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Fault-tolerant hamiltonian connectivity of the WK-recursive networks

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ABSTRACT

Many research on the WK-recursive network has been published during the past several years due to its favorite properties. In this paper, we consider the fault-tolerant hamiltonian connectivity of the WK-recursive network. We use $K(d,t)$ to denote the WK-recursive network of level t, each of which basic modules is a d-vertex complete graph, where $d > 1$ and $t \geqslant 1$. The *fault-tolerant hamiltonian connectivity* $\mathcal{H}_f^\kappa(G)$ is defined to be the maximum integer k such that G is k fault-tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise. In this paper, we prove that $\mathcal{H}_f^\kappa(K(d,t)) = d - 4$ if $d \geqslant 4$.

1. Introduction

As is customary in structure studies of parallel architectures, we restrict our attention to a set of identical processors, and we view the architectures of the underlying interconnection networks as graphs. The vertices of a graph represent the processors of an architecture, and the edges of the graph represent the communication links between processors. There are many mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum from all aspects. One has to design a suitable network depending on the requirements of its properties. The hamiltonian property is one of the major requirements in designing the topology of a network. Fault-tolerance is also desirable in massive parallel systems.

In this paper, a network is represented as a loopless undirected graph. For graph definitions and notations we follow [1]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. Let S be a subset of V. The subgraph of G *induced by* S is the graph $G[S]$ with $V(G[S]) = S$ and $E(G[S]) = \{(u, v)|(u, v) \in E,\ \text{and}\ \{u, v\} \subset S\}$. The *complement* $\overline{G}$ of a graph G with the same vertex set $V(G)$ defined by $(u, v) \in E(\overline{G})$ if and only if $(u, v) \notin E(G)$. We use $\bar{e}$ to denote $|E(\overline{G})|$. The *degree* of a vertex u of G, $\deg_G(u)$, is the number of edges incident with u. A graph G is k-regular if $\deg_G(x) = k$ for any vertex x in G. A *path*, $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, is an ordered list of distinct vertices such that v_i and v_{i+1} are adjacent for $1 \leqslant i \leqslant k-1$. A path is a *hamiltonian path* if its vertices are distinct and span V.

In [5], the performance of the hamiltonian property in faulty networks is discussed. In [10], Huang et al. define a parameter on fault-tolerant hamiltonicity. A hamiltonian graph G is k *fault-tolerant hamiltonian* if $G - F$ remains hamiltonian for

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every $F \subset V(G) \cup E(G)$ with $|F| \leqslant k$. The *fault-tolerant hamiltonicity* $\mathcal{H}_f(G)$ is defined to be the maximum integer k such that G is k fault-tolerant hamiltonian if G is hamiltonian and is undefined otherwise. Clearly, $\mathcal{H}_f(G) \leqslant \delta(G) - 2$ if $\mathcal{H}_f(G)$ is defined. They also introduce the concept of fault-tolerant hamiltonian connectivity. A graph G is *hamiltonian connected* if there exists a hamiltonian path joining any two vertices of G. All hamiltonian connected graphs except the complete graphs K_1 and K_2 are hamiltonian. A graph G is *k fault-tolerant hamiltonian connected* if $G - F$ remains hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant k$. The *fault-tolerant hamiltonian connectivity* $\mathcal{H}_f^\kappa(G)$ is defined to be the maximum integer k such that G is k fault-tolerant hamiltonian connected if G is hamiltonian connected and is undefined otherwise. There are a lot of study on fault-tolerant hamiltonicity and fault-tolerant hamiltonian connectivity [6–11,14]. It can be checked that $\mathcal{H}_f^\kappa(G) \leqslant \delta(G) - 3$ only if $\mathcal{H}_f^\kappa(G)$ is defined and $|V(G)| \geqslant 4$.

In this paper, we consider the fault-tolerant hamiltonian connectivity of the WK-recursive network. The WK-recursive network is proposed by [15]. We use $K(d,t)$ to denote the WK-recursive network of level t, each of which basic modules is a d-vertex complete graph, where $d > 1$ and $t \geqslant 1$. It offers a high degree scalability, which conforms very well to a modular design and implementation of distributed systems involving a large number of computing elements. A transputer implementation of a 15-vertex WK-recursive network has been realized at the Hybrid Computing Center, Naples, Italy. In this implementation, each vertex is implemented with the IMS T414 Transputer [12]. Recently, the WK-recursive network has received much attention due to its many favorable properties. In particular, it is proved that $K(d,t)$ is hamiltonian connected [2] and $\mathcal{H}_f(K(d,t)) = d - 3$ [4]. In this paper, we prove that $\mathcal{H}_f^\kappa(K(d,t)) = d - 4$.

In the following section, we give the definition of WK-recursive network. In Section 3, we give some preliminaries for the discussion on the fault-tolerant hamiltonian connectivity of the WK-recursive network. In Section 4, we prove that $\mathcal{H}_f^\kappa(K(d,t)) = d - 4$.

2. WK-recursive networks

The WK-recursive network can be constructed hierarchically by grouping basic modules. A complete graph of any size d can serve as the basic modules. We use $K(d,t)$ to denote a WK-recursive network of level t, each of whose basic modules is a d-vertex complete graph, where $d > 1$ and $t \geqslant 1$. The structures of $K(5,1), K(5,2)$, and $K(5,3)$ are shown in Fig. 1. $K(d,t)$ is defined in terms of a graph as follows:

Each vertex of $K(d,t)$ is labeled as a t-digit radix d number. Vertex $a_{t-1}a_{t-2}\ldots a_1a_0$ is adjacent to (1) $a_{t-1}a_{t-2}\ldots a_1b$, where $b \neq a_0$ and (2) $a_{t-1}a_{t-2}\ldots a_{j+1}a_{j-1}(a_j)^{j-1}$ if $a_j \neq a_{j-1}$ and $a_{j-1} = a_{j-2} = \ldots = a_0$, where $(a_j)^{j-1}$ denotes $j-1$ consecutive a_js. An *open edge* is incident with $a_{t-1}a_{t-2}\ldots a_0$ if $a_{t-1} = a_{t-2} = \cdots = a_0$. The open edge is reserved for further expansion. Hence, its other end vertex is unspecified. The *open vertex set* O_v of $K(d,t)$ is the set $\{a_{t-1}a_{t-2}\ldots a_0 | a_i = a_{i+1}$ for $0 \leqslant i \leqslant t-2\}$. In other words, O_v contains those vertices with open edges.

Obviously, $K(d,1)$ is a d-vertex complete graph augmented with d open edges. For $t \geqslant 1, K(d,t+1)$ consists d copies of $K(d,t)$, say $K_1(d,t), K_2(d,t), \ldots, K_d(d,t)$. Thus, we consider $K_i(d,t)$ as the ith component of $K(d,t+1)$. Let $I = \{w_1, \ldots, w_q\}$ be any q subset of $\{1,2,\ldots,d\}$, we define graph $K_I(d,t)$ is the subgraph of $K(d,t+1)$ induced by $\bigcup_{i=1}^{q} V(K_{w_i}(d,t))$. For $t \geqslant 2$, the open vertices of $K_i(d,t)$ can be labeled as $o_{i,0}$ and $o_{i,j}$ for $1 \leqslant i \neq j \leqslant d$ where $o_{i,0}$ is the only open vertex of $K(d,t+1)$ in $K_i(d,t)$ and $o_{i,j}$ is the vertex in $K_i(d,t)$ joining with the vertex $o_{j,i}$ in $K_j(d,t)$ with an open edge. Note that $(o_{i,j}, o_{j,i})$ is the only edge joining $K_i(d,t)$ to $K_j(d,t)$.

Now, we define the *extended WK-recursive network* $\widetilde{K}^i(d,t)$ as $V(\widetilde{K}^i(d,t)) = V(K(d,t)) \cup \{x\}$ and $E(\widetilde{K}^i(d,t)) = E(K(d,t)) \cup \{(o_{\alpha,0}, x) | \alpha \in \{1,2,\ldots,d\} - \{i\}\}$. For example, $\widetilde{K}^2(5,1)$ and $\widetilde{K}^3(5,2)$ are illustrated in Fig. 2. Obviously, $\widetilde{K}^i(d,t)$ is isomorphic to $\widetilde{K}^j(d,t)$ for $1 \leqslant i \neq j \leqslant d$.

3. Preliminaries

The following theorem is proved by Ore [13].

Theorem 1 [13]. *Assume that G is an n-vertex graph with $n \geqslant 4$. Then G is hamiltonian if $\bar{e} \leqslant n-3$, and is hamiltonian connected if $\bar{e} \leqslant n-4$.*

Corollary 1. *Assume that $n \geqslant 4$. Then K_n is $(n-3)$ fault-tolerant hamiltonian and $(n-4)$ fault-tolerant hamiltonian connected.*

Proof. Let F be any subset of $V(K_n) \cup E(K_n)$. We use F_v to denote $F \cap V(K_n)$. Then $K_n - F$ is isomorphic to $K_{n-|F_v|} - F'$ where F' is a subset of edges in the subgraph of K_n induced by $\{1,2,\ldots,n\} - F_v$. Obviously, $|F'| \leqslant |F| - |F_v| \leqslant n - 3 - |F_v|$. Since $n - |F_v|$ is the number of vertices of $K_{n-|F_v|} - F'$, the lemma follows from Theorem 1. □

Let $F \subset V(K(d,t+1)) \cup E(K(d,t+1))$ with $|F| \leqslant d-4$. For $1 \leqslant q \leqslant d$, we use F_q to denote $F \cap (V(K_q(d,t)) \cup E(K_q(d,t)))$. Note that it is possible $F - \cup_{q=1}^{d} F_q \neq \emptyset$. For example, it is possible $(o_{1,2}, o_{2,1}) \in F$ but $(o_{1,2}, o_{2,1}) \notin \cup_{q=1}^{d} F_q$.

Now, we construct another graph $H(F)$ from the complete graph K_d with vertex set $\{1,2,\ldots,d\}$ by considering vertex i corresponds to the i-component of $K(d,t+1)$ for every i. Let $F' = \{(\alpha,\beta) | o_{\alpha,\beta} \in F, o_{\beta,\alpha} \in F,$ or $(o_{\alpha,\beta}, o_{\beta,\alpha}) \in F\}$. We set $H(F) = K_d - F'$. Since $|F| \leqslant d-4$, by Corollary 1, $H(F)$ is hamiltonian connected. This result will help us to find a hamiltonian

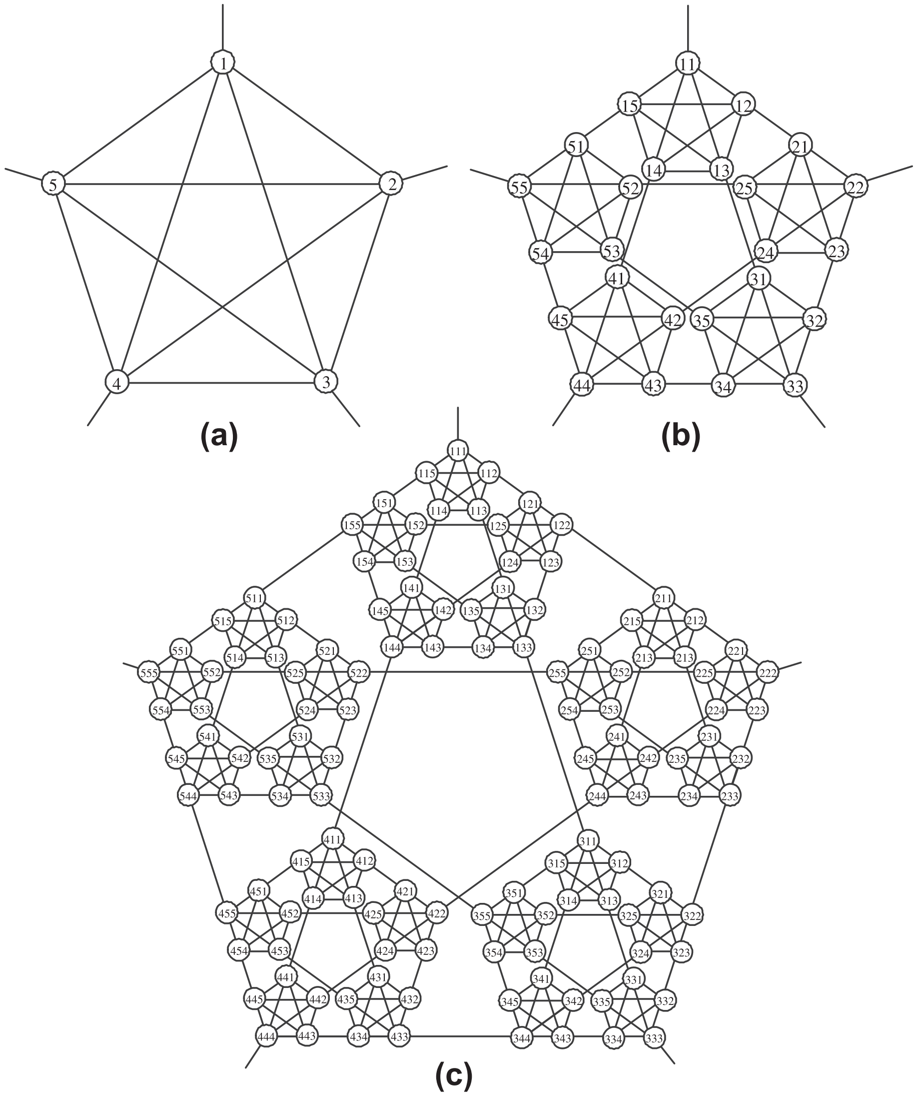

Fig. 1. The graphs (a) $K(5,1)$, (b) $K(5,2)$, and (c) $K(5,3)$.

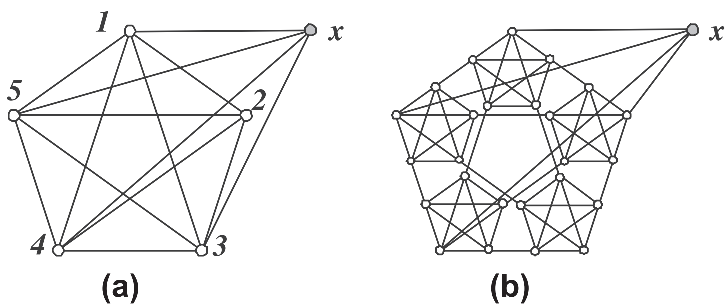

Fig. 2. The graphs (a) $\widetilde{K}^2(5,1)$ and (b) $\widetilde{K}^3(5,2)$.

path between any two vertices in $K(d,t+1)-F$. However, there are several problems need to be conquered. Let us consider the following example.

Assume that u is a vertex in $K_i(d,t)$ and v is a vertex in $K_j(d,t)$ with $1 \leqslant i \neq j \leqslant d$. Let $\langle i = w_1, w_2, \ldots, w_d = j\rangle$ be a hamiltonian path of $H(F)$. Let P_i be a hamiltonian path of $K_i(d,t) - F_i$ joining u to o_{i,w_2}, let P_q be a hamiltonian path of $K_{w_q}(d,t) - F_{w_q}$

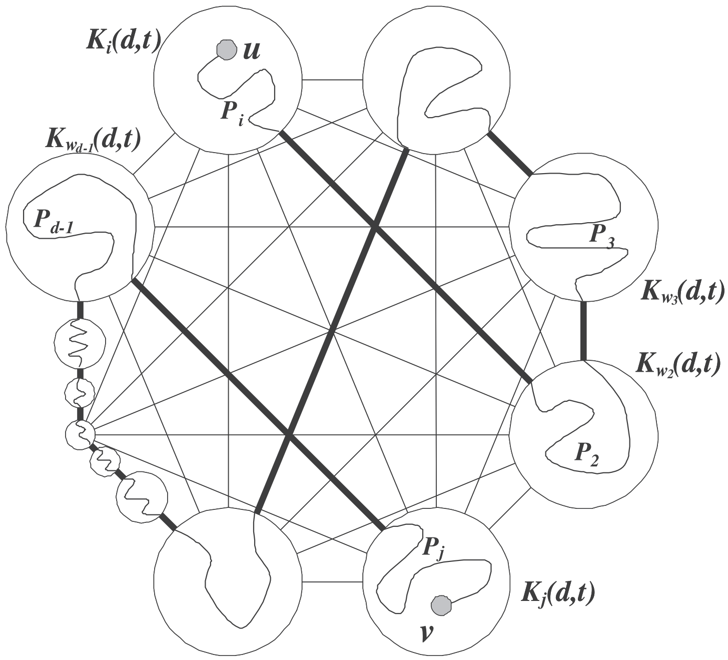

Fig. 3. Finding a hamiltonian path of $K(d,t+1)-F$ between u and v with a hamiltonian path of $H(F)$ between i and j.

joining $o_{w_q,w_{q-1}}$ to $o_{w_q,w_{q+1}}$ for $2 \leqslant q \leqslant d-1$, and let P_j be a hamiltonian path of $K_j(d,t)-F_j$ joining $o_{j,w_{d-1}}$ to v. Obviously, $\langle P_i, P_2, \ldots, P_j \rangle$ forms a hamiltonian path of $K(d,t+1)-F$ joining u to v. See Fig. 3 for illustration.

Yet, we need to guarantee the existence of required paths in each component. Later, we will prove $K(d,t)$ is $(d-4)$ fault-tolerant hamiltonian connected by induction. In the induction step, we assume $K(d,t)$ is $(d-4)$ fault-tolerant hamiltonian connected and prove that $K(d,t+1)$ is $(d-4)$ fault-tolerant hamiltonian connected. With the assumption, the required hamiltonian path P_q exists for $2 \leqslant q \leqslant d-1$. However, we cannot find P_i if $u = o_{i,w_2}$. Similarly, we cannot find P_j if $o_{j,w_{d-1}} = v$. To solve the problem, we can find another hamiltonian path $\langle i = z_1, z_2, \ldots, z_d = j \rangle$ of $H(F)$ to meet the boundary conditions that $u \neq o_{i,z_2}$ and $v \neq o_{j,z_{d-1}}$. As a conclusion, we have the following lemma.

Lemma 1. *Assume that $K(d,t)$ is $(d-4)$ fault-tolerant hamiltonian connected. Let $F \subset V(K(d,t+1)) \cup E(K(d,t+1))$ with $|F| \leqslant d-4$. Let u be a vertex in $K_i(d,t)$ and let v be a vertex in $K_j(d,t)$ with $1 \leqslant i \neq j \leqslant d$. Suppose that $\langle i = w_1, w_2, \ldots, w_d = j \rangle$ be a hamiltonian path of $H(F)$ that satisfies the boundary conditions: $u \neq o_{i,w_2}$ and $v \neq o_{j,w_{d-1}}$. Then there exists a hamiltonian path of $K(d,t+1)-F$ joining u and v.*

From the above discussion, we have three problems to prove that $K(d,t+1)-F$ is hamiltonian connected; i.e., there exists a hamiltonian path of $K(d,t+1)-F$ between any two vertices u and v. First, assume that u is a vertex in $K_i(d,t)$ and v is a vertex in $K_j(d,t)$ with $1 \leqslant i \neq j \leqslant d$. We need to find a hamiltonian path in $H(F)$ that meets the boundary conditions. Second, find a hamiltonian path of $K(d,t+1)-F$ joining u and v if we cannot find a hamiltonian path in $H(F)$ that meets the boundary conditions. Finally, find a hamiltonian path of $K(d,t+1)-F$ joining u and v if both u and v are in $K_i(d,t)$ for some i.

Now, we face the first problem. Let $P_1 = \langle u_1, u_2, \ldots, u_n \rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_n \rangle$ be any two hamiltonian paths of G. We say that P_1 and P_2 are *orthogonal* if $u_1 = v_1, u_n = v_n$, and $u_q \neq v_q$ for $q = 2$ and $q = n-1$. We say a set of hamiltonian paths $\{P_1, P_2, \ldots, P_s\}$ of G are *mutually orthogonal* if any two distinct paths in the set are orthogonal. Suppose there are three mutually orthogonal hamiltonian paths between any two vertices of $H(F)$. By pigeon-hole principle, we can easily find a hamiltonian path with the desired boundary conditions. For this reason, we would like to know all the cases that there are at most two orthogonal hamiltonian paths in $H(F)$. As mentioned above, $H(F)$ is isomorphic to a graph G with n vertices and $\bar{e} \leqslant n-4$.

However, we need some background. Let G and H be two graphs. We use $G+H$ to denote the disjoint union of G and H. We use $G \vee H$ to denote the graph obtained from $G+H$ by joining each vertex of G to each vertex of H. For $1 \leqslant m < n/2$, let $C_{m,n}$ be the graph $(\overline{K}_m + K_{n-2m}) \vee K_m$. See Fig. 4 for illustration.

The following theorem is proved in [3].

Theorem 2. *Assume that G is an n-vertex graph with $n \geqslant 4$ and $\bar{e} \leqslant n-4$. Let s and t be any two vertices of G. Then there are at least two orthogonal hamiltonian paths of G between s and t. Moreover, there are at least three mutually orthogonal hamiltonian paths of G between s and t except the following cases:*

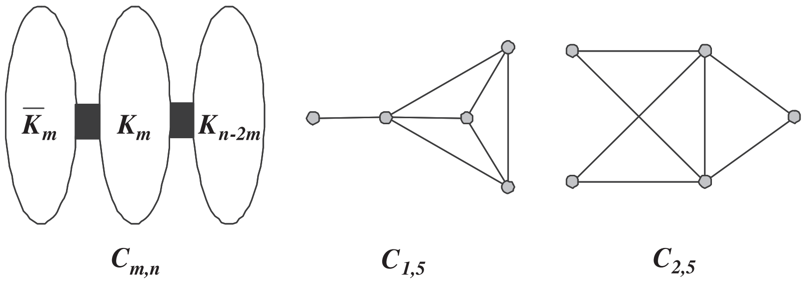

Fig. 4. Illustration of $C_{m,n}$.

C1: *G is isomorphic to K_4 where s and t are any two vertices of G.*
C2: *G is isomorphic to $K_5 - (1,2)$ where s and t are any two vertices except $\{s,t\} = \{1,2\}$.*
C3: *The subgraph H induced by $V(G) - \{s,t\}$ is a complete graph with $n \geqslant 6$ where s is adjacent $V(G) - \{s\}$ and t is adjacent to s and exactly two vertices in H.*
C4: *The subgraph induced by $V(G) - \{s,t\}$ is isomorphic to $C_{2,5}$ where s is adjacent $V(G) - \{s\}$ and t is adjacent to $V(G) - \{t\}$.*
C5: *The subgraph induced by $V(G) - \{s,t\}$ is isomorphic to $C_{1,n-2}$ with $n \geqslant 6$ where s is adjacent $V(G) - \{s\}$ and t is adjacent to $V(G) - \{t\}$.*

To solve the remaining problems, we need some path patterns.

Lemma 2. *Let $d \geqslant 4, t \geqslant 1$, and $I = \{w_1, \ldots, w_n\}$ be any n subset of $\{1,2,\ldots,d\}$. Let u be a vertex in $K_{w_1}(d,t)$ and v be a vertex in $K_{w_n}(d,t)$ such that $u \neq o_{w_1,w_2}$ and $v \neq o_{w_n,w_{n-1}}$. Let F be any subset of $V(K(d,t+1)) \cup E(K(d,t+1))$ such that (1) there exists an edge between $K_{w_q}(d,t) - F_q$ and $K_{w_{q+1}}(d,t) - F_{q+1}$ for $1 \leqslant q \leqslant n-1$ (2) $K_{w_q}(d,t) - F_q$ is hamiltonian connected for $1 \leqslant q \leqslant n$. Then there is a hamiltonian path P of $K_I(d,t) - F$ joining u to v.*

Proof. Since there exists an edge between $K_{w_q}(d,t) - F_q$ and $K_{w_{q+1}}(d,t) - F_{q+1}$ for $1 \leqslant q \leqslant n-1$, the edge $(o_{w_q,w_{q+1}}, o_{w_{q+1},w_q})$ and the vertices $o_{w_q,w_{q+1}}, o_{w_{q+1},w_q}$ are not in F for $1 \leqslant q \leqslant n-1$. Since $K_{w_q}(d,t) - F_q$ is hamiltonian connected for $1 \leqslant q \leqslant n$, there exists a hamiltonian path P_1 of $K_{w_1}(d,t) - F_1$ joining u to o_{w_1,w_2}, there exists a hamiltonian path P_q of $K_{w_q}(d,t) - F_q$ joining $o_{w_q,w_{q-1}}$ to $o_{w_q,w_{q+1}}$ for $2 \leqslant q \leqslant n-1$, there exists a hamiltonian path P_n of $K_{w_n}(d,t) - F_n$ joining $o_{w_n,w_{n-1}}$ to v. Therefore $P = \langle u, P_1, P_2, \ldots, P_n, v\rangle$ is a hamiltonian path of $K_I(d,t) - F$ joining u to v. □

Lemma 3. *Let $d \geqslant 4$ and $t \geqslant 1$. Assume that u and v are two vertices of $K(d,t)$. Let r and s be any two open vertices such that $|\{u,v\} \cap \{r,s\}| = 0$. Then there exist two disjoint paths R and S such that (1) R joins u to one of the vertex in $\{r,s\}$, say r, (2) S joins v to s, and (3) $R \cup S$ spans $K(d,t)$.*

Proof. We prove this lemma by induction on t. The lemma is obviously true for $t = 1$ because $K(d,1)$ is isomorphic to the complete graph K_d with $O_v = V(K(d,1))$. Thus, we assume that this lemma holds for $K(d,n)$ for every $1 \leqslant n \leqslant t$. We claim the statement holds for $K(d,t+1)$.

Let $u \in V(K_i(d,t)), v \in V(K_j(d,t)), r \in V(K_k(d,t))$, and $s \in V(K_l(d,t))$. Since there is only one open vertex in each component, we have $k \neq l$. Now, we consider the following cases.

Case 1: $i \neq j$.

Subcase 1.1: $|\{i,j\} \cap \{k,l\}| = 0$. Thus, there exists an index in $\{k,l\}$, say k, such that $u \neq o_{i,k}$. Let $I_1 = \{i,k\}$ and $I_2 = \{w_1, w_2, \ldots, w_{d-2}\} = \{1,2,\ldots,d\} - I_1$ such that $w_1 = j, w_{d-2} = l$, and $v \neq o_{j,w_2}$. By Lemma 2, there exists a hamiltonian path R of $K_{I_1}(d,t)$ joining u to r; there exists a hamiltonian path S of $K_{I_2}(d,t)$ joining v to s. Obviously, R and S are the required paths.

Subcase 1.2: $|\{i,j\} \cap \{k,l\}| = 1$. Without loss of generality, we assume that $i = k$. Let R be a hamiltonian path of $K_i(d,t)$ joining u to r. Let $I = \{w_1, w_2, \ldots, w_{d-1}\} = \{1,2,\ldots,d\} - \{i\}$ such that $w_1 = j$ and $w_{d-1} = l$. By Lemma 2, there exists a hamiltonian path S of $K_I(d,t)$ joining v to s. Obviously, R and S are the required paths.

Subcase 1.3: $|\{i,j\} \cap \{k,l\}| = 2$. Without loss of generality, we assume that $i = k$ and $j = l$. By assumption, $|\{u,v\} \cap \{r,s\}| = 0$. Let $I = \{w_2, \ldots, w_{d-1}\} = \{1,2,\ldots,d\} - \{i,j\}$. By induction, we can find two disjoint paths S_{j1} and S_{j2} such that (1) S_{j1} joins v to o_{j,w_2}, (2) S_{j2} joins $o_{j,w_{d-1}}$ to s, and (3) $S_{j1} \cup S_{j2}$ spans $K_j(d,t)$. Let R be a hamiltonian path of $K_i(d,t)$ joining u and r. By Lemma 2, there exists a hamiltonian path S' of $K_I(d,t)$ joining $o_{w_2,j}$ to $o_{w_{d-1},j}$. Obviously, R and $S = \langle v, S_{j1}, S', S_{j2}^{-1}, s\rangle$ are the required paths.

Case 2: $i = j$.

Subcase 2.1: $i \notin \{k, l\}$. Let $I = \{w_1, w_2, \ldots, w_{d-2}\} = \{1, 2, \ldots, d\} - \{i, k\}$ such that $w_{d-2} = l$. By induction, we can find two disjoint paths S_{i1} and S_{i2} of $K_i(d,t)$ such that (1) S_{i1} joins u to $o_{i,k}$, (2) S_{i2} joins v to o_{i,w_1}, and (3) $S_{i1} \cup S_{i2}$ spans $K_i(d,t)$. Let R' be a hamiltonian path of $K_k(d,t)$ joining $o_{k,i}$ and r. By Lemma 2, there exists a hamiltonian path S' of $K_I(d,t)$ joining $o_{w_1,i}$ to s. Obviously, $R = \langle u, S_{i1}, R', r\rangle$ and $S = \langle v, S_{i2}, S', s\rangle$ are the required paths.

Subcase 2.2: $i \in \{k, l\}$. Without loss of generality, we assume that $i = k$. Let $I = \{w_1, w_2, \ldots, w_{d-1}\} = \{1, 2, \ldots, d\} - \{i\}$ such that $w_{d-1} = l$. By induction, we can find two disjoint paths S_{i1} and S_{i2} of $K_i(d,t)$ such that (1) S_{i1} joins u to r, (2) S_{i2} joins v to o_{i,w_1}, and (3) $S_{i1} \cup S_{i2}$ spans $K_i(d,t)$. By Lemma 2, there exists a hamiltonian path S' of $K_I(d,t)$ joining $o_{w_1,i}$ to s. Obviously, $R = S_{i1}$ and $S = \langle v, S_{i2}, S', s\rangle$ are the required paths. □

4. Fault-tolerant hamiltonian connectivity

Lemma 4. *Both $K(d,t)$ and $\widetilde{K}^i(d,t)$ are $(d-4)$ fault-tolerant hamiltonian connected for $d \geqslant 4, t \geqslant 1$, and $1 \leqslant i \leqslant d$.*

Proof. Since $\widetilde{K}^i(d,t)$ is isomorphic to $\widetilde{K}^j(d,t)$ for $1 \leqslant i \neq j \leqslant d$, we consider $\widetilde{K}^1(d,t)$ in the following.

Suppose $t = 1$. Note that $K(d,1)$ is isomorphic to K_d and $\widetilde{K}^1(d,1)$ is isomorphic to $K_{d+1} - e$ where e is any edge in K_{d+1}. By Corollary 1, K_d and $K_{d+1} - e$ are $(d-4)$ fault-tolerant hamiltonian connected.

Assume that this lemma holds for $K(d,q)$ and $\widetilde{K}^1(d,q)$ for every $1 \leqslant q \leqslant t$. We will claim that $K(d,t+1)$ and $\widetilde{K}^1(d,t+1)$ are also $(d-4)$ fault-tolerant hamiltonian connected.

First, we show that $K(d,t)$ is $(d-4)$ fault-tolerant hamiltonian connected. Since $K(d,t)$ is hamiltonian connected, $K(4,t)$ is 0 fault-tolerant hamiltonian connected. Thus, we assume that $d \geqslant 5$. Let u be a vertex in $K_i(d,t)$, v be a vertex in $K_j(d,t)$ with $u \neq v$ and F be the faulty set with $|F| \leqslant d-4$. We need to find a hamiltonian path of $K(d,t+1) - F$ joining u to v.

Case A1: $i \neq j$. Assume that there exists a hamiltonian path $\langle i = w_1, w_2, \ldots, w_d = j\rangle$ of $H(F)$ joining i and j that meet the boundary conditions: $u \neq o_{i,w_2}$ and $o_{j,w_{d-1}} \neq v$. By Lemma 1, there exists a hamiltonian path of $K(d,t+1) - F$ joining u and v. By Theorem 2, such hamiltonian path in $H(F)$ that meets the boundary conditions except the cases C2, C3, C4, and C5 of Theorem 2. We will show that this lemma holds for $H(F)$ is isomorphic to $K_5 - (1,2)$, the subgraph N of $H(F)$ induced by $V(H(F)) - \{i,j\}$ is a complete graph; isomorphic to $C_{2,5}$; isomorphic to $C_{1,n-2}$. Since the proof of this part is tedious, we leave this part in Appendix A.

Case A2: $i = j$. Without loss of generality, we assume that $i = j = 1$. Let $A = K_1(d,t) \cup \{(o_{1,r}, o_{r,1}) | 2 \leqslant r \leqslant d\}$, $B = \{o_{r,1} | 2 \leqslant r \leqslant d\}$, and $C = K(d,t+1) - A - B$. We set $F_A = F \cap A, F_B = F \cap B$, and $F_C = F \cap C$.

Suppose that $|F_A| > 0$ or $|F_B| > 0$. We consider the graph $\widetilde{K}_1^1(d,t)$. Let $F' = F_A \cup \{(o_{1,r}, x) | o_{r,1} \in F \text{ or } (o_{1,r}, o_{r,1}) \in F$ for $2 \leqslant r \leqslant d\}$. Obviously, $|F'| \leqslant d-4$. By induction on $\widetilde{K}_1^1(d,t)$, there exists a hamiltonian path P_1 of $\widetilde{K}_1^1(d,t) - F'$ joining u to v. Thus, path P_1 can be written as $\langle u, P_{11}, o_{1,a}, x, o_{1,b}, P_{12}, v\rangle$. Since $|F_A| > 0$ or $|F_B| > 0, |F_C| \leqslant d-5$. Therefore, $H(F) - \{1\}$ is hamiltonian connected. There exists a hamiltonian path $\langle a = w_1, \ldots, w_{d-1} = b\rangle$ of $H(F) - \{1\}$ joining vertex a to vertex b. By Lemma 2, there is a hamiltonian path Q of $K_I(d,t+1) - F$ joining $o_{a,1}$ to $o_{b,1}$ where $I = \{w_1, \ldots, w_{d-1}\}$. Hence, hamiltonian path $\langle u, P_{11}, o_{1,a}, o_{a,1}, Q, o_{b,1}, o_{1,b}, P_{12}, v\rangle$ be the required path.

Suppose that $|F_A| = 0$ and $|F_B| = 0$. Hence, $|F_C| \leqslant d-4$. Therefore, $H(F) - \{1\}$ is hamiltonian. Let $\langle w_1, \ldots, w_{d-1}, w_1\rangle$ be the hamiltonian cycle in $H(F) - \{1\}$ with $|\{o_{1,w_1}, o_{1,w_{d-1}}\} \cap \{u, v\}| = 0$. By Lemma 3, there exist two disjoint paths R and S such that (1) R joins u to one of the vertex in $\{o_{1,w_1}, o_{1,w_{d-1}}\}$, say o_{1,w_1}, (2) S joins v to $o_{1,w_{d-1}}$, and (3) $R \cup S$ spans $K_1(d,t)$. By Lemma 2, there is a hamiltonian path P of $K_I(d,t) - F$ joining $o_{w_1,1}$ to $o_{w_{d-1},1}$ where $I = \{w_1, \ldots, w_{d-1}\}$. Hence, hamiltonian path $\langle u, R, o_{1,w_1}, P, o_{1,w_{d-1}}, S, v\rangle$ be the required path.

Second, we show that $\widetilde{K}^1(d,t)$ is $(d-4)$ fault-tolerant hamiltonian connected for $d \geqslant 4$ and $t \geqslant 2$. Let u and v be any two distinct vertices in $\widetilde{K}^1(d,t)$ and F be the faulty set with $|F| \leqslant d-4$. We want to show that there exists a hamiltonian path of $\widetilde{K}^1(d,t) - F$ joining u to v.

We construct graph $\widetilde{H}^1(F)$ by setting $V(\widetilde{H}^1(F)) = V(H(F)) \cup \{0\}$ and $E(\widetilde{H}^1(F)) = E(H(F)) \cup \{(r,0) | o_{r,0} \notin F$ where $2 \leqslant r \leqslant d\}$. Obviously, $\widetilde{H}^1(F)$ is hamiltonian connected.

Case B1: $u \in K_i(d,t)$ and $v \in K_j(d,t)$ with $i \neq j$. Assume that there exists a hamiltonian path $\langle i = w_1, w_2, \ldots, w_{d+1} = j\rangle$ of $\widetilde{H}^1(F)$ joining i and j that meet the boundary conditions: $u \neq o_{i,w_2}$ and $o_{j,w_d} \neq v$. By Lemma 1, there exists a hamiltonian path of $K(d,t+1) - F$ joining u and v. By Theorem 2, such hamiltonian path in $\widetilde{H}^1(F)$ that meets the boundary conditions except the cases C2, C3, C4, and C5 of Theorem 2. We will show that this lemma holds for $\widetilde{H}^1(F)$ is isomorphic to $K_5 - e$ where e is any edge in K_5, the subgraph N of $\widetilde{H}^1(F)$ induced by $V(\widetilde{H}^1(F)) - \{i,j\}$ is a complete graph; isomorphic to $C_{2,5}$; isomorphic to $C_{1,n-2}$. Since the proof of this part is tedious, we leave this part in Appendix B.

Case B2: $u \in K_i(d,t)$ and v is the vertex x. Since $|F| \leqslant d-4, |F \cap K(d,t+1)| \leqslant d-4$ and there exists at least one vertex $o_{r,x}$ with $\{o_{r,x}, (o_{r,x}, x)\} \cap F = \emptyset$. With above proof, $K(d,t+1) - F$ is hamiltonian connected. There exists a hamiltonian path P_1 of $K(d,t+1) - F$ joining u to $o_{r,x}$. Hence, hamiltonian path $\langle u, P_1, o_{r,x}, x = v\rangle$ be the required path.

Case B3: $u, v \in K_i(d,t)$. Let $A = K_i(d,t) \cup \{(o_{i,r}, o_{r,i}) | 1 \leqslant r \neq i \leqslant d\}$, $B = \{o_{r,i} | 1 \leqslant r \neq i \leqslant d\}$, and $C = K(d,t+1) - \{(o_{i,x}, x)\} - A - B$. We set $F_A = F \cap A, F_B = F \cap B$, and $F_C = F \cap C$.

Suppose that $|F_A| > 0$ or $|F_B| > 0$. Consider the graph $\widetilde{K}_i^i(d,t)$. Let $F' = F_A \cup \{(o_{i,r}, x) | o_{r,i} \in F \text{ or } (o_{i,r}, o_{r,i}) \in F \text{ for } 1 \leqslant r \neq i \leqslant d\}$. Obviously, $|F'| \leqslant d-4$. By induction on $\widetilde{K}_i^i(d,t)$, there exists a hamiltonian path P_i of $\widetilde{K}_i^i(d,t) - F'$ joining u to v. Path P_i can be written as $\langle u, P_{i1}, o_{i,a}, x, o_{i,b}, P_{i2}, v\rangle$. Since $|F_A| > 0$ or $|F_B| > 0, |F_C| \leqslant d-4$. Therefore, $\widetilde{H}^1(F) - \{i\}$ is hamiltonian connected. There exists a hamiltonian path $\langle a = w_1, \ldots, b = w_d\rangle$ of $\widetilde{H}^1(F) - \{i\}$ joining vertex a to vertex b. By Lemma 2, there is a hamiltonian path Q of $K_I(d, t+1) - F$ joining $o_{a,i}$ to $o_{b,i}$ where $I = \{w_1, \ldots, w_d\}$. Hence, hamiltonian path $\langle u, P_{i1}, o_{i,a}, o_{a,i}, Q, o_{b,i}, o_{i,b}, P_{i2}, v\rangle$ be the required path.

Suppose that $|F_A| = 0$ and $|F_B| = 0$. Hence, $|F_C| \leqslant d-3$. Therefore, we know that $\widetilde{H}^1(F) - \{i\}$ is hamiltonian. Let $\langle w_1, \ldots, w_d, w_1\rangle$ be the hamiltonian cycle in $\widetilde{H}^1(F) - \{i\}$ with $|\{o_{i,w_1}, o_{i,w_d}\} \cap \{u, v\}| = 0$. By Lemma 3, there exist two disjoint paths R and S such that (1) R joins u to one of the vertex in $\{o_{i,w_1}, o_{i,w_d}\}$, say o_{i,w_1}, (2) S joins v to o_{i,w_d}, and (3) $R \cup S$ spans $K_i(d,t)$. By Lemma 2, there is a hamiltonian path P of $K_I(d,t) - F$ joining $o_{w_1,i}$ to $o_{w_d,i}$ where $I = \{w_1, \ldots, w_d\}$. Hence, hamiltonian path $\langle u, R, o_{i,w_1}, P, o_{i,w_d}, S, v\rangle$ be the required path. □

Theorem 3. *$\mathcal{H}_f^\kappa(K(d,t)) = d-4$ for $d \geqslant 4$ and $t \geqslant 1$.*

Proof. Since $\delta(K(d,t)) = d-1, \mathcal{H}_f^\kappa(K(d,t)) \leqslant d-4$. By Lemma 4, $K(d,t)$ is $(d-4)$ fault-tolerant hamiltonian connected. Therefore, $\mathcal{H}_f^\kappa(K(d,t)) = d-4$ for $d \geqslant 4$ and $t \geqslant 1$. □

Appendix A. Remaining part of Case A1 in Lemma 4

Subcase A1.1: $H(F)$ is isomorphic to the complete graph $K_5 - (1,2)$ and $\{i,j\} \neq \{1,2\}$. Obviously, $d = 5$ and $|F| = 1$. Thus, exactly one of $o_{1,2}, o_{2,1}$, or $(o_{1,2}, o_{2,1})$ is fault. By the symmetric property of $H(F)$, we may assume that $(i,j) = (1,3)$ or $(i,j) = (5,3)$.

(i) $(i,j) = (1,3)$. Obviously, $\langle i,5,4,2,j\rangle$ and $\langle i,4,2,5,j\rangle$ form two orthogonal hamiltonian paths of $H(F)$. By Lemma 1, we can construct a hamiltonian path between u and v of $K(d,t+1) - F$ unless (1) ($u = o_{i,4}$ and $v = o_{j,2}$) or (2) ($u = o_{i,5}$ and $v = o_{j,5}$).
Suppose that $u = o_{i,4}$ and $v = o_{j,2}$. Obviously, $\langle i,5,2,4,j\rangle$ is a hamiltonian path in $H(F)$ satisfying the boundary conditions: $u \neq o_{i,5}$ and $v \neq o_{j,4}$. Suppose that $u = o_{i,5}$ and $v = o_{j,5}$. Since exactly one of $o_{1,2}, o_{2,1}$, or $(o_{1,2}, o_{2,1})$ is fault, $K_j(d,t) - \{o_{j,5}\}$ is hamiltonian connected. Let P_j be the hamiltonian path of $K_j(d,t) - \{o_{j,5}\}$ joining $o_{j,4}$ to $o_{j,2}$. By induction, $K_q(d,t) - F$ is hamiltonian connected for $q \in \{i,5,4,2\}$. Let P_i be the hamiltonian path of $K_i(d,t) - F$ joining u to $o_{i,4}$; let P_5 be the hamiltonian path of $K_5(d,t)$ joining $o_{5,2}$ to $o_{5,j}$; let P_4 be the hamiltonian path of $K_4(d,t)$ joining $o_{4,i}$ to $o_{4,j}$; let P_2 be the hamiltonian path of $K_2(d,t)$ joining $o_{2,j}$ to $o_{2,5}$. Therefore, path $\langle u, P_i, P_4, P_j, P_2, P_5, v\rangle$ is the required path.

(ii) $(i,j) = (5,3)$. Obviously, $\langle i,1,4,2,j\rangle$ and $\langle i,2,4,1,j\rangle$ form two orthogonal hamiltonian paths of $H(F)$. By Lemma 1, we can construct a hamiltonian path between u and v of $K(d,t+1) - F$ unless (1) ($u = o_{i,1}$ and $v = o_{j,1}$) or (2) ($u = o_{i,2}$ and $v = o_{j,2}$).
The case (1) is similar to (2). We consider (1) only. Let $u = o_{i,1}$ and $v = o_{j,1}$. Since exactly one of $o_{1,2}$, $o_{2,1}$, or $(o_{1,2}, o_{2,1})$ is fault, $K_j(d,t) - \{o_{j,1}\}$ is hamiltonian connected. Let P_j be the hamiltonian path of $K_j(d,t) - \{o_{j,1}\}$ joining $o_{j,i}$ to $o_{j,2}$. By induction, $K_q(d,t) - F$ is hamiltonian connected for $q \in \{i,1,2,4\}$. Let P_i be the hamiltonian path of $K_i(d,t)$ joining u to $o_{i,j}$; let P_1 be the hamiltonian path of $K_1(d,t)$ joining $o_{1,4}$ to $o_{1,j}$; let P_4 be the hamiltonian path of $K_4(d,t)$ joining $o_{4,2}$ to $o_{4,1}$; let P_2 be the hamiltonian path of $K_2(d,t)$ joining $o_{2,j}$ to $o_{2,4}$. Therefore, path $\langle u, P_i, P_j, P_2, P_4, P_1, v\rangle$ is the required path.

Subcase A1.2: The subgraph N of $H(F)$ induced by $V(H(F)) - \{i,j\}$ is a complete graph; vertex i is adjacent to j and all the vertices in N; j is adjacent to i and exactly two vertices 1 and 2 in N. We label the remaining vertices in N as $3, \ldots, d-2$. See Fig. 5(a) for illustration. It is easy to see that $\langle i,2,3,\ldots,d-2,1,j\rangle$ and $\langle i,3,\ldots,d-2,1,2,j\rangle$ form two orthogonal hamiltonian paths of $H(F)$ between i and j. By Lemma 1, we can construct a hamiltonian path of $K(d,t+1) - F$ joining u to v unless (1) ($u = o_{i,2}$ and $v = o_{j,2}$) or (2) ($u = o_{i,3}$ and $v = o_{j,1}$).
Suppose that $u = o_{i,2}$ and $v = o_{j,2}$. Obviously, $\langle i,3,2,4,\ldots,d-2,1,j\rangle$ is a hamiltonian path in $H(F)$ satisfying the boundary conditions: $u \neq o_{i,3}$ and $v \neq o_{j,1}$. Suppose that $u = o_{i,3}$ and $v = o_{j,1}$. Thus, the hamiltonian path $\langle i,1,d-2,\ldots,3,2,j\rangle$ in $H(F)$ satisfying the boundary conditions: $u \neq o_{i,1}$ and $v \neq o_{j,2}$. By Lemma 1, we can construct a hamiltonian path of $K(d,t+1) - F$ joining u to v

Subcase A1.3: The subgraph N of $H(F)$ induced by $V(H(F)) - \{i,j\}$ is isomorphic to $C_{2,5}$; vertex i is adjacent to j and all the vertices in N; j is adjacent to i and all the vertices in N. We label the vertices in N as $1,2,\ldots,5$. See Fig. 5(b) for illustration. Thus, $d = 6$ and $|F| = 3$. Obviously, $|F_i| = |F_j| = 0$. Moreover, $\langle i,1,2,3,4,5,j\rangle$ and $\langle i,5,4,3,2,1,j\rangle$ form two orthogonal hamiltonian paths of $H(F)$ between i and j. By Lemma 1, we can construct a hamiltonian path of $K(d,t+1)$ joining u to v unless (1) ($u = o_{i,1}$ and $v = o_{j,1}$) or (2) ($u = o_{i,5}$ and $v = o_{j,5}$). By the symmetric property of $H(F)$, we only consider the case $u = o_{i,1}$ and $v = o_{j,1}$.

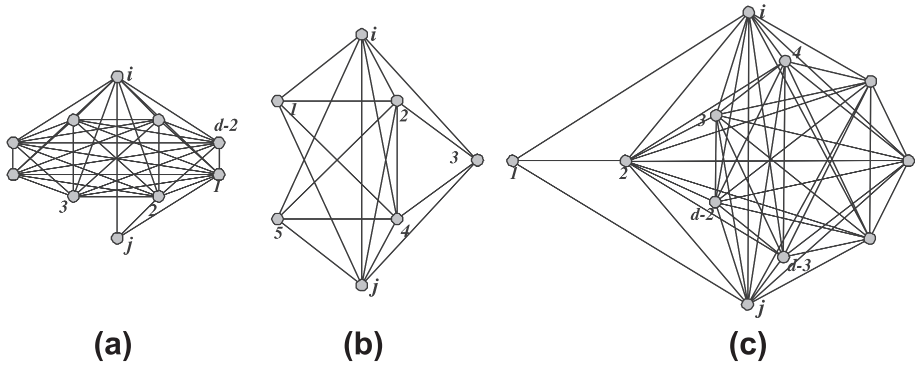

Fig. 5. Illustrations for Appendix A, (a) Subcase A1.2, (b) Subcase A1.3, and (c) Subcase A1.4.

Since $|F_i| = |F_j| = 0, K_j(d,t) - (F_j \cup \{o_{j,1}\})$ is hamiltonian connected. Let P_j be the hamiltonian path of $K_j(d,t) - (F_j \cup \{o_{j,1}\})$ joining $o_{j,5}$ to $o_{j,3}$. By induction, $K_q(d,t) - F_q$ is hamiltonian connected for $q \in \{i, 1, \ldots, 5\}$. Let P_i be the hamiltonian path of $K_i(d,t) - F_i$ joining u to $o_{i,2}$; let P_1 be the hamiltonian path of $K_1(d,t) - F_1$ joining $o_{1,4}$ to $o_{1,j}$; let P_2 be the hamiltonian path of $K_2(d,t) - F_2$ joining $o_{2,i}$ to $o_{2,5}$; let P_3 be the hamiltonian path of $K_3(d,t) - F_3$ joining $o_{3,j}$ to $o_{3,4}$; let P_4 be the hamiltonian path of $K_4(d,t) - F_4$ joining $o_{4,3}$ to $o_{4,1}$; let P_5 be the hamiltonian path of $K_5(d,t) - F_5$ joining $o_{5,2}$ to $o_{5,j}$. Therefore, path $\langle u, P_i, P_2, P_5, P_j, P_3, P_4, P_1, v\rangle$ is the required path.

Subcase A1.4: The subgraph N of $H(F)$ induced by $V(H(F)) - \{i,j\}$ is isomorphic to $C_{1,n-2}$; vertex i is adjacent to j and all the vertices in N; j is adjacent to i and all the vertices in N. We label the vertices in N as $1, 2, \ldots, d-2$. See Fig. 5(c) for illustration. Obviously, $\langle i, 1, 2, \ldots, d-2, j\rangle$ and $\langle i, d-2, d-3, \ldots, 1, j\rangle$ form two orthogonal hamiltonian paths of $H(F)$ between i and j. By Lemma 1, we can construct a hamiltonian path of $K(d, t+1)$ joining u to v unless (1) ($u = o_{i,d-2}$ and $v = o_{j,d-2}$) or (2) ($u = o_{i,1}$ and $v = o_{j,1}$).

Suppose that $u = o_{i,d-2}$ and $v = o_{j,d-2}$. Obviously, $\langle i, 3, d-2, \ldots, 4, 2, 1, j\rangle$ is a hamiltonian path in $H(F)$ satisfying the boundary conditions: $u \neq o_{i,3}$ and $v \neq o_{j,1}$. By Lemma 1, we can construct a hamiltonian path of $K(d, t+1)$ joining u to v.

Suppose that $u = o_{i,1}$ and $v = o_{j,1}$. Since $|F_i| = |F_j| = 0, K_j(d,t) - \{o_{j,1}\}$ is hamiltonian connected. Let P_j be the hamiltonian path of $K_j(d,t) - \{o_{j,1}\}$ joining $o_{j,3}$ to $o_{j,4}$. By induction, $K_q(d,t) - F_q$ is hamiltonian connected for $q \in \{i, 2, \ldots, d-2\}$. Let P_i be the hamiltonian path of $K_i(d,t)$ joining u to $o_{i,3}$; let P_3 be the hamiltonian path of $K_3(d,t) - F_3$ joining $o_{3,i}$ to $o_{3,j}$; let P_4 be the hamiltonian path of $K_4(d,t) - F_4$ joining $o_{4,j}$ to $o_{4,5}$ if $d \geqslant 7$ and let P_4 be the hamiltonian path of $K_4(d,t) - F_4$ joining $o_{4,j}$ to $o_{4,2}$ if $d = 6$; let P_q be a hamiltonian path of $K_q(d,t) - F_q$ joining $o_{q,q-1}$ to $o_{q,q+1}$ for $4 \leqslant q \leqslant d-3$; let P_{d-2} be the hamiltonian path of $K_{d-2}(d,t) - F_{d-2}$ joining $o_{d-2,d-3}$ to $o_{d-2,2}$; let P_2 be a hamiltonian path of $K_2(d,t) - F_2$ joining $o_{2,d-2}$ to $o_{2,1}$; let P_1 be a hamiltonian path of $K_1(d,t) - F_1$ joining $o_{1,2}$ to $o_{1,j}$. Therefore, $\langle u, P_i, P_3, P_j, P_4, \ldots, P_{d-2}, P_2, P_1, v\rangle$ is the required path.

Appendix B. Remaining part of Case B1 in Lemma 4

In these cases, $|F| = |\overline{E(\widetilde{H}^1(F))}| = d-4$. Thus, F contains exactly one of $o_{\alpha,\beta}$, $o_{\beta,\alpha}$, or $(o_{\alpha,\beta}, o_{\beta,\alpha})$ if $(\alpha,\beta) \notin E(\widetilde{H}^1(F))$.

Subcase B1.1: $\widetilde{H}^1(F)$ is isomorphic to the complete graph $K_5 - e$ for any edge e in K_5. We label the vertices in K_5 as $0, 1, \ldots, 4$. See Fig. 6(a) for illustration. By the definition on $\widetilde{H}^1(F)$, $\{i,j\} \neq \{0,1\}$. Obviously, $d = 4$ and $|F| = 0$. By the symmetric property of $\widetilde{H}^1(F)$, we may assume that $(i,j) = (1,2)$ or $(i,j) = (4,2)$.

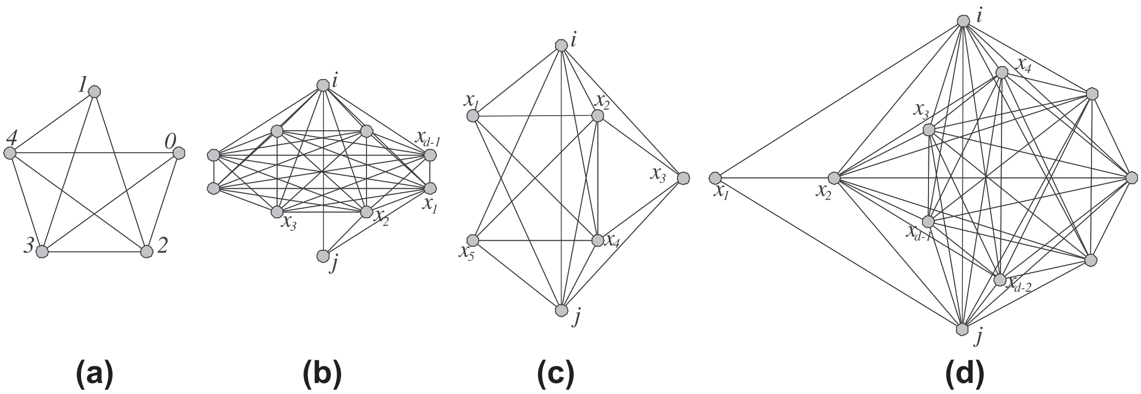

Fig. 6. Illustrations for Appendix B, (a) Subcase B1.1, (b) Subcase B1.2, (c) Subcase B1.3, and (d) Subcase B1.4.

(i) $(i,j)=(1,2)$. Obviously, $\langle i,4,0,3,j\rangle$ and $\langle i,3,0,4,j\rangle$ form two orthogonal hamiltonian paths of $\widetilde{H}^1(F)$. By Lemma 1, we can construct a hamiltonian path between u and v of $\widetilde{K}^1(d,t+1)-F$ unless (1) ($u=o_{i,4}$ and $v=o_{j,4}$) or (2) ($u=o_{i,3}$ and $v=o_{j,3}$).
Suppose that $u=o_{i,4}$ and $v=o_{j,4}$. Obviously, $\langle i,3,4,0,j\rangle$ is a hamiltonian path in $\widetilde{H}^1(F)$ satisfying the boundary conditions: $u\neq o_{i,3}$ and $v\neq o_{j,0}$. Suppose that $u=o_{i,3}$ and $v=o_{j,3}$. Obviously, $\langle i,4,3,0,j\rangle$ is a hamiltonian path in $\widetilde{H}^1(F)$ satisfying the boundary conditions: $u\neq o_{i,4}$ and $v\neq o_{j,0}$.

(ii) $(i,j)=(4,2)$. Obviously, $\langle i,0,3,1,j\rangle$ and $\langle i,1,3,0,j\rangle$ form two orthogonal hamiltonian paths of $\widetilde{H}^1(F)$. By Lemma 1, we can construct a hamiltonian path between u and v of $\widetilde{K}^1(d,t+1)-F$ unless (1) ($u=o_{i,0}$ and $v=o_{j,0}$) or (2) ($u=o_{i,1}$ and $v=o_{j,1}$).
Suppose that $u=o_{i,0}$ and $v=o_{j,0}$. Let P_i be the hamiltonian path of $K_i(d,t)-\{u\}$ joining $o_{i,3}$ to $o_{i,1}$; let P_3 be the hamiltonian path of $K_3(d,t)$ joining $o_{3,0}$ to $o_{3,2}$; let P_1 be the hamiltonian path of $K_1(d,t)$ joining $o_{1,2}$ to $o_{1,4}$; P_j be the hamiltonian path of $K_j(d,t)$ joining $o_{j,1}$ to v. Therefore, path $\langle u,x,P_3,P_i,P_1,P_j,v\rangle$ is the required path. Suppose that $u=o_{i,1}$ and $v=o_{j,1}$. Let P_i be the hamiltonian path of $K_i(d,t)-\{u\}$ joining $o_{i,3}$ to $o_{i,0}$; let P_3 be the hamiltonian path of $K_3(d,t)$ joining $o_{3,1}$ to $o_{3,2}$; let P_1 be the hamiltonian path of $K_1(d,t)$ joining $o_{1,i}$ to $o_{1,3}$; P_j be the hamiltonian path of $K_j(d,t)$ joining $o_{j,0}$ to v. Therefore, path $\langle u,P_1,P_3,P_i,x,P_j,v\rangle$ is the required path.

Subcase B1.2: The subgraph N of $\widetilde{H}^1(F)$ induced by $V(\widetilde{H}^1(F))-\{i,j\}$ is a complete graph; vertex i is adjacent to j and all vertices in N; j is adjacent to i and exactly two vertices, say x_1 and x_2, in N. Since $\deg_{\widetilde{H}^1(F)}(0)<d$, x_1 or x_2 is not vertex 0. We label the remaining vertices in N as $x_3,\ldots,x_{d-1}$. See Fig. 6(b) for illustration. It is easy to see that $\langle i,x_2,x_3,\ldots,x_{d-1},x_1,j\rangle$ and $\langle i,x_3,\ldots,x_{d-1},x_1,x_2,j\rangle$ form two orthogonal hamiltonian paths of $\widetilde{H}^1(F)$ between i and j. By Lemma 1, we can construct a hamiltonian path between u and v of $\widetilde{K}^1(d,t+1)-F$ unless (1) ($u=o_{i,x_2}$ and $v=o_{j,x_2}$) or (2) ($u=o_{i,x_3}$ and $v=o_{j,x_1}$).

Suppose that $u=o_{i,x_2}$ and $v=o_{j,x_2}$. Obviously, $\langle i,x_3,x_2,x_4,\ldots,x_{d-1},x_1,j\rangle$ is a hamiltonian path in $\widetilde{H}^1(F)$ satisfying the boundary conditions: $u\neq o_{i,x_3}$ and $v\neq o_{j,x_1}$. Suppose that $u=o_{i,x_3}$ and $v=o_{j,x_1}$. The hamiltonian path $\langle i,x_1,x_{d-1},\ldots,x_3,x_2,j\rangle$ in $\widetilde{H}^1(F)$ satisfying the boundary conditions: $u\neq o_{i,x_1}$ and $v\neq o_{j,x_2}$. By Lemma 1, we can construct a hamiltonian paths between u and v of $\widetilde{K}^1(d,t+1)-F$.

Subcase B1.3: The subgraph N of $\widetilde{H}^1(F)$ induced by $V(\widetilde{H}^1(F))-\{i,j\}$ is isomorphic to $C_{2,5}$; vertex i is adjacent to j and all the vertices in N; j is adjacent to i and all the vertices in N. We label the vertices of $C_{2,5}$ as indicated in Fig. 6(c). Obviously, $d=6$ and $|F|=2$. Thus, $|E(\overline{N})|=3$ and $\deg_{\overline{N}}(x_1)=\deg_{\overline{N}}(x_3)=\deg_{\overline{N}}(x_5)=2$. It is easy to see that $\langle i,x_1,x_2,x_3,x_4,x_5,j\rangle$ and $\langle i,x_5,x_4,x_3,x_2,x_1,j\rangle$ form two orthogonal hamiltonian paths of $\widetilde{H}^1(F)$ between i and j. By Lemma 1, we can construct a hamiltonian paths between u and v of $\widetilde{K}^1(d,t+1)-F$ unless (1) ($u=o_{i,x_1}$ and $v=o_{j,x_1}$) or (2) ($u=o_{i,x_5}$ and $v=o_{j,x_5}$). By the symmetric property of $\widetilde{H}^1(F)$, we only consider the case $u=o_{i,x_1}$ and $v=o_{j,x_1}$.

Since $|F_i|=|F_j|=0$, $K_j(d,t)-(F_j\cup\{o_{j,x_1}\})$ is hamiltonian connected. Let P_j be the hamiltonian path of $K_j(d,t)-(F_j\cup\{o_{j,x_1}\})$ joining o_{j,x_5} to o_{j,x_3}. Let l be the index that x_l is vertex 0 in N. By induction, $K_q(d,t)-F_q$ is hamiltonian connected for $q\in\{i,x_1,\ldots,x_5\}-\{x_l\}$. Let P_i be the hamiltonian path of $K_i(d,t)-F_i$ joining u to o_{i,x_2}; let P_1 be the hamiltonian path of $K_1(d,t)-F_1$ joining o_{x_1,x_4} to $o_{x_1,j}$ if $l\neq 1$ and $P_1=\{x\}$ if otherwise; let P_2 be the hamiltonian path of $K_2(d,t)-F_2$ joining $o_{x_2,i}$ to o_{x_2,x_5}; let P_3 be the hamiltonian path of $K_3(d,t)-F_3$ joining $o_{x_3,j}$ to o_{x_3,x_4} if $l\neq 3$ and $P_3=\{x\}$ if otherwise; let P_4 be the hamiltonian path of $K_4(d,t)-F_4$ joining o_{x_4,x_3} to o_{x_4,x_1}; let P_5 be the hamiltonian path of $K_5(d,t)-F_5$ joining o_{x_5,x_2} to $o_{x_5,j}$ if $l\neq 5$ and $P_5=\{x\}$ if otherwise. Therefore, path $\langle u,P_i,P_2,P_5,P_j,P_3,P_4,P_1,v\rangle$ is the required path.

Subcase B1.4: The subgraph N of $\widetilde{H}^1(F)$ induced by $V(\widetilde{H}^1(F))-\{i,j\}$ is isomorphic to $C_{1,n-2}$; vertex i is adjacent to j and all the vertices in N; j is adjacent to i and all the vertices in N. We label the vertices of $C_{1,n-2}$ as indicated in Fig. 6(d). Obviously, $E(\overline{N})=d-3$ and $\deg_{\overline{N}}(1)=d-3$. It is easy to see that $\langle i,x_1,x_2,\ldots,x_{d-1},j\rangle$ and $\langle i,x_{d-1},x_{d-2},\ldots,x_1,j\rangle$ form two orthogonal hamiltonian paths of $\widetilde{H}^1(F)$ between i and j. By Lemma 1, we can construct a hamiltonian paths between u and v in $\widetilde{K}^1(d,t+1)-F$ unless (1) ($u=o_{i,x_{d-1}}$ and $v=o_{j,x_{d-1}}$) or (2) ($u=o_{i,x_1}$ and $v=o_{j,x_1}$).

Suppose that $u=o_{i,x_{d-1}}$ and $v=o_{j,x_{d-1}}$. Obviously, $\langle i,x_3,x_{d-1},\ldots,x_4,x_2,x_1,j\rangle$ is a hamiltonian path in $\widetilde{H}^1(F)$ satisfying the boundary conditions: $u\neq o_{i,x_3}$ and $v\neq o_{j,x_1}$. By Lemma 1, we can construct a hamiltonian paths between u and v in $\widetilde{K}^1(d,t+1)-F$.

Suppose that $u=o_{i,x_1}$ and $v=o_{j,x_1}$. Since $|F_i|=|F_j|=0$, $K_j(d,t)-\{o_{j,x_1}\}$ is hamiltonian connected. Let P_j be the hamiltonian path of $K_j(d,t)-\{o_{j,x_1}\}$ joining o_{j,x_3} to o_{j,x_4}. Let l be the index that x_l is vertex 0 in N. By induction, $K_q(d,t)-F_q$ is hamiltonian connected for $q\in\{i,x_2,\ldots,x_{d-1}\}-\{x_l\}$. Let P_i be the hamiltonian path of $K_i(d,t)$ joining u to o_{i,x_3}; let P_3 be the hamiltonian path of $K_3(d,t)-F_3$ joining $o_{x_3,i}$ to $o_{x_3,j}$ if $l\neq 3$ and $P_3=\{x\}$ if otherwise. Suppose $l\neq 4$, let P_4 be the hamiltonian path of $K_4(d,t)-F_4$ joining $o_{x_4,j}$ to o_{x_4,x_5} if $d\geqslant 7$ and let P_4 be the hamiltonian path of $K_4(d,t)-F_4$ joining $o_{x_4,j}$ to o_{x_4,x_2} if $d=6$. Suppose $l=4$, let $P_4=\{x\}$. Let P_q be a hamiltonian path of $K_q(d,t)-F_q$ joining $o_{x_q,x_{q-1}}$ to $o_{x_q,x_{q+1}}$ for $4\leqslant q\leqslant d-2$ if $l\neq q$ and $P_q=\{x\}$ if otherwise; let P_{d-1} be the hamiltonian path of $K_{d-1}(d,t)-F_{d-1}$ joining $o_{x_{d-1},x_{d-2}}$ to o_{x_{d-1},x_2} if $l\neq d-1$ and $P_{d-1}=\{x\}$ if otherwise; let P_2 be a hamiltonian path of $K_2(d,t)-F_2$ joining $o_{x_2,x_{d-1}}$ to o_{x_2,x_1}; let P_1 be a hamiltonian path of $K_1(d,t)-F_1$ joining o_{x_1,x_2} to $o_{x_1,j}$ if $l\neq 1$ and $P_1=\{x\}$ if otherwise. Therefore, path $\langle u,P_i,P_3,P_j,P_4,\ldots,P_{d-1},P_2,P_1,v\rangle$ is the required path.

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