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Maximally local connectivity and connected components of augmented cubes ☆

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ABSTRACT

The *connectivity* of a graph is an important issue in graph theory, and is also one of the most important factors in evaluating the *reliability* and *fault tolerance* of a network. It is known that the *augmented cube* AQ_n is *maximally connected*, i.e. $(2n-1)$-connected, for $n \geqslant 4$. By the classic *Menger's Theorem*, every pair of vertices in AQ_n is connected by $2n-1$ *vertex-disjoint paths* for $n \geqslant 4$. A routing with parallel paths can speed up transfers of large amounts of data and increase fault tolerance. Motivated by research on networks with faults, we obtained the result that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 2n-7$ for $n \geqslant 4$, each pair of non-faulty vertices, denoted by u and v, in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $AQ_n - F$, respectively. Moreover, we demonstrate that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 4n-9$ for $n \geqslant 4$, there exists a large connected component with at least $2^n - |F| - 1$ vertices in $AQ_n - F$, which improves on the results of Ma et al. (2008) who show this for $n \geqslant 6$.

1. Introduction

Interconnection networks have been widely studied recently. The architecture of an interconnection network is usually denoted as an undirected graph G. For the graph definition and notation we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a,b) | (a,b)\ (a \neq b)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. The interconnection network topology is usually represented by a graph $G = (V, E)$, where vertices represent processors and edges represent links between processors. The *neighborhood* of vertex v, denoted by $N(v)$, is $\{x | (v, x) \in E\}$. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of vertices in $N(v)$. A graph G is *k-regular* if $\deg(v) = k$ for every vertex $v \in V$. For the purpose of connecting hundreds or thousands of processing elements, many interconnection network topologies have been proposed in the literature. Graph theory can be used to analyze network reliability, so we use the terminology *graphs* and *networks* synonymously.

The *reliability* and *fault tolerance* of a network with respect to processor failures is directly related to the *connectivity* of the corresponding graph. Connectivity is one of the important factors for evaluating the fault tolerance of a network [3,4,14]. The connectivity of G, written $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and

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$\kappa(G) = |V(G)| - 1$ otherwise. Traditional connectivity only considers how many faulty vertices there can be before the network fails. It is known that $\kappa(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G. For the most part, even if the number of faulty vertices is higher than that specified by network connectivity standards, the network remains connected or at least a large part of it remains connected. Many measures of fault tolerance of networks are related to the maximal size of the connected components of networks with faulty vertices, so it is essential to estimate the maximally connected component of the network with the faulty vertices [1]. Yang et al. [15–17] have proposed a way to determine the maximally connected component of the *n-dimensional hypercube*.

A distributed system is useful because it offers the advantage of improved connectivity. Menger's Theorem [10] shows that if a network G is k-connected, every pair of vertices in G is connected by k vertex-disjoint (parallel) paths. Efficient routing can be achieved using vertex-disjoint paths, providing parallel routing and high fault tolerance, increasing the efficiency of data transmission, and decreasing transmission time. Saad and Schultz [12] studied the n vertex-disjoint parallel paths of an n-dimensional hypercube Q_n. Day and Tripathi [7] discussed the $n-1$ vertex-disjoint parallel paths of an n-dimensional star graph S_n for any two vertices of S_n.

Many useful topologies have been proposed to balance performance and cost parameters. Among them, the binary *hypercube* Q_n [5,12] is one of the most popular topologies, and has been studied for parallel networks. *Augmented cubes* are derivatives of the hypercubes with good geometric features that retain some favorable properties of the hypercubes, such as vertex symmetry, maximum connectivity, best possible wide diameter, routing, and broadcasting procedures with linear time complexity. The augmented cube of dimension n, denoted by AQ_n, is a Cayley graph, $(2n-1)$-regular, $(2n-1)$-connected, and has diameter $\lceil n/2 \rceil$ [6]. In this paper, we demonstrate a tight result that for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 2n-7$ for $n \geqslant 4$, each pair of non-faulty vertices u and v in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $AQ_n - F$, respectively. In addition, we consider the maximally connected component of the augmented cube with faulty vertices. In 2008, Ma et al. showed that for $n \geqslant 6$, for any faulty vertex set $F \subset V(AQ_n)$ and $|F| \leqslant 4n-9$, the maximally connected component of $AQ_n - F$ has at least $2^n - |F| - 1$ vertices. We improve this result by demonstrating it for $n \geqslant 4$.

In the next section, we give the definition of the augmented cube AQ_n for $n \geqslant 1$. Section 3 deals with the maximally connected component of $AQ_n - F$ with $|F| \leqslant 4n-9$ for $n \geqslant 4$. Section 4 studies the vertex-disjoint fault-free paths in $AQ_n - F$ with $|F| \leqslant 2n-7$ for $n \geqslant 4$.

2. The augmented cube AQ_n

The definition of the *n-dimensional augmented cube* is stated as the following. Let $n \geqslant 1$ be a positive integer. The n-dimensional augmented cube [6,8], denoted by AQ_n, is a vertex transitive and $(2n-1)$-regular graph with 2^n vertices. Each vertex is labeled by an n-bit binary string and $V(AQ_n) = \{u_n u_{n-1} \ldots u_1 | u_i \in \{0,1\}\}$. AQ_1 is the complete graph K_2 with vertex set $\{0,1\}$ and edge set $\{(0,1)\}$. As for $n \geqslant 2$, AQ_n consists of (1) two copies of $(n-1)$-dimensional augmented cubes, denoted by AQ^0_{n-1} and AQ^1_{n-1}; and (2) 2^n edges (two perfect matchings of AQ_n) between AQ^0_{n-1} and AQ^1_{n-1}. AQ_n can be written as $AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ for $n \geqslant 2$. $V(AQ^0_{n-1}) = \{0u_{n-1}u_{n-2}\ldots u_1 \mid u_i \in \{0,1\}\}$ and $V(AQ^1_{n-1}) = \{1v_{n-1}v_{n-2}\ldots v_1 \mid v_i \in \{0,1\}\}$. Vertex $u = 0u_{n-1}u_{n-2}\ldots u_1$ of AQ^0_{n-1} is joined to vertex $v = 1v_{n-1}v_{n-2}\ldots v_1$ of AQ^1_{n-1} if and only if either.

(i) $u_i = v_i$ for $1 \leqslant i \leqslant n-1$; in this case, (u, v) is called a *hypercube edge* and we set $v = u^h$, or
(ii) $u_i = \bar{v}_i$ for $1 \leqslant i \leqslant n-1$; in this case, (u, v) is called a *complement edge* and we set $v = u^c$.

The augmented cubes AQ_1, AQ_2, and AQ_3 are illustrated in Fig. 1. Let the hypercube edge set of AQ_n be E^h_n and the complement edge set of AQ_n be E^c_n. Thus, $E^h_n = \{(u, u^h) | \ u \in V(AQ^0_{n-1})\}$ and $E^c_n = \{(u, u^c) | \ u \in V(AQ^0_{n-1})\}$. Obviously, each of E^h_n and E^c_n is a perfect matching between the vertices of AQ^0_{n-1} and AQ^1_{n-1}. Then, both $|E^h_n|$ and $|E^c_n|$ are equal to 2^{n-1}.

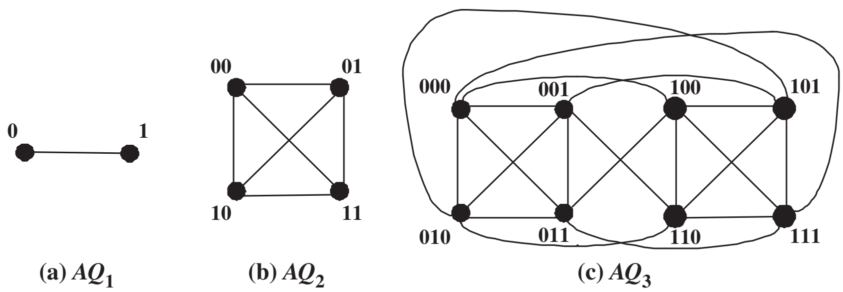

Fig. 1. The augmented cubes AQ_1, AQ_2, and AQ_3.

3. Maximally connected component

Before proving our main results, we show some properties of the augmented cubes in the following two lemmas.

Lemma 1. *Assume n is an integer with* $n \geqslant 3$. *Let* $AQ_n = AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ *be an n-dimensional augmented cube, and* u, v *be any two vertices in* AQ^0_{n-1}. *Then, the vertices u and v have totally two distinct neighborhoods in* AQ^1_{n-1} if $u = 0a_{n-1}\dots a_1$ *and* $v = 0\overline{a_{n-1}\dots a_1}$ *with* $a_i \in \{0,1\}$ *for* $1 \leqslant i \leqslant n-1$. *Otherwise, the vertices u and v have totally four distinct neighborhoods in* AQ^1_{n-1}. *That is,*

$$|(N(u) \cup N(v)) \cap V(AQ^1_{n-1})| = \begin{cases} 2 & \text{if } u = 0\,a_{n-1}\dots a_1 \text{ and } v = 0\overline{a_{n-1}\dots a_1}, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. We first suppose that $u = 0a_{n-1}\dots a_1$ and $v = 0\overline{a_{n-1}\dots a_1}$ with $a_i \in \{0,1\}$ for $1 \leqslant i \leqslant n-1$. Then, vertices u and v have two distinct neighborhoods $1a_{n-1}\dots a_1$ and $1\overline{a_{n-1}\dots a_1}$ in AQ^1_{n-1}. Otherwise, suppose that $u = 0a_{n-1}\dots a_1$ and $v = 0b_{n-1}\dots b_1$, where $a_{n-1}\dots a_1 \neq \overline{b_{n-1}\dots b_1}$. Then, vertices u and v have four distinct neighborhoods $1a_{n-1}\dots a_1$, $1\overline{a_{n-1}\dots a_1}$, $1b_{n-1}\dots b_1$, and $1\overline{b_{n-1}\dots b_1}$ in AQ^1_{n-1}. As a result, this lemma follows. □

By the structure of the augmented cubes, every pair of vertices in the augmented cube has at most four common neighborhoods, as has been proved in [9] as Lemma 2.

Lemma 2. *For* $n \geqslant 3$, let AQ_n *be an n-dimensional augmented cube, any two vertices u and v* of AQ_n *have at most four common neighborhoods. That is,* $|N(u) \cap N(v)| \leqslant 4$.

Choudum and Sunitha [6] have shown that the augmented cube AQ_n is maximally connected for $n \geqslant 4$, as the following lemma.

Lemma 3. $\kappa(AQ_1) = 1,\ \kappa(AQ_2) = 3,\ \kappa(AQ_3) = 4$, *and* $\kappa(AQ_n) = 2n-1$, *where* $n \geqslant 4$.

For ease of the proof of Lemma 5 and Theorem 1, we need the following lemma.

Lemma 4. *Assume that n is an integer with* $n \geqslant 2$. Let $AQ_n = AQ^0_{n-1} \diamondsuit AQ^1_{n-1}$ *be an n-dimensional augmented cube,* $F \subset V(AQ_n)$ *be a set of vertices of* AQ_n, *and* $F_1 = F \cap V(AQ^1_{n-1})$ *with* $|F_1| \leqslant 1$. *Then,* $AQ_n - F$ *is still a connected graph which contains* $2^n - |F|$ *vertices.*

Proof. According to Lemma 3, $AQ^1_{n-1} - F_1$ is a connected component with $2^{n-1} - |F_1|$ vertices. For each vertex $v \in V(AQ^0_{n-1})$, at least one of its two neighborhoods, which is located in $V(AQ^1_{n-1})$, is fault-free since $|F_1| \leqslant 1$. Therefore, $AQ_n - F$ is connected, and its cardinality of the fault-free vertex set is $2^n - |F|$. This lemma is completed. □

The following lemma is the base case for Theorem 1. We note that for $n = 4$ in the context of the following lemma, $4n - 9 = 7$ and $2^n - |F| - 1 = 8$.

Lemma 5. *For a 4-dimensional augmented cube* AQ_4, let $F \subset V(AQ_4)$ *be a faulty vertex set with* $|F| = 7$. *Then,* $AQ_4 - F$ *has a connected component containing at least eight vertices.*

Proof. Let $F_0 = F \cap V(AQ^0_3)$ and $F_1 = F \cap V(AQ^1_3)$, thus $F = F_0 \cup F_1$. Without loss of generality, we may assume that $|F_0| \geqslant |F_1|$. Thus, $|F_0| \geqslant 4,\ |F_1| \leqslant 3$, and $AQ^1_3 - F_1$ is connected by Lemma 3. In the following, we divide the proof according to the cardinality of F_0.

Case 1: $|F_0| \geqslant 6$.

Since $|F_0| \geqslant 6,\ |F_1| \leqslant 1$. By Lemma 4, $AQ_4 - F$ is a connected component containing 9 vertices, and this case follows.

Case 2: $4 \leqslant |F_0| \leqslant 5$.

Let C be the connected component with minimal cardinality in $AQ^0_3 - F_0$. First, suppose C consists of only one vertex, say vertex u. Then, $N_{AQ^0_3}(u) \subset F$ and $|F_0| = 5$. Then, $AQ^0_3 - (F_0 \cup \{u\})$ is a connected component with 2 vertices, and thus $AQ_4 - F$ has a connected component containing at least 8 vertices.

Second, suppose C consists of two vertices, then either $F_0 = \{0000, 0011, 0101, 0110\}$ or $F_0 = \{0001, 0010, 0100, 0111\}$. Thus, $AQ^0_3 - F_0$ is composed of two connected components with two vertices respectively. The vertex set of $AQ^0_3 - F_0$ is either $\{0001, 0010, 0100, 0111\}$ or $\{0000, 0011, 0101, 0110\}$. In addition, $AQ^1_3 - F_1$ is connected. Each of the two connected components in $AQ^0_3 - F_0$ has four distinct neighborhoods in AQ^1_3, and is connected to $AQ^1_3 - F_1$. Therefore, $AQ_4 - F$ is a connected component containing 9 vertices.

Now, suppose C consists of three or four vertices, it is easy to see that $AQ_3^0 - F_0$ is connected. Let u, v be two vertices in C such that $u = 0a_3a_2a_1$ and $v \neq 0\overline{a_3a_2a_1}$. Hence, by Lemma 1, u or v has at least one fault-free neighborhood in AQ_3^1. Therefore, $AQ_4 - F$ is a connected component containing 9 vertices, and this lemma follows. □

We now show the maximally connected component of the augmented cube with faulty vertices.

Theorem 1. *Let AQ_n be an n-dimensional augmented cube with $n \geqslant 4$, and $F \subset V(AQ_n)$ be a faulty vertex set with $|F| = 4n - 9$. Then, $AQ_n - F$ has a large connected component containing at least $2^n - |F| - 1$ vertices.*

Proof. We prove this theorem by induction on n. For $n = 4$, it is already proved by Lemma 5 that $AQ_4 - F$ has a connected component containing at least eight vertices. By the induction hypothesis, we may assume that the result is true for AQ_{n-1} with $|F| = 4 \times (n-1) - 9 = 4n - 13$. Now we consider AQ_n with $|F| = 4n - 9$ and show that $AQ_n - F$ has a connected component containing at least $2^n - |F| - 1$ vertices.

Let $F_0 = F \cap V(AQ_{n-1}^0)$ and $F_1 = F \cap V(AQ_{n-1}^1)$. Without loss of generality, we may assume that $|F_0| \geqslant |F_1|$. Thus $|F_0| \geqslant 2n - 4$, $|F_1| \leqslant 2n - 5$, and $AQ_{n-1}^1 - F_1$ is connected according to Lemma 3. In the following, we divide the proof into three cases according to the cardinality of F_0.

Case 1: $|F_0| \geqslant 4n - 10$.

Since $|F_0| \geqslant 4n - 10$, $|F_1| \leqslant 1$. According to Lemma 4, $AQ_n - F$ is a connected component containing $2^n - |F|$ vertices, and this case follows.

Case 2: $4n - 11 \geqslant |F_0| \geqslant 4n - 12$. Let $AQ_{n-1}^0 - F_0$ be composed of connected components $C_1, C_2, \ldots, C_x$, and let $|V(C_1)| \leqslant |V(C_2)| \leqslant \ldots \leqslant |V(C_x)|$ with $x \geqslant 1$. Now, we shall show that **(1)** $|V(C_i)| \geqslant 2$ for $2 \leqslant i \leqslant x$; and **(2)** For each $|V(C_i)| \geqslant 2$ where $1 \leqslant i \leqslant x, C_i$ is connected to $AQ_{n-1}^1 - F_1$. With **(1)** and **(2)** holds, $AQ_n - F$ contains a connected component containing at least $2^n - |F| - 1$ vertices, and this case follows.

Proof of (1): Suppose (1) is incorrect, then $|V(C_1)| = |V(C_2)| = 1$, we denote that $V(C_1) = \{u\}$, $V(C_2) = \{v\}$, and $(u, v) \notin E(AQ_n)$. Because any two vertices have at most four common neighborhoods by Lemma 2, $|F_0| \geqslant |N_{AQ_{n-1}^0}(u) \cup N_{AQ_{n-1}^0}(v)| \geqslant (2(n-1) - 1) \times 2 - 4 = 4n - 10$, which contradicts to our assumption that $4n - 11 \geqslant |F_0| \geqslant 4n - 12$.

Proof of (2): First, suppose $|V(C_i)| = 2$. Let (u, v) be the edge of C_i. By Lemma 1, $|(N(u) \cup N(v)) \cap V(AQ_{n-1}^1)|$ is either 2 or 4. Suppose $|(N(u) \cup N(v)) \cap V(AQ_{n-1}^1)| = 2$, u and v will have at most two common neighborhoods in AQ_{n-1}^0 according to Lemma 2. Thus, $|F_0| \geqslant |(N_{AQ_{n-1}^0}(u) \cup N_{AQ_{n-1}^0}(v)) - \{u, v\}| \geqslant (2(n-1) - 2) \times 2 - 2 = 4n - 10$, which is a contradiction to our assumption that $4n - 11 \geqslant |F_0| \geqslant 4n - 12$. Suppose $|(N(u) \cup N(v)) \cap V(AQ_{n-1}^1)| = 4$. Because $|F_1| \leq 3$, there is at least one fault-free edge, i.e., two vertices of the edge are fault-free, between C_i and $AQ_{n-1}^1 - F_1$. Therefore, C_i is connected to $AQ_{n-1}^1 - F_1$.

Now, suppose $|V(C_i)| \geqslant 3$. Each C_i in AQ_{n-1}^0 exists two vertices u and v such that $u = 0a_{n-1} \ldots a_1$ and $v \neq 0\overline{a_{n-1} \ldots a_1}$. Note that $|F_1| \leqslant 3$. Hence, according to Lemma 1, u or v has at least one fault-free neighborhood in AQ_{n-1}^1. As a result, C_i is connected to $AQ_{n-1}^1 - F_1$.

Case 3: $|F_0| \leqslant 4n - 13$. By the induction hypothesis, $AQ_{n-1}^0 - F_0$ has a connected component containing at least $2^{n-1} - |F_0| - 1$ vertices with $|F_0| = 4n - 13$. Then, it is obviously that if $|F_0| < 4n - 13, AQ_{n-1}^0 - F_0$ also has a connected component containing at least $2^{n-1} - |F_0| - 1$ vertices. Moreover, $AQ_{n-1}^1 - F_1$ is a connected component with $2^{n-1} - |F_1|$ vertices. Suppose $n = 4$, $|F| = 4n - 9 = 7$ and $|F_0| \leqslant 4n - 13 = 3$, which contradicts to our assumption that $|F_0| \geqslant |F_1|$. Without loss of generality, we may assume $n \geqslant 5$. Now, $AQ_{n-1}^0 - F_0$ is connected to $AQ_{n-1}^1 - F_1$ since $(2^{n-1} - |F_0| - 1) + (2^{n-1} - |F_1|) > |E_n^h| = |E_n^c| = 2^{n-1}$, where $n \geq 5$. Therefore, $AQ_n - F$ has a connected component containing at least $(2^{n-1} - |F_0| - 1) + (2^{n-1} - |F_1|) = 2^n - |F| - 1$ vertices and the proof is complete. □

Corollary 1. *For an n-dimensional augmented cube AQ_n with $n \geqslant 4$, let $F \subset V(AQ_n)$ be any vertex set with $|F| \leqslant 4n - 9$. Then, $AQ_n - F$ has a connected component containing at least $2^n - |F| - 1$ vertices.*

4. Vertex-disjoint paths

Menger's Theorem [10] is a classic result in connectivity and states that if a network G is k-connected, every pair of vertices in G is connected by k vertex-disjoint paths. A k-regular graph G is *strongly Menger-connected* if for any copy $G - F$ of G with at most $k - 2$ vertices removed, each pair u and v of $G - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths

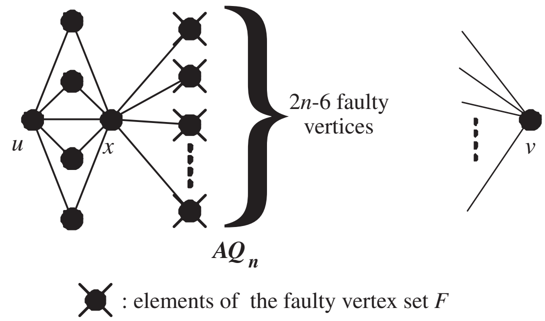

Fig. 2. An example that $|F| = 2n - 6$.

in $G - F$, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $G - F$, respectively [11,13]. It has been proved that *star graphs* are strongly Menger-connected [11].

However, augmented cubes are not strongly Menger-connected according to their structures. We can see this in Fig. 2. For an n-dimensional augmented cube AQ_n with $n \geqslant 4, AQ_n$ is $(2n-1)$-regular. Let (u,x) be an edge of AQ_n such that $|N(u) \cap N(x)| = 4, F$ be a faulty vertex set such that $F = N(x) - (N(u) \cup \{u\})$, and v be a vertex of AQ_n such that $v \in V(AQ_n) - (N(u) \cup N(x))$. Note that $|F| = 2n - 6$. As a result, the vertices u and v are **not** connected by $\min\{\deg_f(u), \deg_f(v)\} = 2n - 1$ vertex-disjoint fault-free paths in $AQ_n - F$.

Now, we give the definition of *maximally local connectivity*. Given a graph G and a vertex set $F \subset V(G), (G,F)$ is said to be maximally local connected if and only if for each pair of vertices, denoted by u and v, of $G - F$, u and v are connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths in $G - F$, where $\deg_f(u)$ and $\deg_f(v)$ are the degree of u and v in $G - F$, respectively. For the vertex-disjoint fault-free paths of AQ_n under a set of faulty vertices with $|F| \leqslant 2n - 7$, a tight result is stated and proved in Theorem 2. For the proof of Theorem 2, the following lemma is needed.

Lemma 6. *For an n-dimensional augmented cube AQ_n with $n \geqslant 4$, let $F = F' \cup \{(u,v)\}$ where $F' \subset V(AQ_n)$ is any vertex set with $|F'| \leqslant 4n - 10$ and (u,v) is any edge in AQ_n. Then, $AQ_n - F$ has a connected component containing at least $2^n - |F| - 1$ vertices.*

Proof. By Corollary 1, $AQ_n - (F' \cup \{u\})$ has a connected component containing at least $2^n - |F| - 1$ vertices. It is clear that this connected component is also a connected component of $AQ_n - F$ that containing at least $2^n - |F| - 1$ vertices, so this lemma is proved. □

Theorem 2. *For an n-dimensional augmented cube AQ_n with $n \geqslant 4$, let $F \subset V(AQ_n)$ be a set of faulty vertices with $|F| \leqslant 2n - 7$. Then, each pair of vertices u and v in $AQ_n - F$ is connected by $\min\{\deg_f(u), \deg_f(v)\}$ vertex-disjoint fault-free paths in $AQ_n - F$.*

Proof. We shall prove this theorem by contradiction. Let u, v be two distinct vertices in $AQ_n - F$ and let $m = \min\{\deg_f(u), \deg_f(v)\}$. Suppose that there do not exist m vertex-disjoint fault-free paths connecting u and v in $AQ_n - F$. Firstly, if $(u,v) \notin E(AQ_n)$, by Menger's Theorem, vertices u and v are disconnected in $(AQ_n - F) - F'$ for some faulty vertex set $F' \subset V(AQ_n - F)$ and $|F'| = m - 1$. Now, if $(u,v) \in E(AQ_n)$, by Menger's Theorem, vertices u and v are disconnected in $(AQ_n - F) - F'$ for some faulty vertex set $F' = V_f \cup \{(u,v)\}$ where $V_f \subset V(AQ_n - F)$ and $|V_f| = m - 2$. Thus, the total number of faulty elements in AQ_n is $|F| + |F'| \leqslant (2n-7) + (m-1) \leqslant (2n-7) + (2n-1-1) = 4n - 9$. By Corollary 1 and Lemma 6, $(AQ_n - F) - F'$ has a large connected component containing at least $2^n - (4n-9) - 1$ vertices. That is, if $(AQ_n - F) - F'$ is disconnected, it consists of two connected components and one of which is an isolated vertex. Note that u and v are disconnected in $(AQ_n - F) - F'$, thus F' consists of all the neighborhoods of vertex u or v of $AQ_n - F$. Hence, $|F'| \geqslant m$, which is a contradiction to our assumption that $|F'| = m - 1$. Consequently, this theorem is proved. □

Corrollary 2. *For an n-dimensional augmented cube AQ_n* with $n \geqslant 4$, *let $F \subset V(AQ_n)$ be any vertex set with $|F| \leqslant 2n - 7$*. Then, *(AQ_n, F) is maximally local connected.*

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