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Evaluating corporate bonds with complicated liability structures and bond provisions ☆

Chuan-Ju Wang [a,1], Tian-Shyr Dai [b,2], Yuh-Dauh Lyuu [c,*,3]

[a] Department of Computer Science, University of Taipei, No. 1, Aiguo W. Rd., Taipei 100, Taiwan
[b] Department of Information and Finance Management, Institute of Information Management and Institute of Finance, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan
[c] Department of Finance and Department of Computer Science & Information Engineering, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei 106, Taiwan

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ABSTRACT

This paper presents a general and numerically accurate lattice methodology to price risky corporate bonds. It can handle complex default boundaries, discrete payments, various asset sales assumptions, and early redemption provisions for which closed-form solutions are unavailable. Furthermore, it can price a portfolio of bonds that accounts for their complex interaction, whereas traditional approaches can only price each bond individually or a small portfolio of highly simplistic bonds. Because of the generality and accuracy of our method, it is used to investigate how credit spreads are influenced by the bond provisions and the change in a firm's liability structure due to bond repayments.

1. Introduction

A risk-free bond can be priced by simply summing all the discounted future cash flows, independently of other outstanding bonds of the same issuer. A risky bond in sharp contrast must be priced simultaneously with other outstanding bonds of the same issuer because the issuance and *each* repayment of a bond changes the financial status of the issuer and thus the likelihood of default of all the bonds, even more senior ones. This point is most serious for an issuer with multiple bonds outstanding and a complex liability structure. This complex interaction among the bonds due to their payment schedules and provisions makes pricing them beyond the reach of analytic approaches (see Lando, 2004).

Given a firm's liability structure, a credit model is needed to price risky bonds (Chiarella, Fanelli, & Musti, 2011; Onorato & Altman, 2005; Saunders, Xiouros, & Zenios, 2007; Westgaard & Van der Wijst, 2001). One of such models, the structural model, specifies the evolution of the firm's asset value and the conditions leading to default (see Merton, 1974), from which the change in a firm's liability structure follows naturally. The bond repayment financed by selling the firm's asset, for example, is modeled by a downward move in the firm's asset value. So structural models make explicit the connection between default and the firm's assets and liabilities. This paper will focus on structural models.

Merton (1974) assumes the firm's asset value follows a lognormal diffusion process and default can only occur at the single bond's maturity date when the firm's asset value cannot meet its payment obligations. Therefore, equities can be viewed as call options on the firm's asset and can be priced by the Black–Scholes formula (see Black & Scholes, 1973). Black and Cox (1976) develop the first-passage model, which assumes the firm issues only one bond and it defaults once the asset value hits an exogenous default boundary. The single-bond case is clearly too restricted for practical applications. Geske (1977) is the first to price a risky bond in the presence of other outstanding bonds. He considers a portfolio consisting of a senior bond with a maturity date of T_1 and a subordinated one with a later maturity date of T_2. Then he applies the compound-option framework to price both bonds. In summary, the analytical methods can only price each bond individually or a very small portfolio of highly simplistic bonds (see Ericsson & Reneby, 1998; Glasserman & Nouri, 2012). Generalizing them to more complicated liability structures remains elusive.

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* Corresponding author.
E-mail addresses: cjwang@utaipei.edu.tw (C.-J. Wang), d88006@csie.ntu.edu.tw (T.-S. Dai), lyuu@csie.ntu.edu.tw (Y.-D. Lyuu).
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Bond provisions such as restrictions on asset sales, exogenous default boundaries, seniorities of bonds, and early redemption (like put provisions) affect risky bond prices profoundly. We now go over each of them briefly. The value of a risky bond depends strongly on the assumptions regarding asset sales (see Lando, 2004). To protect the bond holders, bond provisions may prohibit equity holders from selling the firm's asset to finance bond repayments or dividend payouts. This no-asset-sales assumption is often needed for closed-form solutions (see Leland, 1994). But allowing asset sales is more common in the real world. To loosen the restriction on asset sales while keeping the problem analytically solvable, some papers adopt the proportional-asset-sales assumption, which allows the firm to sell a proportion of its asset (see Kim, Ramaswamy, & Sundaresan, 1993; Leland, 1994; Hilberink & Rogers, 2002). Besides the two aforementioned assumptions, Merton (1974) and Brennan and Schwartz (1978) assume the payout can be fully financed by selling the firm's asset. We call this third assumption the total-asset-sales assumption. This assumption significantly increases the difficulty to price the bonds, analytically or otherwise. This is because a *fixed* amount of the firm's asset is sold to finance the repayments, which is essentially the well-known problem faced by option pricing with fixed dividends (see Dai, 2009).

We now move onto exogenous default boundaries. The positive-net-worth covenant forces the firm into bankruptcy if its asset value hits an exogenous default boundary that depends on the firm's liability structure (see Brennan & Schwartz, 1978; Kim et al., 1993; Longstaff & Schwartz, 1995; Nielsen, Saá-Requejo, & Santa-Clara, 2001; Briys & De Varenne, 1997). Note that a complex liability structure entails a complex exogenous default boundary. We follow Leland (1994) in calling a bond with an exogenous default boundary a protected bond. The default boundary can also be determined endogenously based on assumptions on asset sales. For example, under the no-asset-sales and proportional-asset-sales assumptions, the firm defaults if the equity holders fail to raise enough equity capital to meet the bond payments (see Leland, 1994), whereas under the proportional-asset-sales and total-asset-sales assumptions, the firm defaults when the firm's asset is insufficient to cover the payments. Note that the default boundary for a protected bond is shaped by both the exogenous and the endogenous default boundaries. In contrast, the default boundary of a bond without protection from the positive-net-worth covenant is simply the endogenous default boundary. We follow Leland (1994) in calling this bond an unprotected bond.

Seniority refers to payment priority in the event of bankruptcy. When the issuer goes bankrupt, senior bonds are repaid before subordinated ones. But a subordinated bond may still affect the risk of the senior ones. This is because when a firm is allowed to sell its asset to finance the bond repayments, the repayment of a subordinated bond before the maturity of the senior ones increases the risk of the latter as the asset sale changes the financial status of the firm.

Finally, we discuss early redemptions. The putable provision provides some protection for the bond holders against the increase in interest rate, which reduces the bond value. Our paper will also show that it can provide some protection against the issuer's credit risk. This is because bond holders can exercise the putable right before the firm's financial status is weakened due to scheduled bond repayments.

In summary, real-world bond provisions and the complex interaction among bonds make pricing risky bonds infeasible, in most cases, for analytical approaches and challenging for numerical ones. To rectify the situation, this paper develops a general lattice methodology for pricing corporate bonds with complicated liability structures and bond provisions under the structural model. A lattice is a popular numerical method. It divides a certain time interval into n time steps and the pricing results converge to the theoretical price as $n \to \infty$ (see Duffie, 1996). However, some provisions such as exogenous default boundaries will cause naive implementations to experience price oscillations as Fig. 1 shows. To eliminate the oscillations, we incorporate the techniques of Dai and Lyuu (2010) to makes certain nodes or price levels on the lattice align with the exogenous default boundaries. In addition, the trinomial structure of Dai (2009) is used to handle the discontinuities in the firm's asset value resulting from asset sales. Backward induction then handles bond provisions such as seniority and embedded options.

With our proposed lattice method in place, the paper explores how credit spreads are influenced by the bond provisions and change in the firm's liability structure due to each bond repayments. Numerical results reveal that they greatly affect bond prices, sometimes in unexpected ways. Complex scenarios such as this are hard to analyze by the traditional approaches, but they pose no difficulties for our lattice. Finally, our methodology is flexible enough to make it applicable to other complex option-related problems such as real options (see Ho & Yi, 2004; Zmeškal, 2010).

Our paper is organized as follows. The model, lattice constructions, and the oscillation problem are introduced in Section 2. Section 3 describes how our lattice is constructed to cope with complicated liability structures and various bond provisions. Section 4 details how bond provisions are handled in backward induction. Section 5 analyzes the price behaviors of risky bonds for complicated liability structures and various bond provisions. Section 6 concludes.

2. Basic terms and preliminaries

2.1. The dynamics of the firm's asset value

Denote the firm's asset value at time t as V_t, whose dynamics follows the following process (see Merton, 1974),

$$dV_t = (rV_t - P)\,dt + \sigma V_t\,dz. \tag{1}$$

Above, r is the risk-free rate, P denotes the firm's payout financed by selling the firm's asset per annum, σ denotes the volatility, and dz is a standard Brownian motion (see Black & Scholes, 1973; Osborne, 1959). P can depend on V_t and t.

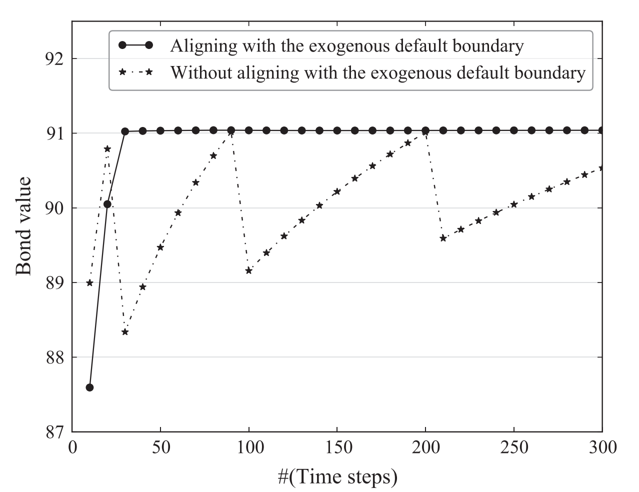

Fig. 1. The price oscillation phenomenon. The firm's asset value is assumed to follow the lognormal diffusion process. The firm's initial asset value is $100, the risk-free interest rate is 1%, and the volatility of the asset value is 25%. The firm issues a zero-coupon bond with one-year maturity and face value $95. The exogenous default boundary is set to $90. The prices oscillate significantly if the lattice does not align with the exogenous boundary.

The liability structure of the bond issuer ("the firm") should be considered when evaluating a risky bond B. This is because a new issuance or repayments of other outstanding bonds will affect the firm's capacity to repay B. But how a bond repayment influences the firm's asset and equity values depends on the assumptions regarding asset sales. Let us ignore the tax shelter benefit for the time being. Suppose the payout is made continuously and the payout per annum is C. Under the no-asset-sales assumption, P is set to 0; so the bond payout C is financed by selling equities and will not affect the firm's asset value. Suppose the firm is allowed to finance the payout by selling the firm's asset. Under the total-asset-sales assumption, $P \equiv C$. Under the proportional-asset-sales assumption, a proportion D of the asset value is sold to finance the payout, so $P \equiv DV_t$. Under this assumption, if $P > C$, the extra payment $P - C$ goes to the equity holders as dividends (see Fan & Sundaresan, 2000); otherwise, the equity holders will either finance the shortfall by selling equities or let the firm default. D may depend on the payout amount such as $D \equiv C/V_0$ (Leland, 1994). Suppose the payout is made discretely at time t. Hereafter, t^- and t^+ denote the times immediately before and immediately after the payout, respectively. Under the total-asset-sales assumption, with the payout C', we have $V_{t^+} = V_{t^-} - C'$, and under the proportional-asset-sales assumption, with the payout $D'V_{t^-}$, we have $V_{t^+} = V_{t^-} - D'V_{t^-}$. The firm defaults when $V_{t^+} \leqslant 0$. We remark that when the firm issues a bond with a face value of N at time $t, V_{t^+} = V_{t^-} + N$. If the tax shelter benefit is considered, then part of the interest payment is covered by the tax benefit, and the equity holders or the firm only need to pay the remaining amount.

2.2. Lattice constructions and the oscillation problem

2.2.1. Binomial lattice

A lattice is a popular numerical method for discretizing a continuous-time stochastic process. It partitions the time span from time 0 to time T into n equal time steps and specifies the value of the stochastic process at each time step. The length of one time step Δt equals T/n. Each node on a lattice can branch to ℓ nodes at the next time step to form an ℓ-nomial lattice. For example, Fig. 2(a) illustrates a 2-time-step binomial lattice: At each time step, the firm's asset value V can either make an upward move to become Vu with probability P_u or a downward move to become Vd with probability $P_d \equiv 1 - P_u$.

The firm's asset value is assumed to follow the lognormal diffusion process between two repayment dates. Define the V-log-price of the firm's asset value V' as $\ln(V'/V)$ and the log-distance between the firm's asset values V and V' as $|\ln(V) - \ln(V')|$. Under the lognormal diffusion process, the mean (denoted as μ) and variance (denoted as $\mathtt{Var}$) of the V_t-log-price of $V_{t+\Delta t}$ are $(r - \sigma^2/2)\ \Delta t$ and $\sigma^2\Delta t$, respectively. To make the lattice converge to the process for the firm's asset value, the branching probabilities (P_u and P_d) and the multiplicative factors (u and d) are chosen to asymptotically match μ and $\mathtt{Var}$. For example, the CRR lattice of Cox, Ross, and Rubinstein (1979) in Fig. 2(a) adopts the following solution: $u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}, P_u = (e^{r\Delta t} - d)/(u - d), P_d = (e^{r\Delta t} - u)/(d - u)$. The log-distance between any two vertically adjacent nodes on this lattice (e.g., nodes X and Y in Fig. 2(a) is $2\sigma\sqrt{\Delta t}$).

2.2.2. Trinomial structure

Fig. 2(b) illustrates a trinomial structure in which the nodes at time $t + \Delta t'$ are laid out like the CRR lattice with Δt as the length of a time step. Node X at time t is connected to 3 successor nodes (nodes A, B, and C in the figure) at time $t + \Delta t'$ as long as $\Delta t \leqslant \Delta t' < 2\Delta t$. Denote the firm's asset value of node Z as $v(Z)$. The trinomial branches should match the mean μ and variance $\mathtt{Var}$ of the $v(X)$-log-price of $V_{t+\Delta t'}$, where, we recall, $\mu \equiv (r - \sigma^2/2)\Delta t'$ and $\mathrm{Var} \equiv \sigma^2\Delta t'$, with the branching probabilities p_u, p_m, and p_d between 0 and 1.

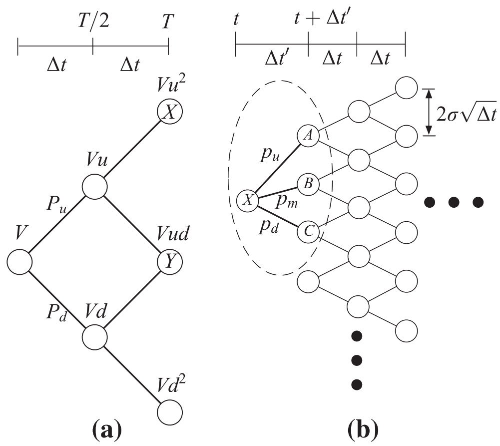

Fig. 2. Binomial lattice and trinomial structure. Fig. 2(a) illustrates a 2-time-step binomial lattice. Fig. 2(b) illustrates the unique trinomial structure for an arbitrary node X at time t to connect to 3 successor nodes A, B, and C at time $t + \Delta t'$.

Recall that the log-distance between the $v(X)$-log-prices of any two adjacent nodes at the same time step of the CRR lattice is $2\sigma\sqrt{\Delta t}$ (see Fig. 2 (b)). Therefore, at time $t + \Delta t'$, there must exist a unique node B whose $v(X)$-log-price $\hat{\mu}$ lies in the interval $[\mu - \sigma\sqrt{\Delta t}, \mu + \sigma\sqrt{\Delta t})$. We select node B and its two adjacent nodes A and C to construct a trinomial structure from node X. The branching probabilities from node X (i.e., p_u, p_m, p_d) can be obtained by asymptotically matching μ and Var (see Dai & Lyuu, 2010).

2.2.3. Price oscillation and nonlinearity error

Although the prices generated by the CRR lattice converge to the theoretical value of a contingent claim as $n \to \infty$ (Duffie, 1996), the prices may oscillate significantly. According to Figlewski and Gao (1999), this is due to the error introduced by the nonlinearity of the value function of the contingent claim. The nonlinearity error can be much reduced by making a node or a price level of the lattice coincide with the critical locations where the value function of the contingent claim is highly nonlinear (see Dai & Lyuu, 2010). For the structural model, critical locations occur along the exogenous default boundary and at the time points when bond or dividend payouts occur. The next section will show that, by incorporating the above lattice constructions, our lattice can make a node, a time step, or a price level of the lattice coincide with the critical locations. The result is drastically suppressed nonlinearity error (review Fig. 1).

3. Lattice construction

This section describes how to combine the binomial and the trinomial structures to construct a lattice that can cope with complicated liability structures and such bond provisions as various assumptions on asset sales and positive net-worth covenants. Other bond provisions, like seniority and early redemption, will be addressed in Section 4. The example lattice given in Section 3.1 focuses on the total-asset-sales assumption, the most difficult assumption on asset sales to handle. The extensions to the proportional-asset-sales assumption, the no-asset-sales assumption, and time-varying default boundaries are straightforward and will be described in Section 3.2.

3.1. Lattice construction

This subsection uses a generic example to illustrate how to build a lattice for the total-asset-sales assumption and with an exogenous default boundary. Consider two zero-coupon bonds with face values F_1 and F_2 and maturities $T/2$ and T, respectively. We will refer to Fig. 3 throughout this subsection. Because bond repayments are fully financed by selling the firm's asset, the firm is bankrupt once its asset value falls to the exogenous default boundary. This subsection assumes the exogenous default boundary is a constant proportion κ of the sum of the face values of all outstanding bonds (see Kim et al., 1993; Longstaff & Schwartz, 1995; Nielsen et al., 2001). Thus it is $\kappa(F_1+F_2)$ (the thick black line) for the time interval $[0,T/2]$ and κF_2 (the thick gray line) for the time interval $(T/2,T]$. To suppress oscillations, the lattice aligns two price levels with these two default boundaries. Finally, since the firm defaults when its asset value hits F_2 at time T, there should be a node (node F) to coincide with the face value F_2 at time T to suppress oscillations (but this alignment is not that critical numerically).

The lattice is composed of two parts: one covers the interval $[0,T/2]$ and the other covers $(T/2,T]$. We focus on the former part first. The firm's initial asset value at time 0 is associated with node X. The lattice emanating from node X is required to have a price level that coincides with the exogenous default boundary $\kappa(F_1+F_2)$, which can be accomplished via the technique in Dai and Lyuu (2010). The trinomial branches from node X can be constructed by the procedure in Section 2.2.2. The exogenous default boundary passes through nodes D and E, and the firm defaults once the firm's asset value hits the exogenous default boundary.

At time $T/2$, the vertical, downward jumps from the white nodes to gray ones by the amount of F_1 reflect the fact that the firm sells its asset to repay the bond obligation. The evolution of the firm's asset value for the second part of the lattice begins from the gray nodes. The gray nodes will be connected to a truncated CRR lattice beginning at time $T/2+\Delta t''$ by trinomial structures. To suppress the nonlinearity error, this part of the lattice has a price level that coincides with the exogenous default boundary κF_2 and a node that coincides with the bond obligation F_2 (node F) at time T. The truncated CRR lattice emanates from nodes J, K, L, M, N, and O and ends at time T. The firm defaults once its asset value hits the exogenous default boundary or when the firm cannot meet the bond obligation F_2 at time T.

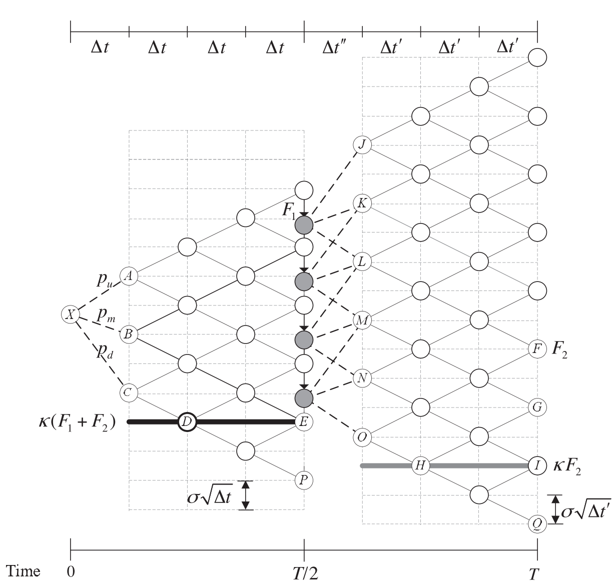

Fig. 3. Firm's asset value with exogenous default boundaries under the total-asset-sales assumption. At time $T/2$, the nodes above the exogenous default boundary jump downward to gray nodes by the amount of F_1 as a result of the face value F_1 payout. To reduce the nonlinearity error, the lattice has two price levels that coincide with the two exogenous default boundaries (the thick black and gray lines) and a node (node F) that coincides with the bond obligation F_2 at time T. Nodes A, B, and C are selected for constructing the trinomial structure. The branching probabilities from node X are p_u, p_m, and p_d.

The effects of tax benefits and bankruptcy costs can be easily incorporated into our lattice. Suppose a coupon C is due at time t. The firm's asset value jumps downward by the amount of the "after-tax coupon" $C-\tau C$ at time t, where τ denotes the corporate tax rate and τC denotes the tax benefit of repaying the coupon. If the firm defaults, the bankruptcy cost is subtracted from the firm's asset value and the residual value is paid to the bond holders. The lattice also allows the firm's asset value to be checked against the exogenous default boundary only at certain time points. For instance, if the firm's asset value is only checked at the bond's repayment dates $T/2$ and T, then the firm defaults once its asset value hits nodes E, P, G, I, or Q. This reflects the reality that the firm's asset value is often only monitored regularly at the release of financial reports.

3.2. Extensions to other assumptions on asset sales and exogenous default boundaries

Our lattice can be easily extended to deal with the proportional-asset-sales assumption, the no-asset-sales assumption, and time-varying default boundaries. Fig. 4(a) illustrates how to deal with the two above-mentioned assumptions regarding asset sales. The firm's asset value at node X at time step 0 will move to either node A or B at time step 1. P_1 and P_2 denote the payouts financed by selling the firm's asset at nodes A and B, respectively. Under the proportional-asset-sales assumption, the firm sells a fixed proportion D of its asset value (i.e., $P=DV_t$ in Eq. (1)) so $P_1=DVu\Delta t$ and $P_2=DVd\Delta t$. On the other hand, under the no-asset-sales assumption, $P_1=P_2=0$. In both cases, the nodes associated with the firm's asset values after the asset sales (i.e., nodes a and b) will be connected to the nodes at the next time step via the trinomial structures described in Section 2.2.2.

Fig. 4(b) depicts the time-varying exogenous default boundary as a curve. The lattice starts by placing gray nodes on the curve to reduce the nonlinearity error. All the other nodes are then laid out from the gray nodes upward, and the log-distance between any two vertically adjacent nodes remains $2\sigma\sqrt{\Delta t}$. This setting helps us construct trinomial branches from any node at time step i to the three successor nodes at time step $i+1$ via the procedure in Section 2.2.2. Start from node X for the firm's initial asset value at time 0. Its successor nodes will be selected from the nodes at time step 1, the successor nodes of these 3 nodes will be selected from the nodes at time step 2, and so on. Note that the firm defaults once its asset value reaches the gray nodes on the exogenous default boundary. The gray nodes have no need for successor nodes.

4. Pricing with backward induction

The equity value ($\mathbb{E}$), bond value ($\mathbb{B}$), tax benefit ($\mathbb{TB}$), and bankruptcy cost ($\mathbb{BC}$) can be viewed as contingent claims on the firm's asset value. They can be computed on the lattice by backward induction. The levered firm value, or $\mathbb{E}+\mathbb{D}$, equals $V+\mathbb{TB}-\mathbb{BC}$ (see Leland, 1994). The default events and such bond provisions as seniority and early redemptions can all be handled during backward induction. This section focuses on a generic case where the firm issues two bonds: $\mathbb{B}_1$ matures at time T_1 with face value F_1 and $\mathbb{B}_2$ matures at time T_2 with F_2, where $T_1<T_2$. Bond $\mathbb{B}_i$ pays at a rate of C_i dollars per annum continuously, $i=1,2$. Thus the coupon payout for $\mathbb{B}_i$ over a time step with duration Δt is $C_i\Delta t$. Backward induction at time t is divided into four cases: (1) $t=T_2$, (2) $T_1<t<T_2$, (3) $t=T_1$, and (4) $0\leqslant t<T_1$. If the firm

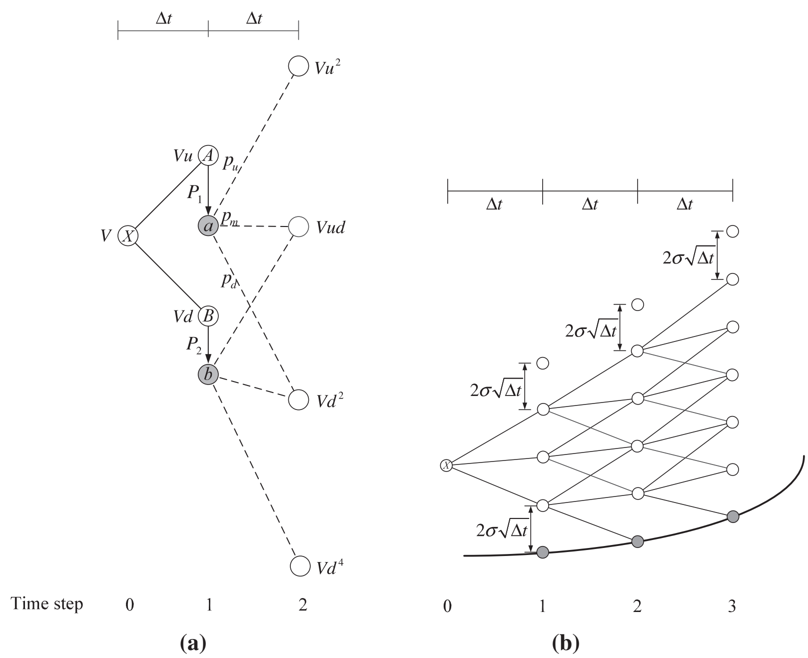

Fig. 4. Extensions to other assumptions on asset sales and exogenous default boundaries. In Fig. 4(a), the white nodes and the gray nodes at the first time step denote the firm's asset values before and after the asset sales, respectively. The asset value for each white node is listed next to the node. P_1 and P_2 denote the payouts financed by selling the firm's asset at nodes *A* and *B*, respectively. The probabilities of the branches from node *a* are p_u, p_m, and p_d. Fig. 4(b) has a time-varying exogenous default boundary marked by the curve. The firm defaults when its asset value reaches the gray nodes.

issues more than two bonds, more cases need to be added. Let $\mathbb{E}(t,V), \mathbb{B}_1(t,V)$, and $\mathbb{B}_2(t,V)$ denote the values of equity, bond $\mathbb{B}_1$, and bond $\mathbb{B}_2$ at time *t*, respectively, given asset value *V*. $\mathtt{PV}(t,\mathbb{B}_i)$ represents the discounted present value at time *t* of all future repayments of bond $\mathbb{B}_i$ (see Black & Cox, 1976). Let $\mathbb{B}_i \prec \mathbb{B}_j$ mean $\mathbb{B}_j$ is more senior than $\mathbb{B}_i$. Finally, α is the bankruptcy cost as a percentage of the asset value, and τ, as before, is the tax rate.

If there is an exogenous default boundary, then the firm defaults the moment the firm's asset value hits it. Default can also be triggered endogenously. Under the no-asset-sales and proportional-asset-sales assumptions, the firm defaults if the equity holders fail to raise enough equity capital to meet the firm's bond obligations, whereas under the proportional-asset-sales and total-asset-sales assumptions, the firm defaults when the firm's asset value is insufficient to cover the repayments. Clearly, the default boundary is shaped by both the exogenous default boundary and the endogenous one.

Backward induction is used to obtain the value of a contingent claim at time *t* by its values at the next time step, at time $t+\Delta t$. The firm's asset value at each time step should be adjusted if the firm sells its asset to finance the bond payout at that time step. Recall that t^- and t^+ denote the times immediately before and immediately after the payout, respectively. Let V^- and V^+ denote the firm's asset values immediately before and immediately after the payout, respectively. The value of any contingent claim on the asset value V^+ at time t^+, denoted as $f(t^+,V^+)$, can be expressed as the discounted expected value of that contingent claim at time t^+, where applicable. Take node *a* at time *t* in Fig. 4(a) for example. We have

$$f(t^+, Vu-P_1) \equiv e^{-r\Delta t}\Big[p_u f((t+\Delta t)^-, Vu^2) + p_m f((t+\Delta t)^-, Vud) + p_d f((t+\Delta t)^-, Vd^2)\Big].$$

Case 1: $t = T_2$

At $\mathbb{B}_2$'s maturity date T_2, default occurs when the firm's asset value *V* cannot meet $\mathbb{B}_2$'s face value F_2 plus the after-tax coupon over one time step: $(1-\tau)C_2\Delta t$. Thus,

$$\mathbb{E}(T_2,V) = \begin{cases} V - F_2 - (1-\tau)C_2\Delta t, & \text{if } V \geqslant F_2 + (1-\tau)C_2\Delta t,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\mathbb{B}_2(T_2,V) = \begin{cases} F_2 + C_2\Delta t, & \text{if } V \geqslant F_2 + (1-\tau)C_2\Delta t,\\ (1-\alpha)V, & \text{otherwise.}\end{cases}$$

Recall that $\tau C_2\Delta t$ is the tax benefit from repaying the coupon. TB and BC are equal to 0 and αV, respectively, if the firm defaults at time T_2; otherwise, TB and BC are equal to $\tau C_2\Delta t$ and 0, respectively.

Case 2: $T_1 < t < T_2$

The value of a contingent claim at t^- depends on its values at t^+. Thus $\mathbb{E}(t^-,V^-)$ can be calculated as follows:

$$\mathbb{E}(t^-,V^-) = \begin{cases} \mathbb{E}(t^+,V^+) - S, & \text{if default does not occur at time } t^-,\\ 0, & \text{if default occurs at time } t^-,\end{cases} \tag{2}$$

where *S* denotes the amount of equities sold to finance the payout. *S* and the evolution of the firm's asset value depend on the assumptions on asset sales. Under the no-asset-sales assumption, the equity holders sell additional equities to finance the after-tax coupon; so the firm's asset value does not change. Under the proportional-asset-sales assumption, the firm is allowed to sell a predetermined proportion *D* of the firm's asset to finance the after-tax coupon. If the said proportion of the firm's asset value is insufficient to meet the payout, the equity holders will try to finance the shortfall by selling additional equities. Under the total-asset-sales assumption, the firm is allowed to sell its asset to finance the payout to the extent the asset value allows it; therefore, the firm's asset value jumps downward by the amount of the payout. In summary,

$$\begin{array}{lll} \text{No-asset-sales}: & S=(1-\tau)C_2\Delta t, & V^+=V^-,\\ \text{Propotional-asset-sales}: & S=(1-\tau)C_2\Delta t - DV^-\Delta t, & V^+=V^- - DV^-\Delta t,\\ \text{Total-asset-sales}: & S=0, & V^+=V^- - (1-\tau)C_2\Delta t.\end{array}$$

Note that $(1-\tau)C_2\Delta t = S + (V^- - V^+)$; in other words, the after-tax coupon is financed by the sum of equities sold (S) and the firm's asset value sold. Under the proportional-asset-sales assumption, a negative S says the payout from selling the firm's asset exceeds the coupon payment, and the extra payout goes to the equity holders as cash dividends.

$\mathtt{B}_2(t^-, V^-)$ can be calculated by the following formulas:

$$\mathtt{B}_2(t^-,V^-)=\begin{cases}\mathtt{B}_2(t^+,V^+)+C_2\Delta t, & \text{if default does not occur at time } t^-,\\ (1-\alpha)V^-, & \text{if default occurs at time } t^-.\end{cases}$$

$\mathtt{TB}$ and $\mathtt{BC}$ are equal to 0 and αV^-, respectively, if the firm defaults; otherwise, $\mathtt{TB}$ and $\mathtt{BC}$ are both calculated by backward induction.

Case 3: $t = T_1$

The firm is required to repay $\mathtt{B}_1$'s face value and coupon at time T_1. The equity value $E(t^-, V^-)$ can be calculated by Eq. (2) as in case 2, but S and the evolution of the firm's asset value are different:

$$\begin{aligned}&\text{No-asset-sales}: \quad S=(1-\tau)(C_1+C_2)\Delta t+F_1, \quad V^+=V^-,\\ &\text{Propotional-asset-sales}: \quad S=(1-\tau)(C_1+C_2)\Delta t+F_1-DV^-\Delta t,\\ &V^+=V^--DV^-\Delta t,\\ &\text{Total-asset-sales}: \quad S=0, \quad V^+=V^--(1-\tau)(C_1+C_2)\Delta t-F_1.\end{aligned}$$

Note that $(1-\tau)(C_1+C_2)\Delta t + F_1 = S + (V^- - V^+)$; in other words, the after-tax coupons and the face value of $\mathtt{B}_1$ are financed by the sum of equities sold (S) and the firm's asset value sold.

The relative seniorities of $\mathtt{B}_1$ and $\mathtt{B}_2$ determine the payouts to their bond holders when default occurs. Thus we have

$$\mathtt{B}_2(t^-,V^-)=\begin{cases}\mathtt{B}_2(t^+,V^+)+C_2\Delta t,\\ \text{if default does not occur at time } t^-,\\ \max(\min((1-\alpha)V^--(F_1+C_1\Delta t),\\ \mathtt{PV}(t^-,\mathtt{B}_2)),0),\\ \text{if default occurs at time } t^-\text{and } \mathtt{B}_2\prec\mathtt{B}_1,\\ \min((1-\alpha)V^-,\mathtt{PV}(t^-,\mathtt{B}_2)),\\ \text{if default occurs at time } t^- \text{ and } \mathtt{B}_1\prec\mathtt{B}_2,\end{cases} \tag{3}$$

and

$$\mathtt{B}_1(t^-,V^-)=\begin{cases}F_1+C_1\Delta t,\\ \text{if default does not occur at time } t^-,\\ \min((1-\alpha)V^-,F_1+C_1\Delta t),\\ \text{if default occurs at time } t^-\text{and } \mathtt{B}_2\prec\mathtt{B}_1,\\ \max(\min((1-\alpha)V^--\mathtt{PV}(t^-,\mathtt{B}_2),\\ \mathtt{PV}(t^-,\mathtt{B}_1)),0),\\ \text{if default occurs at time } t^- \text{ and } \mathtt{B}_1\prec\mathtt{B}_2.\end{cases}$$

$\mathtt{TB}$ and $\mathtt{BC}$ are equal to 0 and αV^-, respectively, if the firm defaults; otherwise, $\mathtt{TB}$ and $\mathtt{BC}$ are both calculated by backward induction.

Case 4: $0 \leqslant t < T_1$

$\mathtt{E}(t^-, V^-)$ can be calculated by Eq. (2) as in case 2, but S and the evolution of the firm's asset value are different:

$$\begin{aligned}&\text{No-asset-sales}: \quad S=(1-\tau)(C_1+C_2)\Delta t, \quad V^+=V^-,\\ &\text{Propotional-asset-sales}: \quad S=(1-\tau)(C_1+C_2)\Delta t-DV^-\Delta t,\\ &V^+=V^--DV^-\Delta t,\\ &\text{Total-asset-sales}: \quad S=0, \quad V^+=V^--(1-\tau)(C_1+C_2)\Delta t.\end{aligned}$$

Note that $(1-\tau)(C_1+C_2)\Delta t = S + (V^- - V^+)$; in other words, the after-tax coupons are financed by the sum of equities sold (S) and the firm's asset value sold.

$\mathtt{B}_2(t^-, V^-)$ and $\mathtt{B}_1(t^-, V^-)$ can be priced by the following formulas:

$$\mathtt{B}_2(t^-,V^-)=\begin{cases}\mathtt{B}_2(t^+,V^+)+C_2\Delta t,\\ \text{if default does not occur at time } t^-,\\ \max(\min((1-\alpha)V^--\mathtt{PV}(t^-,\mathtt{B}_1),\\ \mathtt{PV}(t^-,\mathtt{B}_2)),0),\\ \text{if default occurs at time } t^- \text{ and } \mathtt{B}_2\prec\mathtt{B}_1,\\ \min((1-\alpha)V^-,\mathtt{PV}(t^-,\mathtt{B}_2)),\\ \text{if default occurs at time } t^- \text{ and } \mathtt{B}_1\prec\mathtt{B}_2,\end{cases}$$

and

$$\mathtt{B}_1(t^-,V^-)=\begin{cases}\mathtt{B}_1(t^+,V^+)+C_1\Delta t,\\ \text{if default does not occur at time } t^-,\\ \min((1-\alpha)V^-,\mathtt{PV}(t^-,\mathtt{B}_1)),\\ \text{if default occurs at time } t^- \text{ and } \mathtt{B}_2\prec\mathtt{B}_1,\\ \max(\min((1-\alpha)V^--\mathtt{PV}(t^-,\mathtt{B}_2),\\ \mathtt{PV}(t^-,\mathtt{B}_1)),0),\\ \text{if default occurs at time } t^- \text{ and } \mathtt{B}_1\prec\mathtt{B}_2.\end{cases}$$

$\mathtt{TB}$ and $\mathtt{BC}$ are equal to 0 and αV^-, respectively, if the firm defaults; otherwise, $\mathtt{TB}$ and $\mathtt{BC}$ are both calculated by backward induction.

Early redemption can be easily handled by backward induction. For example, a putable bond's holder will sell the bond back to the issuer at time t if it is more beneficial to do so than keeping it. Similar arguments can be applied to callable bonds. The case that the coupon is paid discretely can be implemented without difficulty, too.

5. Numerical results

This section examines how the prices of risky bonds are influenced by bond provisions and changes in the firm's liability structure. First, we confirm the robustness and generality of our lattice by showing it can accommodate many popular structural models. We then show how bond values change under various bond provisions. We also show that our numerical method can be used to analyze the optimal capital structures under realistic bond-provision settings. Then we examine how the repayment of one outstanding bond affects the value of another bond under various bond provisions. In summary, our lattice will be used to analyze (1) the restriction on asset sales to finance the repayments, (2) the positive net-worth covenant, (3) the embedded put option, (4) the seniority, and (5) the combined effect of the above provisions, none of which admits analytical formulas.

5.1. Comparison with Merton (1974), Black and Cox (1976), Geske (1977), and Leland (1994)

Our lattice can accurately price risky bonds under many popular structural models. Table 1 compares the bond values by our lattice and the analytical formulas for 4 popular structural models: Merton (1974), Black and Cox (1976), Geske (1977), and Leland (1994). In Merton (1974), the firm issues a zero-coupon bond with a one-year maturity and a face value of \$3000. The bond can be priced by the Black and Scholes (1973) formula. Black and Cox (1976) extend Merton's model by assuming that the firm can default prior to the bond's maturity date of one year if its asset value hits this default boundary $3000 \times e^{-0.04(1-t)}$. The bond can then be priced by the closed-form formula for barrier options. Geske (1977) assumes the firm issues two zero-coupon bonds: (1) a shorter-term, senior bond and (2) a longer-term, subordinated bond. Under the no-asset-sales assumption, the compound option pricing formula can price both. The Geske (1977) column assumes that bond $\mathtt{B}_1$ with a face value of \$500 matures at year 0.5, bond $\mathtt{B}_2$ with a face value of \$2500 matures at year 1, and $\mathtt{B}_2 \prec \mathtt{B}_1$. The de-

Table 1
Accuracy and generality of our lattice. The firm's initial asset value is $5000, the risk-free interest rate r is 2%. The term σ denotes the volatility of the firm's asset value. The values under "Lattice" and "Formula" denote the bond values generated by our lattice and by the relevant analytical formula, respectively. The length of one time step Δt for the lattices are 0.001 (year). The values within the parentheses denote relative pricing errors. In the Merton (1974), Black and Cox (1976), and Geske (1977) columns, the bankruptcy cost α and tax rate τ are assumed to be 0. In the Geske (1977) column, the firm issues two zero-coupon bonds: B_1 and B_2, where $B_2 \prec B_1$. Only the values of B_2 are shown. In the Leland, 1994 column, the tax rate τ and the bankruptcy cost α are assumed to be 0.35 and 0.5, respectively; in addition, the optimal coupon per year $C = \$112.259$ with the maximum levered firm value $5766.65 is used when $\sigma = 0.25$ and $C = \$159.403$ with the maximum levered firm value $5557.91 is used when $\sigma = 0.4$.

σ	Merton (1974)		Black and Cox (1976)		Geske (1977)		Leland (1994)	
	Lattice	Formula	Lattice	Formula	Lattice	Formula	Lattice	Formula
0.25	2934.82 (0.00003%)	2934.82	2940.03 (0.00002%)	2940.03	2449.81 (−0.00082%)	2449.79	3419.57 (−0.00556%)	3419.38
0.4	2875.60 (0.00014%)	2875.60	2935.53 (−0.00010%)	2935.53	2425.37 (−0.00742%)	2425.55	2941.41 (−0.02788%)	2942.23

fault event occurs when the firm fails to raise sufficient equity capital to meet the repayment of B_1 at year 0.5 or when the firm's asset value fails to meet the repayment of B_2 at year 1. The Leland (1994) column assumes the firm issues a consol bond with an endogenously defined optimal coupon payment that maximizes the levered firm value. The default is triggered, endogenously, if the firm fails to raise sufficient equity capital to meet bond obligations. Although our lattice is a numerical method for bonds with a finite maturity, it can accurately price the consol bond by setting the finite maturity of the bond T to a large number (200 years in Table 1). (Brennan & Schwartz (1978) use a similar technique.) The numerical closeness for the 4 models attests to the robustness and generality of our lattice.

The numerical examples in Sections 5.2 and 5.3 will show that our lattice can be easily extended to even more complicated liability structures and bond provisions; furthermore, none can be efficiently or accurately priced by other methods.

5.2. Assumptions on asset sales and exogenous default boundaries

Numerical results in Fig. 5(a) illustrate the levered firm values generated by our lattice under various leverage ratios, asset-sales assumptions, and default boundaries. Recall that a bond with an exogenous default boundary is a protected bond whereas that without one is an unprotected bond. The firm is assumed to issue a one-year coupon bond with face value F. The x-axis denotes the leverage ratio B/v, where B denotes the bond value and v denotes the levered firm value. The y-axis denotes the levered firm value. The levered value for the firm issuing the protected bond is lower than the value for the firm issuing an otherwise identical unprotected bond since the presence of the exogenous default boundary $0.9 \times F$ increases the likelihood of default and, as a result, the bankruptcy cost.

The impact of the assumptions regarding asset sales on the levered firm value depends on the presence of the exogenous default boundary and the leverage ratio as Fig. 5(a) shows. For the protected bonds, since the firm's asset value under the no-asset-sales assumption tends to be higher than its value under the total-asset-sales assumption, the firm's asset value under the former assumption is less likely to hit the exogenous default boundary $0.9 \times F$. Therefore, the levered firm value under the former assumption is larger than the value under the latter one for the protected bonds. However, the case with the unprotected bonds is rather different. For this case, there are two opposite forces that affect the levered firm value: (1) The no-asset-sales assumption tends to increase the firm's asset value, which improves the firm's financial status, but (2) the equity holders under this assumption are obligated to raise equity capital to meet the coupon payments, which might motivate the equity holders to let the firm default. When the leverage ratio is low, (1) dominates (2), and the levered firm value under the no-asset-sales assumption is higher than that under the total-asset-sales one. But this relation is reversed when the leverage ratio becomes high.

Fig. 5(b) magnifies part of Fig. 5(a) to pinpoint the maximum levered firm values under various bond provisions. The maximum levered firm values for the unprotected bonds are slightly higher

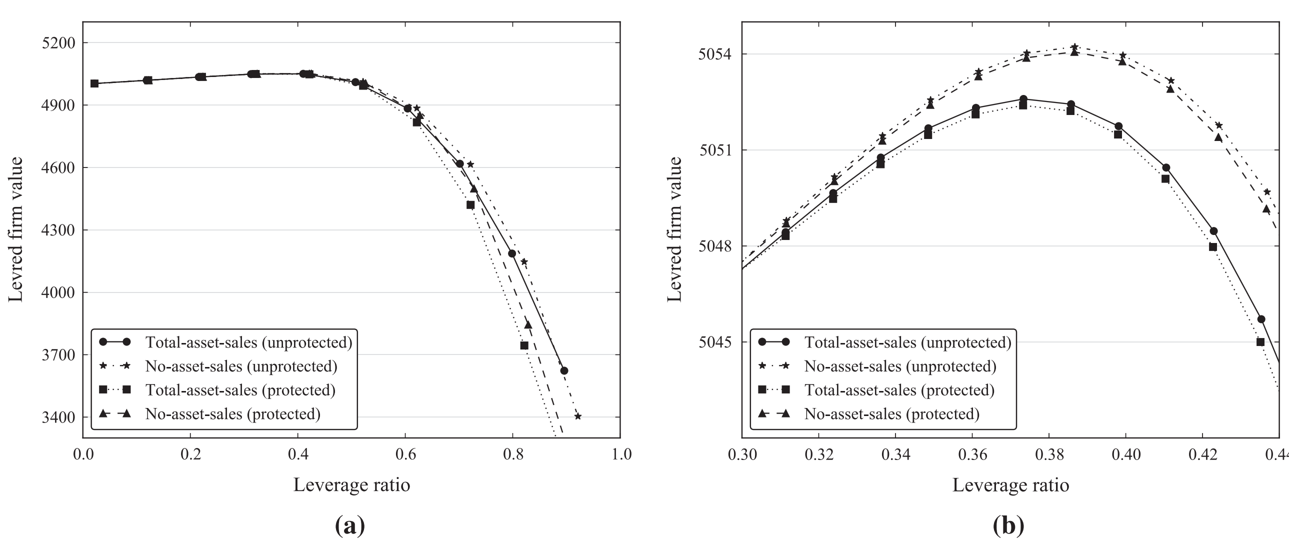

Fig. 5. Maximum levered firm value. The firm's initial asset value is $5000, the risk-free interest rate is $r = 2\%$, the volatility of the firm's asset value is $\sigma = 40\%$, the bankruptcy cost is $\alpha = 0.5$, and the tax rate is $\tau = 0.35$. The firm issues a coupon bond with one-year maturity, face value F, and annualized coupon rate 10%. The leverage ratio is B/v, where B denotes the bond value and v denotes the levered firm value. For the protected bonds, the exogenous default boundary is set to $0.9 \times F$. The firm's asset value is checked against the exogenous default boundary once every quarter, and the coupon is paid quarterly. Fig. 5(b) magnifies part of Fig. 5(a) to show the maximum levered firm values under various bond provisions. The length of one time step Δt for the lattices are 0.001 (year).

than those for the protected bonds since default is less likely for the former. Moreover, the maximum levered firm values under the no-asset-sales assumption are higher than those under the total-asset-sales assumption. Therefore, the highest maximum levered firm value is achieved with the unprotected bond under the no-asset-sales assumption.

5.3. Handling complicated liability structures

A risky bond cannot be priced independently of other outstanding bonds because the presence of these other bonds may change the likelihood of default for the bond under consideration. Such interdependency is, in general, complex and difficult to tackle via analytical methods. But it poses no problems for our lattice. In fact, once the lattice for the asset price is built, the equity and the bonds become derivatives and they can be priced by backward induction on the lattice, fully incorporating their relative seniorities and bond provisions.

We will analyze the values of a bond under generic liability structures with our lattice and compare them with analytical approaches wherever applicable. The firm issues two zero-coupon bonds. Bond $\mathtt{B}_1$ matures at year 3 with a face value of $2500. The face value of the other bond, $\mathtt{B}_2$, is $500. The results for $\mathtt{B}_2$ appear in Table 2. Our lattice can easily handle more than two bonds.

We first price an unprotected $\mathtt{B}_2$ under the no-asset-sales assumption. The bond repayments are financed by selling additional equities, and the default event is triggered endogenously if the firm fails to raise sufficient equity capital to meet bond obligations. Table 2 lists the values of $\mathtt{B}_2$ calculated by known analytical formulas and our lattice. We remark that the total-asset-sales assumption is not considered here because no analytical formula is known. Different seniorities for $\mathtt{B}_2$ are considered. Recall that $\mathtt{B}_i \prec \mathtt{B}_j$ mean $\mathtt{B}_j$ is more senior than $\mathtt{B}_i$. The value of $\mathtt{B}_2$ when $\mathtt{B}_2 \prec \mathtt{B}_1$ is lower than that when $\mathtt{B}_1 \prec \mathtt{B}_2$, as expected. Under $\mathtt{B}_2 \prec \mathtt{B}_1$, shortening the maturity of $\mathtt{B}_2$ from 3.5 years to 2.5 years (which is less than $\mathtt{B}_1$'s maturity of 3 years) significantly increases $\mathtt{B}_2$'s value from 339.08 to 366.23. This is because repaying the subordinated bond $\mathtt{B}_2$ prior to the maturity of the more senior bond $\mathtt{B}_1$ reduces the credit risk of $\mathtt{B}_2$, at the expense of $\mathtt{B}_1$. Compare this increase with the increase from 466.12 to 475.59 when $\mathtt{B}_1 \prec \mathtt{B}_2$ instead.

Not every scenario in Table 2 can be priced by analytical formulas. The Merton (1974) formula can only price the shorter-term bond and only when it is more senior than the longer-term one. To price the longer-term, subordinated bond, the compound option approach of Geske (1977) can be used. In the degenerate case where the maturities of the two bonds are identical, both senior and subordinated bonds can be priced by the Black–Scholes formula (see Lando, 2004). Clearly, our lattice generates prices for $\mathtt{B}_2$ close to those by the relevant analytical formulas. No analytical formulas exist to price (1) longer-term, senior bonds and (2) shorter-term, subordinated bonds, even in the simplest two-bond case (the "x" entries in Table 2). But neither case presents difficulties for our lattice.

Table 2
Pricing unprotected bonds under the no-asset-sales assumption. The firm's initial asset value is $5000, the risk-free interest rate is $r = 2\%$, the volatility of the firm's asset value is $\sigma = 40\%$, the bankruptcy cost is $\alpha = 0$, and the tax rate is $\tau = 0$. The firm issues two zero-coupon bonds $\mathtt{B}_1$ and $\mathtt{B}_2$. $\mathtt{B}_1$ matures at year 3 with a face value of $2500. The face value of $\mathtt{B}_2$ is $500 and the maturities are in the first column. The analytical prices generated by the closed-form formulas are in the `Formula` columns. M, G, and L denote the formulas of Merton (1974), Geske (1977), and Lando (2004), respectively. The prices generated by our lattice are under `Lattice`. The length of one time step Δt for the lattices are 0.001 (year).

Maturity of $\mathtt{B}_2$	Prices of $\mathtt{B}_2$			
	$\mathtt{B}_1 \prec \mathtt{B}_2$		$\mathtt{B}_2 \prec \mathtt{B}_1$	
	`Formula`	`Lattice`	`Formula`	`Lattice`
2.5	475.59 (M)	475.59	x	366.23
3	470.80 (L)	470.80	342.27 (L)	342.27
3.5	x	466.12	339.06 (G)	339.08

The impacts of the firms liability structures and various bond provisions on a protected bond's price are assessed in Table 3, in which all cases cannot be calculated by known analytical formulas. As in Table 2, the firm issues two zero-coupon bonds except that, now, both bonds are protected by an exogenous default boundary set to 80% of the sum of the face values of the outstanding bonds. In the first column, the two maturities of $\mathtt{B}_2$, 2.917 years (35 months) and 3.083 years (37 months), tightly sandwich year 3, the maturity of $\mathtt{B}_1$, to pinpoint the impacts of liability structures on the credit spreads of $\mathtt{B}_2$. The credit spread of a bond $\mathtt{B}$ is defined as $-\ln(B(\mathtt{B})/F(\mathtt{B}))/T - r$, where $B(\mathtt{B})$ denotes the price of $\mathtt{B}$, $F(\mathtt{B})$ denotes the face value of $\mathtt{B}$, and T denotes the time to maturity of $\mathtt{B}$.

In the upper panel of Table 3, the firm is allowed to sell its asset to finance bond repayments under the total-asset-sales assumption. Thus the firm defaults only when its asset value hits the exogenous default boundary or when its asset is insufficient to cover the bond repayments. Suppose $\mathtt{B}_1 \prec \mathtt{B}_2$. The credit spread of the nonputable $\mathtt{B}_2$ jumps from about 0 bps to 42.26 bps when the repayment date of $\mathtt{B}_2$ moves from being slightly earlier than the repayment date of $\mathtt{B}_1$ to being slightly later than the repayment date of $\mathtt{B}_1$. Obviously, repaying $\mathtt{B}_1$ earlier than $\mathtt{B}_2$ by selling the firm's asset significantly increases the risk of $\mathtt{B}_2$ even though $\mathtt{B}_2$ is more senior. However, the addition of the put provision to $\mathtt{B}_2$ provides some protection for $\mathtt{B}_2$ against this risk; as a result, its credit spread with a maturity of 3.083 years is reduced from 42.26 bps to 0.19 bps. Suppose $\mathtt{B}_2 \prec \mathtt{B}_1$ instead. The credit spreads of $\mathtt{B}_2$ are now much higher than those under $\mathtt{B}_1 \prec \mathtt{B}_2$, as expected, because $\mathtt{B}_2$ has a lower priority than $\mathtt{B}_1$ given bankruptcy, which significantly increases $\mathtt{B}_2$'s risk. But the addition of the put provision to the subordinated $\mathtt{B}_2$ now significantly reduces that risk since $\mathtt{B}_2$ can be redeemed prior to the repayment of $\mathtt{B}_1$ in case the issuer's financial status deteriorates. On the other hand, the possibility of selling asset to finance $\mathtt{B}_1$'s repayment prior to $\mathtt{B}_2$'s increases the risk of $\mathtt{B}_2$. Therefore, the credit spread of $\mathtt{B}_2$ jumps upward from 1549 bps to 1622 bps for the nonputable case and from 35 bps to 42 bps for the putable case when the maturity date of $\mathtt{B}_2$ moves from year 2.917 to year 3, the maturity date of the more senior $\mathtt{B}_1$.

Table 3
Effects of complicated liability structures and bond provisions on protected bonds. The firm's initial asset value is $5000, the risk-free interest rate is $r = 2\%$, the volatility of the firm's asset value is $\sigma = 40\%$, the bankruptcy cost is $\alpha = 0$, and the tax rate is $\tau = 0$. The firm issues two zero-coupon bonds $\mathtt{B}_1$ and $\mathtt{B}_2$. Bond $\mathtt{B}_1$ matures at year 3 with a face value of $2500. The face value of $\mathtt{B}_2$ is $500 and the maturities of $\mathtt{B}_2$ are listed in the first column. The put price at year t for $\mathtt{B}_2$ is $500 \times e^{-0.04(T-t)}$, where T is the maturity of $\mathtt{B}_2$. The firm's asset value is checked against an exogenous default boundary which is assumed to be $0.8 \times F_t$ at each time step, where F_t is the sum of the face values of the outstanding bonds at time t. The length of one time step Δt for the lattices are 0.001 (year).

Maturity of $\mathtt{B}_2$	The credit spreads (bps) of $\mathtt{B}_2$			
	$\mathtt{B}_1 \prec \mathtt{B}_2$		$\mathtt{B}_2 \prec \mathtt{B}_1$	
	Nonputable	Putable	Nonputable	Putable
Total-asset-sales				
2.917	0.00000	0.00000	1549.01731	35.26993
3	0.00189	0.00189	1622.31542	42.51885
3.083	42.26309	0.19148	1577.32038	37.65838
No-asset-sales				
2.917	0.00000	0.00000	1618.31594	35.27188
3	0.00189	0.00189	1622.31542	42.53319
3.083	0.00000	0.00000	1575.47098	39.04945

The behaviors of B_2's credit spreads under the no-asset-sales assumption in the lower panel of Table 3 are different from those in the upper panel. Under $B_1 \prec B_2$, repaying the senior B_2 after the maturity of B_1 no longer significantly increases the risk of B_2 whether B_2 is putable or not. This is because disallowing asset sales to finance the repayment of the subordinated B_1 provides some protection for the senior B_2. Unlike the total-asset-sales assumption above, the put provision of B_2 is almost worthless here. Now suppose $B_2 \prec B_1$. The credit spreads of B_2 are much higher than those under $B_1 \prec B_2$, as expected. Unlike the $B_1 \prec B_2$ case, the addition of the put provision to the subordinated B_2 significantly reduces its credit spreads, since B_2 can be redeemed early. Moving the maturity of nonputable B_2 from being slightly earlier than the maturity of the more senior B_1 to being slightly later decreases the credit spread of B_2 from 1618 to 1575.

6. Conclusions

This paper proposes a general numerical methodology for pricing corporate bonds under complicated liability structures. The resulting methodology can tackle realistic assumptions such as complex default boundaries, discrete payments, asset sales assumptions, and early redemption provisions for which closed-form solutions are unavailable. Furthermore, the proposed method can price a portfolio of bonds that account for their complex interaction. The numerical results confirm not only the above claims but also the lattice's ability to accurately price risky bonds with nontrivial liability structures and bond provisions.

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