}$$

where $V \in \mathbb{R}$ represents the membrane potential of the neuron, and $\boldsymbol{w} \in \mathbb{R}^{n-1}$ are currents taking into the account the transport of ions across the membrane via the ion channels. $\boldsymbol{p} \in \mathbb{R}^k$ are the parameters describing various types of neurons, and f_1, f_2 are some continuous functions of V, $\boldsymbol{w}$, and $\boldsymbol{p}$.

In this paper, we study the chemically coupled network of N neurons for which each single neuronal dynamics is governed by equations having the form of Eq. (1). The equations of motion then read, for $i = 1, \ldots, N$,

$$\dot{V}_i = f_{i,1}(V_i, \boldsymbol{w}_i, \boldsymbol{p}_i) - (V_i - v)\sum_{j=1}^{N} g_{ij}c_{ij}\Gamma(V_j),$$
$$\dot{\boldsymbol{w}}_i = f_{i,2}(V_i, \boldsymbol{w}_i, \boldsymbol{p}_i), \tag{2}$$

where $V_i \in \mathbb{R}$ and $\boldsymbol{w}_i \in \mathbb{R}^{n_i-1}$. Here, g_{ij} is the strength of the synaptic coupling from neuron j to neuron i, $c_{ij} \in \{-1, 0, 1\}$, v is the synaptic reversal potential with $v > V_i(t)$ for all time t, and Γ is the synaptically coupling function. Here, $c_{ij} = 1$ (respectively, -1) if neuron i receives synaptic current from excitatory (respectively, inhibitory) neuron j, otherwise $c_{ij} = 0$. Let

$$\boldsymbol{V} = (V_1, \ldots, V_N)^T,$$

and

$$\boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N)^T.$$

Then in vector-matrix form, Eq. (2) becomes

$$\dot{\boldsymbol{V}} = \boldsymbol{F}_1(\boldsymbol{V}, \boldsymbol{w}, \boldsymbol{p}) + g_s(v\boldsymbol{I}_N - \mathrm{diag}(\boldsymbol{V}))\boldsymbol{C}\Gamma(\boldsymbol{V}),$$
$$\dot{\boldsymbol{w}} = \boldsymbol{F}_2(\boldsymbol{V}, \boldsymbol{w}, \boldsymbol{p}), \tag{3}$$

where $\Gamma(\boldsymbol{V}) := (\Gamma(V_1), \ldots, \Gamma(V_N))^T$, $\boldsymbol{F}_i(\boldsymbol{V}, \boldsymbol{w}, \boldsymbol{p}) := (f_{i,1}(V_1, \boldsymbol{w}_1, \boldsymbol{p}_1), \ldots, f_{i,N}(V_N, \boldsymbol{w}_N, \boldsymbol{p}_N))$, $i = 1, 2$, $\boldsymbol{p} := (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)^T$, $\boldsymbol{I}_N$ is the identity matrix of size $N \times N$, g_s is the normalized strength of the synaptic coupling of the system, and $\boldsymbol{C} := \left(\frac{g_{ij}}{g_s}c_{ij}\right) := (\bar{c}_{ij})$. Note that $\boldsymbol{C}$ depicts the synaptically coupled network, and the element $\bar{c}_{ij} > 0$ implies that the current injected from neuron j to i is positive (excitatory) and hence depolarization occurs in neuron i. On the other hand, $\bar{c}_{ij} < 0$ implies that the current injected from neuron j to i is negative (inhibitory) and hence hyperpolarization occurs in neuron i. Herein, the sign of $\bar{c}_{ij}$ could be either positive or negative, and $\boldsymbol{C}$ could be nonsymmetric.

The objective of this paper is to study the synchrony phenomena in neurons. Since Eq. (2) consists of possibly nonidentical systems, we are led to consider the notion of cluster synchronization of the model. To this end, we assume the model of N neurons is divided into m types of neurons where the neurons of each type are governed by an identical equation. It should be remarked that the governing equations between different types of neurons are allowed to be different. We further assume that the number of neurons in the rth type is N_r ($\sum_{r=1}^{m} N_r = N$). Under above assumptions, the notations in Eq. (2) can be further simplified. The following

notations are so set up that within each type the vector field $\boldsymbol{f}$ and the parameters $\boldsymbol{p}$ are the same and that the superscript is used to distinguish $\boldsymbol{f}$ and $\boldsymbol{p}$ among the different types. Let

$$*^{(1)} \triangleq *_1 = *_2 = \cdots = *_{N_1},$$
$$\vdots$$
$$*^{(r)} \triangleq *_{\sum_{s<r}N_s+1} = *_{\sum_{s<r}N_s+2} = \cdots = *_{\sum_{s\leq r}N_s},$$
$$\vdots$$
$$*^{(m)} \triangleq *_{\sum_{s<m}N_s+1} = *_{\sum_{s<m}N_s+2} = \cdots = *_{\sum_{s\leq m}N_s}, \tag{4}$$

where $*$ denotes parameters $\boldsymbol{p}$ or nonlinearities $\boldsymbol{f} := (f_1, f_2)^T$, as shown in Eq. (1). We are now in a position to give the precise definition of the cluster synchronization.

Definition 1. Let $\boldsymbol{x}_i = (V_i, \boldsymbol{w}_i)^T$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)^T$. Then

(i)   Define the set

$$\Xi = \{\boldsymbol{x} \in \mathbb{R}^{nN} : \boldsymbol{x}_1 = \cdots = \boldsymbol{x}_{N_1},$$
$$\boldsymbol{x}_{N_1+1} = \cdots = \boldsymbol{x}_{N_1+N_2}, \ldots,$$
$$\boldsymbol{x}_{N-N_m+1} = \cdots = \boldsymbol{x}_N\}. \tag{5}$$

Then Ξ is said to be the *CSM*.

(ii)   A trajectory $\boldsymbol{x}(t)$ is said to be *clusteringly synchronized* if

$$\lim_{t\to\infty} d(\boldsymbol{x}(t), \Xi) = \boldsymbol{0},$$

where d is a metric from a point to a set.

(iii) The CSM Ξ is said to be *locally stable* if each trajectory $\boldsymbol{x}(t)$ whose initial value $\boldsymbol{x}(0)$ is sufficiently close to Ξ will stay close to Ξ for all time t and is clusteringly synchronized.

To investigate the cluster synchronization, the first step is to ensure the invariant property of the CSM Ξ. The following proposition gives the sufficient and necessary condition for Ξ to be invariant.

Proposition 1. *Assume that coupled system (2) satisfies Eq. (4). Then its CSM Ξ is invariant if and only if $\boldsymbol{C}$ can be partitioned into the form*

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} & \ldots & \boldsymbol{C}_{1m} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \ldots & \boldsymbol{C}_{2m} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{C}_{m1} & \boldsymbol{C}_{m2} & \ldots & \boldsymbol{C}_{mm} \end{pmatrix}, \tag{6}$$

where $\boldsymbol{C}_{rs} \in \mathbb{R}^{N_r \times N_s}$, and each row sum of $\boldsymbol{C}_{rs}$, $1 \leq r, s \leq m$, is equal to k_{rs}.

The proof of Proposition 1 is clear and hence is omitted. This proposition implies that for CSM to be invariant, each neuron in the rth type of neurons receives the same amount of the total connection weights k_{rs} from any neurons in the sth type.

Remark 1. If there exists an r such that not all the row sum of $\boldsymbol{C}_{rr}$ are the same, then $\boldsymbol{C}_{rr}$ can be further decoupled into smaller blocks until Eq. (6) is satisfied. This classification also indicates that the distinction of neuron types not

just depends on the parameters p and the vector field f but also on the total connection weights k_{rs}.

We next describe the dynamics on the invariant manifold Ξ. As defined in Eq. (5), the dynamics on Ξ is determined by $\sum_{r=1}^{m} n_r$ variables. Let $(V_c^{(r)}, w_c^{(r)}) \triangleq x_{\sum_{s<r} N_s + 1} = x_{\sum_{s<r} N_s + 2} = \ldots = x_{\sum_{s\leq r} N_s}$, $r = 1, \ldots, m$. Then, the dynamics on Ξ is governed by the following equations:

$$\dot{V}_c = F_{c,1}(V_c, w_c, p_c) + g_s(vI_m - \mathrm{diag}(V_c))K\Gamma(V_c),$$
$$\dot{w}_c = F_{c,2}(V_c, w_c, p_c), \tag{7}$$

where $V_c = (V_c^{(1)}, \ldots, V_c^{(m)})^T$, $w_c = (w_c^{(1)}, \ldots, w_c^{(m)})^T$, $p_c = (p^{(1)}, \ldots, p^{(m)})^T$, $F_{c,i}(V_c, w_c, p_c) = (f_i^{(1)}(V_c^{(1)}, w_c^{(1)}, p^{(1)}), \ldots, f_i^{(m)}(V_c^{(m)}, w_c^{(m)}, p^{(m)}))^T$, $i=1,2$, and $K = (k_{rs}) \in \mathbb{R}^{m\times m}$. In the special case that $k_{rs}=0$, $\forall r \neq s$, Eq. (7) can be further reduced to m separable subsystems.

Proposition 2. *Suppose $k_{rs} = 0$, $\forall r \neq s$. Then, Eq. (7) can be written as the following form:*

$$\dot{V}_c^{(r)} = f_1^{(r)}(V_c^{(r)}, w_c^{(r)}, p^{(r)}) + k_{rr}g_s(v - V_c^{(r)})\Gamma(V_c^{(r)}),$$
$$\dot{w}_c^{(r)} = f_2^{(r)}(V_c^{(r)}, w_c^{(r)}, p^{(r)}), \tag{8}$$

for $r = 1, \ldots, m$.

The proof of Proposition 2 follows directly from Eq. (7), and hence is omitted. Note that Eq. (8) implies that the dynamics of each type of neurons is independent of other types of neurons on the CSM Ξ.

III. STABILITY OF CLUSTER SYNCHRONIZATION MANIFOLD

In this section, some sufficient criteria for the local stability of CSM Ξ will be derived. We begin with the derivation of the variational equations of Eq. (2) along the CSM Ξ. The vector equations read, for $r = 1, \ldots, m$,

$$\delta\dot{V}_r = \frac{\partial}{\partial V}f_1^{(r)}(u_c^{(r)})\delta V_r + \frac{\partial}{\partial w}f_1^{(r)}(u_c^{(r)})\delta w_r$$
$$-g_s\sum_{s=1}^{m}k_{rs}\Gamma(V_c^{(s)})\delta V_r$$
$$+g_s(v - V_c^{(r)})\sum_{s=1}^{m}\Gamma'(V_c^{(s)})C_{rs}\delta V_s, \tag{9}$$
$$\delta\dot{w}_r = \frac{\partial}{\partial V}f_2^{(r)}(u_c^{(r)})\delta V_r + \frac{\partial}{\partial w}f_2^{(r)}(u_c^{(r)})\delta w_r,$$

where $u_c^{(r)} = (V_c^{(r)}, w_c^{(r)}, p^{(r)})$, $\delta V_r \in \mathbb{R}^{N_r}$, and $\delta w_r \in \mathbb{R}^{(n_r-1)N_r}$. Next, we define matrices

$$E_r = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}_{(N_r-1)\times N_r},$$

and

$$E_r^{\dagger} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{N_r\times(N_r-1)},$$

and let $\xi_r = E_r\delta V_r$ and $\eta_r = E_r\delta w_r$, $r = 1, \ldots, m$. Note that $E_rE_r^{\dagger} = I_{N_r}$. Then from Eq. (9), we have

$$\dot{\xi}_r = \frac{\partial}{\partial V}f_1^{(r)}(u_c^{(r)})\xi_r + \frac{\partial}{\partial w}f_1^{(r)}(u_c^{(r)})\eta_r$$
$$-g_s\sum_{s=1}^{m}k_{rs}\Gamma(V_c^{(s)})\xi_r$$
$$+g_s(v - V_c^{(r)})\sum_{s=1}^{m}\Gamma'(V_c^{(s)})(E_rC_{rs}E_s^{\dagger})\xi_s,$$
$$\dot{\eta}_r = \frac{\partial}{\partial V}f_2^{(r)}(u_c^{(r)})\xi_r + \frac{\partial}{\partial w}f_2^{(r)}(u_c^{(r)})\eta_r. \tag{10}$$

By the definitions of ξ_r and η_r, it is clear that the local stability of the CSM Ξ is equivalent to that of the equilibrium 0 of Eq. (10). Now, we derive a sufficient criterion for the stability of 0 of Eq. (10).

Theorem 1. *Let C_{rs} be the $N_r \times N_s$ matrices as defined in Eq. (6). Assume that all rows of the matrices C_{rs} with $r < s$, are the same, and $k_{rs} = 0$, $\forall r \neq s$. Let the equilibrium 0 of each of the following linear systems, $r = 1, \ldots, m$,*

$$\dot{\xi}_r = \left(\frac{\partial}{\partial V}f_1^{(r)}(u_c^{(r)}) - g_sk_{rr}\Gamma(V_c^{(r)})\right)\xi_r + \frac{\partial}{\partial w}f_1^{(r)}(u_c^{(r)})\eta_r$$
$$+g_s(v - V_c^{(r)})\Gamma'(V_c^{(r)})(E_rC_{rr}E_r^{\dagger})\xi_r,$$
$$\dot{\eta}_r = \frac{\partial}{\partial V}f_2^{(r)}(u_c^{(r)})\xi_r + \frac{\partial}{\partial w}f_2^{(r)}(u_c^{(r)})\eta_r, \tag{11}$$

is globally exponentially asymptotically stable. Then, the CSM Ξ is locally stable.

The proof of Theorem 1 is given in Appendix. The condition that all rows of C_{rs} are the same means either there is no connection from the sth type of neurons to the rth type of neurons or each rth type of neurons receives the same totally synaptic coupling weights from the sth type of neurons. Note that the system of equations in Eq. (11) is uncoupled. Consequently, the error dynamics of the rth type of neurons is independent of sth type of neurons under the assumptions. Moreover, the advantage of such decoupling is that some well developed methods such as the construction of Lyapunov functions[1,2] or the monotone dynamics approach[14] can be applied directly to show the stability of Eq. (11) with respect to the equilibrium 0 and hence the local stability of Ξ.

Remark 2. In this Remark, we compare our problem and results with some related work in the recent literatures. To the best of our knowledge, most work concerning cluster synchronization are formulated as the (electrically) diffusively coupled network[15–19] rather than the (chemically) synaptically coupled network as given in Eq. (2). Nevertheless, we shall compare the network topology considered there with ours. In

Ref. [16], Cao and Li considered the case where $\boldsymbol{E}_r\boldsymbol{C}_{rs}\boldsymbol{E}_s^\dagger = \boldsymbol{0}, \forall 1 \leq r,s \leq m$. But, we only require that the above equalities be true for $r < s$ in our work. In Ref. [17], Lu, Liu, and Chen considered the case where $\boldsymbol{C}$ satisfies the common intercluster coupling condition, i.e., if $\boldsymbol{C}_{rs} \neq \boldsymbol{0}$ then each row of $\boldsymbol{C}_{rs}$ contains at least one nonzero element. But in Theorem 1, we do not need this assumption. In Ref. [18], Wu, Jiao, and Chen considered the assumption where all eigenvalues of $\boldsymbol{C}$ are real. Such limitation is not required in our work. It should be mentioned that in Ref. [16], authors also considered the effect of the delay and that in Refs. [17] and [18], the assumption that all row of $\boldsymbol{C}_{rs}$ are the same is not required there. The most related work to ours is that of Belykh and Hasler.[1,2] They studied the chemical coupling network of the identical Hindmarsh-Rose neurons, and derived some sufficient criteria, which are not completely rigorous for the local stability of Ξ. It should be mentioned that the assumptions between the network topology of theirs and ours are neither mutually exclusive nor inclusive. Specifically, their assumptions are that $k_{rs} \neq 0$ for some r, s but $\boldsymbol{E}_r\boldsymbol{C}_{rs}\boldsymbol{E}_s^\dagger = \boldsymbol{0}, \forall 1 \leq r,s \leq m$.

IV. APPLICATIONS

In this section, some examples are given to illustrate the effectiveness of our theory. The synaptically coupling function Γ under consideration in this section is modeled by the sigmoidal function[20]

$$\Gamma(V_j) = \frac{1}{1 + \exp\{-\lambda(V_j - \theta_s)\}}, \tag{12}$$

where θ_s is the threshold for the neuronal firing and the value is chosen so that every spike of a single neuron can reach the threshold. In the limit $\lambda \to \infty$, the above sigmoid function reduces to a Heaviside step function. In the following examples, the parameters θ and λ and the synaptic reversal potential v are set to be -0.25, 7.5, and 2, respectively.

The connection topology under consideration is column-consistence, i.e., the nonzero entries in the columns of the coupling matrix $\boldsymbol{C}$ are of the same sign. Biologically, such consistence means that *all* the synapses that a neuron projects to other neurons are either excitatory or inhibitory. We also remark that in Ref. [12], a model of neurons for which neurons could make both excitatory and inhibitory synapses with other neurons is considered. It should be mentioned that our model can be applied to such scenarios as well.

A. Coupled HR neurons with different parameters

The Hindmarsh-Rose model was obtained by biological consideration over the response to stimuli of a real neuronal cell. It can be used to describe the bursting or spiking of neurons. The model takes the form

$$\begin{aligned} \dot{x} &= ax^2 - x^3 + y - z + I, \\ \dot{y} &= -y - dx^2 + 1, \\ \dot{z} &= \mu(b(x - x_0) - z). \end{aligned} \tag{13}$$

Here, x denotes the membrane potential of the neuron, $\boldsymbol{w} := (y, z)$ are the currents flowing in/out of neurons through the ion channels, and $\boldsymbol{p} := (a, b, d, \mu, I, x_0)$ are the parameters.

Consider the neural network consisting of 12 Hindmarsh-Rose neurons with 3 different types:

Type 1: The number N_1 of neurons in type 1 is 4, and the parameters $\boldsymbol{p}$ are chosen as $\boldsymbol{p}^{(1)} = (2.8, 20, 4.4, 0.001, 1, -1.11)$. The neuronal dynamics in type 1 represents regular spiking.

Type 2: The number N_2 of neurons in type 2 is 4, and the parameters $\boldsymbol{p}$ are chosen as $\boldsymbol{p}^{(2)} = (3, 4, 5, 0.006, 3, -1.56)$. Its dynamics represents irregular bursting.[21]

Type 3: The number N_3 of neurons in this subpopulation is 4, and the parameters $\boldsymbol{p}$ are chosen as $\boldsymbol{p}^{(3)} = (2.6, 4, 5, 0.01, 4, -1.6)$. Its neuronal dynamics represents regular bursting.[14]

The coupling topology, Fig. [1], of the synaptically coupled network under consideration has the following matrix form:

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} & \boldsymbol{C}_{13} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \boldsymbol{C}_{23} \\ \boldsymbol{C}_{31} & \boldsymbol{C}_{32} & \boldsymbol{C}_{33} \end{pmatrix}, \tag{14}$$

where

$$\boldsymbol{C}_{11} = \boldsymbol{C}_{22} = \boldsymbol{C}_{33} = \frac{1}{10} \begin{pmatrix} 0 & 17 & 0 & -7 \\ 11 & 0 & -1 & 0 \\ 8 & 3 & 0 & -1 \\ 17 & 0 & -7 & 0 \end{pmatrix},$$

$$\boldsymbol{C}_{12} = \boldsymbol{C}_{23} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

$$\boldsymbol{C}_{21} = \boldsymbol{C}_{32} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix},$$

and

$$\boldsymbol{C}_{13} = \boldsymbol{C}_{31} = \boldsymbol{0}.$$

It then follows from Theorem 1 that if the corresponding equations (11) for $r = 1, \ldots, 3$ are all exponentially stable, then the CSM

$$\begin{aligned} \Xi = \{\boldsymbol{x} \in \mathbb{R}^{27} : \boldsymbol{x}_1 = \boldsymbol{x}_2, \boldsymbol{x}_3 = \boldsymbol{x}_4 = \boldsymbol{x}_5, \\ \boldsymbol{x}_6 = \boldsymbol{x}_7 = \boldsymbol{x}_8 = \boldsymbol{x}_9\} \end{aligned} \tag{15}$$

is also locally stable.

We shall apply the Lyapunov function technique developed in Ref. [2] on Eq. (11) with $r = 1$ to show that its equilibrium $\boldsymbol{0}$ is asymptotically stable. In particular, we need to find (γ, g_{s1}) so that the following inequality is satisfied:

$$\begin{aligned} &\left(\gamma(3 - \gamma d^2)x_c^{(1)^2} - \gamma(2a - d)x_c^{(1)} - \frac{1}{4}\right) \\ &+ \gamma g_{s1}\left(k_{11}\Gamma(x_c^{(1)}) - (v - x_c^{(1)})\Gamma'(x_c^{(1)})\lambda_2(\boldsymbol{C}_{11})\right) > 0, \tag{16} \end{aligned}$$
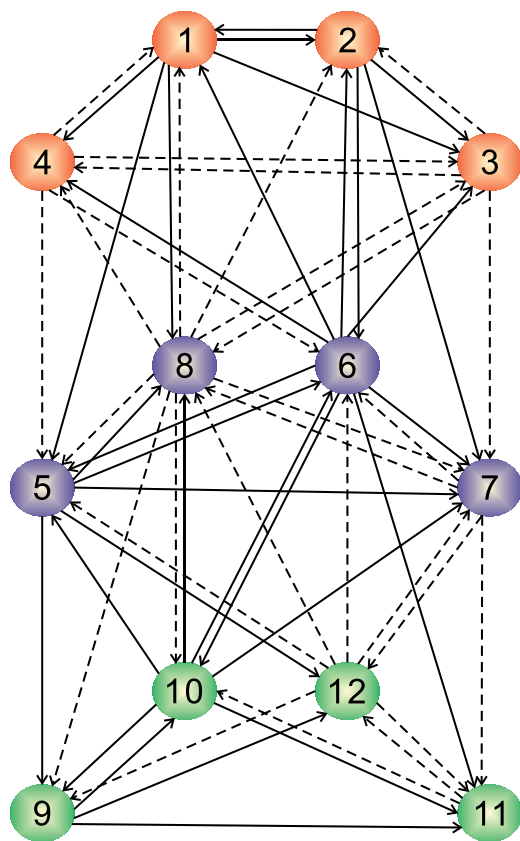
FIG. 1. Network topology describing by C as given in Eq. (14). The solid (dotted) line denotes the connection between two neurons is excitatory (inhibitory).

for $x < v$. Here, $\lambda_2(C_{11}) = -0.2$ is the second maximum eigenvalue of C_{11}. A direct verification would yield that the choice of $(\gamma, g_{s1}^*) = (0.12, 11.81)$ makes the inequality in Eq. (16) hold.

To study Eq. (11) with $r = 2$, 3, we shall use the monotone dynamic technique proposed in Corollary 1 of Ref. 14.

The resulting answers are that if $g_s > g_{s2} \approx 0.87$ and $g_s > g_{s3} \approx 0.87$, then the equilibrium $\mathbf{0}$ of equation (11) for $r = 2$, 3 is asymptotically stable, respectively. Consequently, if $g_s > \max\{11.81, 0.87\}$, then the CSM is locally stable. It should be noted that if one applies the Lyapunov function technique developed in Ref. 2 on Eq. (11) with $r = 2$, 3, then g_{s2} and g_{s3} need to be much larger than 11.81 to have the corresponding inequality in Eq. (16) satisfied. On the other hand, if one applies the monotonic dynamic technique derived in Ref. 14 on Eq. (11) with $r = 1$, then the choice of g_{s1} has to be much larger than 11.81. Our choice of combined methods on studying the stability of Eq. (11) with $r = 1$, 2, 3 gives the better estimate of the coupling strength g_s as compared to applying either one of the single method on Eq. (11).

The clusteringly synchronous dynamics with $g_s = 11.82$ is illustrated in Fig. 2. As one can see in Fig. 2, the first type of neurons achieve the relaxation oscillation synchronization, while the second and the third type of neurons achieve the fixed-point synchronization.

B. Coupled HR and FN neurons

In this example, we consider the network consisting of two Hindmarsh-Rose neurons ($i = 1$, 2) and two FitzHugh-Nagumo neurons ($i = 3$, 4). Here, the dynamics of Hindmarsh-Rose neurons is governed by Eqs. (13) with parameters $\boldsymbol{p}^{(1)} = (2.8, 20, 4.4, 0.001, 1, -1.11)$ and the dynamics of FitzHugh-Nagumo neurons is determined by

$$\begin{aligned} \dot{x} &= -x(x - \alpha)(x - 1) - cy + J \\ \dot{y} &= \nu(x - \beta y). \end{aligned} \tag{17}$$

Where parameters $\boldsymbol{p}^{(2)} = (\alpha, c, \beta, \nu, J) = (0.01, 2, 0.15, 0.008, 0.1)$. Moreover, the network topology under consideration is
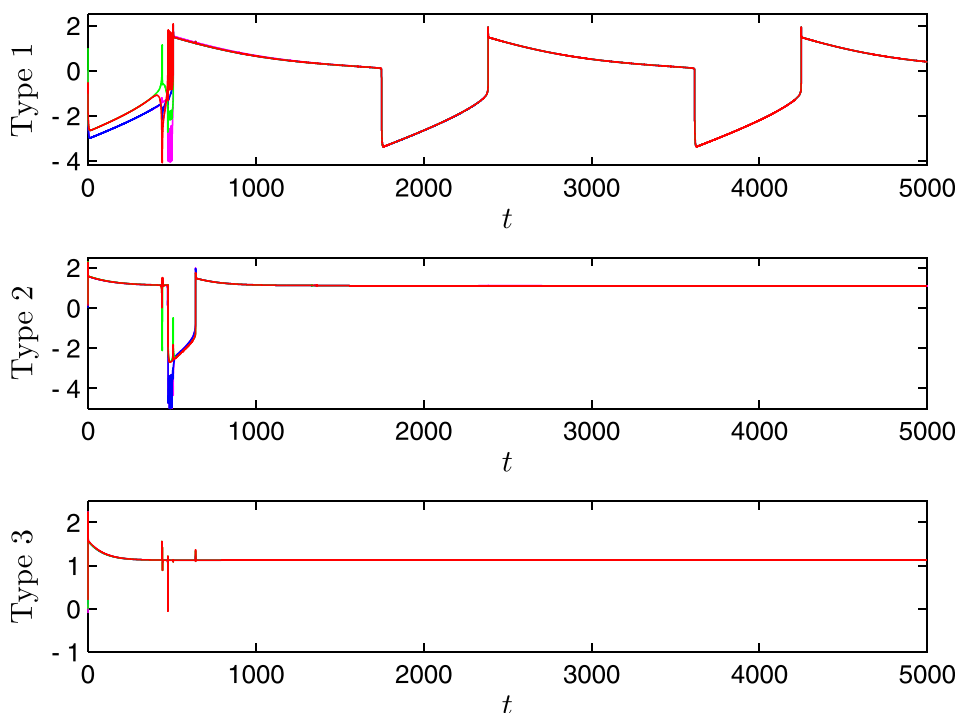

FIG. 2. Time series of $\boldsymbol{x}(t)$ with $g_s = 11.82$. The graph demonstrates the local stability of the CSM Ξ as in Eq. (15).
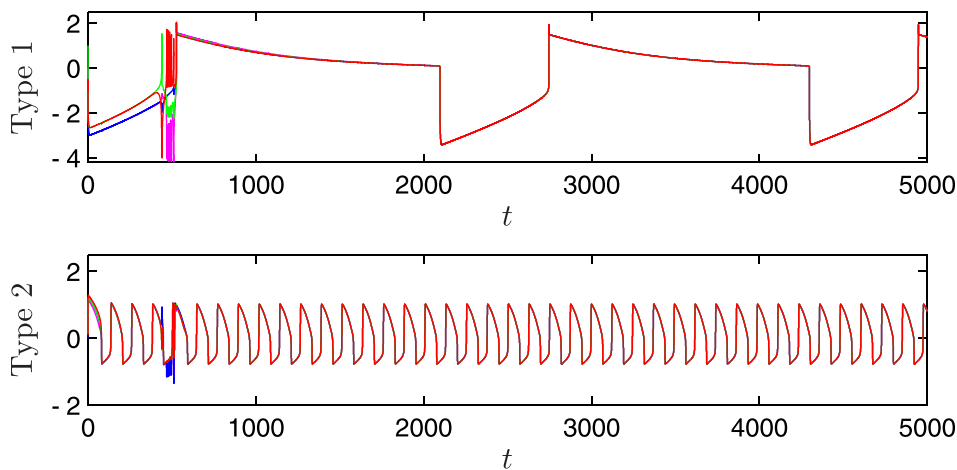
FIG. 3. Time series of $x(t)$ with $g_s = 0.62$. The graph demonstrates the local stability of the CSM Ξ.

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$= \left(\begin{array}{cccc|cccc} 0 & 34 & 0 & -14 & 0 & 1 & 0 & -1 \\ 22 & 0 & -2 & 0 & 0 & 1 & 0 & -1 \\ 16 & 6 & 0 & -2 & 0 & 1 & 0 & -1 \\ 34 & 0 & -14 & 0 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 2 & 0 & -1 \\ 1 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \end{array}\right).$$

Using the Lyapunov function technique,[2] we have that suppose there exist γ and g^* such that

$$\begin{aligned} \left(\gamma(3 - 19.36\gamma)x^2 - 1.2\gamma x - \frac{1}{4}\right) + \gamma g^*\Omega(x) > 0, \\ (3x^2 - 2.02x + 0.01) + g^*\Gamma(x) > 0, \end{aligned} \tag{18}$$

for $x < v$. Here, $\Omega(x) := 20\Gamma(x) + 4(v - x)\Gamma'(x)$. Then, the CSM is locally stable provided $g_s > g^*$. It is numerically estimated that $g^* \approx 0.6$. Hence, we can conclude that the CSM Ξ is locally stable if $g_s > g^* \approx 0.6$ (Fig. 3).

V. CONCLUSION

In the recent years, the study of the cooperative behavior of complex dynamical networks has been shifted toward the cluster synchronization. However, most of theoretical works are networks of synaptically coupled neurons from linearly (gap-junction) coupled neurons. In this paper, the cluster synchronization of nonidentically, chemically, and generally formulated coupled networks is investigated. The sufficient condition, resulting in verification of the stability of the origins of some decoupled linear systems, for the existence of stably synchronous clusters is obtained. Examples of nonidentical systems having different synchronized dynamical behaviors are provided to show the effectiveness of our theoretical prediction.

For the future work, it is worthwhile to study the combined effect of electrically and chemically synaptical couplings in the networks for the stability of the cluster synchronization manifold. Moreover, the extension of our work to the delayed or stochastic neural networks is an interesting and important issue.

APPENDIX: PROOF OF THEOREM 1

Proof. The condition that all rows of C_{rs}, $\forall r < s$, are the same implies that $E_r C_{rs} E_s^\dagger = 0$, $\forall r < s$. Upon using the condition that $k_{rs} = 0$, $\forall r \neq s$, we can rewrite Eq. (10) as the form

$$\dot{\zeta}_r = A_r(t)\zeta_r + \sum_{s < r} B_{rs}(t)\zeta_s, \tag{A1}$$

where $\zeta_r = (\xi_r, \eta_r)^T$, and

$$A_r(t) = \begin{pmatrix} A_{r,11}(t) & A_{r,12}(t) \\ A_{r,21}(t) & A_{r,22}(t) \end{pmatrix},$$

with

$$\begin{aligned} A_{r,11}(t) &= \left(\frac{\partial}{\partial V}f_1^{(r)}(u_c^{(r)}) - g_s k_{rr}\Gamma(V_c^{(r)})\right)I_{N_r} \\ &\quad + g_s(v - V_c^{(r)})\Gamma'(V_c^{(r)})(E_r C_{rr} E_r^\dagger), \end{aligned}$$

$$A_{r,12}(t) = \frac{\partial}{\partial w}f_1^{(r)}(u_c^{(r)}),$$

$$A_{r,21}(t) = \frac{\partial}{\partial V}f_2^{(r)}(u_c^{(r)}),$$

$$A_{r,22}(t) = \frac{\partial}{\partial w}f_2^{(r)}(u_c^{(r)}),$$

and

$$\boldsymbol{B}_{rs}(t) = \begin{pmatrix} g_s(v - V_c^{(r)})\Gamma'(V_c^{(s)})(\boldsymbol{E}_r \boldsymbol{C}_{rs} \boldsymbol{E}_s^\dagger) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}.$$

For $r = 1$, Eq. (A1) becomes $\dot{\zeta}_1 = \boldsymbol{A}_1(t)\zeta_1$. Then by assumption (11) for $r = 1$, we have $\zeta_1(t)$ converges exponentially to **0**. For $r = 2$, Eq. (A1) becomes

$$\dot{\zeta}_2 = \boldsymbol{A}_2(t)\zeta_2 + \boldsymbol{B}_{21}(t)\zeta_1. \tag{A2}$$

Let $\Phi_2(t)$ be the fundamental solution of $\dot{\zeta}_2 = \boldsymbol{A}_2(t)\zeta_2$, then by assumption (11) for $r = 2$, we have $\|\Phi_2(t)\Phi_2^{-1}(s)\| \le c_2 e^{-r_2(t-s)}$, $\forall t \ge s$ for some $c_2, r_2 > 0$. Moreover,

$$
\begin{aligned}
\|\zeta_2(t)\| &= \|\Phi_2(t)\Phi_2^{-1}(T)\zeta_2(T) \\
&\quad + \int_T^t \Phi_2(t)\Phi_2^{-1}(s)\boldsymbol{B}_{21}(s)\zeta_1(s)\,ds\| \\
&\le c_2 e^{-r_2(t-T)}\|\zeta_2(T)\| \\
&\quad + c_2 \int_T^t e^{-r_2(t-s)}ds \cdot \max_{t \ge T}\|\boldsymbol{B}_{21}(t)\zeta_1(t)\|.
\end{aligned}
$$

Since $\zeta_1(t)$ converges exponentially to **0**, we have, via the above inequality, that $\zeta_2(t)$ also converges exponentially to **0**.

Continuing above process for $r = 3, \ldots, m$, we can obtain the conclusion that $\zeta_r(t)$ converge exponentially to **0**, for $r = 1, \ldots, m$. Hence, the statement of Theorem 1 holds true and the proof of the theorem is completed.   □

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