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A Unified Approach to a Characterization of Grassmann Graphs and Bilinear Forms Graphs

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Wilbrink and Brouwer [18] proved that certain semi-partial geometries with some weak restrictions on parameters satisfy the dual of Pasch's axiom. Inspired by their work, a class of incidence structures associated with distance-regular graphs with classical parameters is studied in this paper. As a consequence, the Grassmann graphs and the bilinear forms graphs are characterized simultaneously among distance-regular graphs with classical parameters, together with some extra geometric conditions.

1. INTRODUCTION

By a graph Γ, we shall mean a finite, simple and undirected graph. For x in $V(\Gamma)$, the vertex set of Γ, let $\Gamma_i(x) = \{y \in V(\Gamma) \mid \partial(x, y) = i\}$, where $\partial(x, y)$ is the distance between x and y. If $A, B \subseteq V(\Gamma)$, $\partial(A, B)$ is defined to be the minimum of $\partial(a, b)$ for all $a \in A$ and $b \in B$. A distance-regular graph Γ of diameter d is one for which the parameters $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$, $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ and $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ depend only on the distance $i = \partial(x, y)$. It is clear that $a_i = b_0 - b_i - c_i$. The sequence $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ. Most distance-regular graphs related to classical groups and groups of Lie type have an intersection array the parameters of which can be expressed in terms of the diameter d and three other parameters q, α and β, called the *classical parameters*, as follows:

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right), \qquad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \tag{1a, b}$$

($i = 0, 1, \ldots, d$), where $\left[{n \atop k}\right]$ denotes the Gaussian coefficient with basis q (for $q = 1$, it is the ordinary binomial coefficient). Clearly,

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(\beta - 1 + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \tag{1c}$$

($i = 0, 1, \ldots, d$). Furthermore, the corresponding eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$ can be calculated in terms of the intersection array as follows:

$$\theta_i = \begin{bmatrix} d-i \\ 1 \end{bmatrix}\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) - \begin{bmatrix} i \\ 1 \end{bmatrix} \tag{2}$$

($i = 0, 1, \ldots, d$). Refer to [4, Chapters 6 and 8] for more details.

The Grassmann graphs $J_q(n, d)$ (with parameters $(\alpha, \beta) = (q, \left[{n-d+1 \atop 1}\right] - 1)$) and the bilinear forms graphs $H_q(n, d)$ (with parameters $(\alpha, \beta) = (q-1, q^n - 1)$), defined below, are two typical examples in this family. Distance-regular graphs with classical parameters (d, q, α, β) and $q = 1$ have been characterized by Neumaier and Terwilliger [4, Theorem 6.1.1]. We consider the case of $q \geq 4$ here. Our main purpose in this paper is to prove the following theorem:

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MAIN THEOREM. *Let Γ be a distance-regular graph of diameter $d \geqslant 3$, with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ as in (1a, b) and such that:*
(A.1) $\beta > \alpha[{d \atop 1}]$;
(A.2) *every singular line of Γ has size at least $\alpha + 1$, where $\alpha + 1 \geqslant \max\{5, q\}$;*
(A.3) *for any vertex x and maximal clique m, $|\Gamma_1(x) \cap m| \neq 1$.*
Then q is a prime power and one of the following holds:
(1) $(\alpha, \beta) = (q, [{n-d+1 \atop 1}] - 1)$, $n \geqslant 2d+1$, $q \geqslant 4$, *and Γ is isomorphic to the Grassmann graph $J_q(n, d)$; or*
(2) $(\alpha, \beta) = (q-1, q^n - 1)$, $n \geqslant d+1$, $q \geqslant 5$, *and Γ is isomorphic to the bilinear forms graph $H_q(n, d)$.*

The *Grassmann graph* $J_q(n, d)$ (also called the q-analog of the Johnson graph) is defined on the set of all d-dimensional subspaces of an n-dimensional vector space over $GF(q)$. Two vertices x and y are adjacent whenever $\dim(x \cap y) = d - 1$. The *bilinear forms graph* $H_q(n, d)$ (also called the q-analog of the Hamming graph) is defined on the set of all bilinear forms on $W \times V$, where W and V are vector spaces of dimension d and n ($d \leqslant n$), respectively. Two forms e and f are called adjacent if the rank of $e - f$ is 1. With respect to given bases of W and V, the vertex set of $H_q(n, d)$ is identical to the set of all $d \times n$ matrices over $GF(q)$. Indeed, $J_q(n, d)$ and $H_q(n, d)$ are the collinearity graphs of the (d, q, n)-projective incidence structures [11] and the (d, q, n)-attenuated spaces [10, 12], respectively. The *(d, q, n)-projective incidence structure* is the collection of subspaces of the n-dimensional vector space over $GF(q)$ where subspaces of dimension d are called 'points', those of dimension $d-1$ are called 'lines', and incidence is the usual containment. An alternative interpretation can be found in [2]. Let V be an $(n+d)$-dimensional vector space over $GF(q)$ and let W be a given n-dimensional subspace of V. The *(d, q, n)-attenuated space* is the collection of subspaces U of V with $U \cap W = 0$, where subspaces U of dimension d are called 'points', those of dimension $d-1$ are called 'lines', and incidence is the usual containment. The bilinear forms graph $H_q(n, d)$ can be viewed not only as a subgraph but also as a geometric hyperplane of the Grassmann graph $J_q(n+d, d)$.

The Grassmann graphs $J_q(n, d)$ ($n \geqslant 3d \geqslant 9$, $(q, d) \neq (2, n/3)$) have been characterized by Sprague in [13], and the bilinear forms graphs $H_q(n, d)$ ($n \geqslant 2d \geqslant 6$, $q \geqslant 4$) have been characterized by Huang in [10] (see also Cuypers [7] for a later improvement of that characterization). The above two numerical constraints on n and d, interpreted as $\beta \geqslant \alpha[{2d \atop 1}]$ with $d \geqslant 3$ in terms of the classical parameters, were needed in both cases because both Sprague and Huang used the Bose–Laskar argument, and the condition $q \geqslant 4$ is assumed in [10] because of [5]. In this paper, the Bose–Laskar argument is replaced by the following two theorems, obtained by the technique of graph representations, which provide essential information on intersections of maximal cliques of a distance-regular graph simply in terms of the parameters of the graph. As a consequence, the above constraint is partially improved to $\beta > \alpha[{d \atop 1}]$, i.e. $n \geqslant 2d+1$ for $J_q(n, d)$ and $n \geqslant d+1$ for $H_q(n, d)$. However, the assumption $\alpha + 1 \geqslant \max\{5, q\}$ is still needed because the following Theorem A(ii) is used.

For a vertex x of a graph Γ, let $x^\perp = \Gamma_1(x) \cup \{x\}$. If S is a set of vertices, we let $S^\perp = \bigcap_{x \in S} x^\perp$ and $S^{\perp\perp} = (S^\perp)^\perp$. $\{x, y\}^{\perp\perp}$ is called the *singular line* determined by x and y whenever x and y are adjacent.

THEOREM A [4, p. 160]. *Let Γ be a distance-regular graph of diameter at least* 3 *with second largest eigenvalue $\theta_1 < b_1 - 1$, and let $b^+ = b_1/(\theta_1 + 1)$. Suppose that every*

singular line of Γ has size at least $s+1$. Then:
(i) *If $s \geqslant 3$ and $b^+ \leqslant s^2 - s + 1$, or if $s = 2$ and $b^+ \leqslant 2$, then distinct maximal cliques intersect in a singular line, a point or the empty set.*
(ii) *If $s \geqslant 4$ and $b^+ \leqslant s+1$, then every edge is in at most two maximal cliques.*

THEOREM B [4, p. 160]. *Let Γ be a distance-regular graph of diameter $d \geqslant 2$ with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ and put $b^+ = b_1/(\theta_1 + 1)$ and $b^- = b_1/(\theta_d + 1)$. Then the size of a clique C in Γ is bounded by $|C| \leqslant 1 - k/\theta_d$. If equality holds then every vertex $x \notin C$ is adjacent to either 0 or $b^- + 1 - k/\theta_d$ vertices of C. No vertex of Γ has distance d to C.*

Basing ourselves on Theorems A and B, we derive incidence structures from the distance-regular graph considered in the Main Theorem, regarding classes of maximal cliques as lines. It is worth mentioning that, in these incidence structures, any two points at distance 2 have $(q+1)(\alpha+1)$ (i.e. c_2 in (1b)) common neighbors.

Wilbrink and Brouwer [18] proved that certain semi-partial geometries with some weak restrictions on parameters satisfy the dual of Pasch's axiom, and that two intersecting lines generate a subspace that is a partial geometry. Cuypers recognized that their arguments remain valid in a more general setting (see [6, Section 3]). Inspired by the work of Wilbrink and Brouwer [18], together with that of Cuypers [6], we show in Section 2 that the set of maximal cliques of the graph Γ considered in the Main Theorem can be partitioned into two non-empty subsets such that every edge of Γ is contained in exactly one maximal clique of each class. Moreover, maximal cliques from different classes have fixed but different sizes and intersect in 0 or $\alpha + 1$ vertices, where $\alpha + 1$ is either $q + 1$ or q. Both classes of maximal cliques induce structures of semilinear spaces over Γ. Proposition 2.13 and Corollary 2.13.1 include similar results mentioned in [18]. As a consequence, the Main Theorem is proved in Section 3.

2. BASIC GEOMETRIC STRUCTURES

The graphs $J_q(n, d)$ and $H_q(n, d)$ have many properties in common. For example, they both have two classes of maximal cliques; every edge is contained in exactly two maximal cliques that are in different classes. Throughout the rest of this paper, we assume that Γ is a given distance-regular graph of diameter $d \geqslant 3$ satisfying (A.1), (A.2) and (A.3) of the Main Theorem. We shall show in this section that these common properties of $J_q(n, d)$ and $H_q(n, d)$ hold in Γ, recovering them from the classical parameters of Γ.

LEMMA 2.1. (i) *Distinct maximal cliques of Γ intersect in a singular line, a point or the empty set.*
(ii) *Every edge of Γ is contained in exactly two maximal cliques.*

PROOF. For the given graph Γ, each singular line has size at least $\alpha + 1$, $\alpha \geqslant 4$ by (A.2). From (1a) and (2), we have $b_1 = ([{d \atop 1}] - 1)(\beta - \alpha)$, $\theta_1 = [{d-1 \atop 1}](\beta - \alpha) - 1$, and hence $b_1/(\theta_1 + 1) = q \leqslant \alpha + 1$ (by (A.2)). Theorem A shows that (i) holds and that every edge is contained in at most two maximal cliques. Furthermore, Theorem B shows that the size of any maximal clique is bounded above by $1 - b_0/\theta_d = 1 + \beta$. From (1c), $a_1 + 2 = (\beta - 1 + \alpha([{d \atop 1}] - 1)) + 2 > 1 + \beta$; this shows that every edge is contained in at least two maximal cliques. Thus (ii) follows immediately. □

Let l_{xy} and A_{xy} be the two maximal cliques containing the edge xy. Then $l_{xy} \cap A_{xy}$ must be a singular line and $l_{xy} \cup A_{xy} = (\Gamma_1(x) \cap \Gamma_1(y)) \cup \{x, y\}$ by the previous lemma.

LEMMA 2.2. *If $x, y \in V(\Gamma)$ are adjacent, then any vertices $u \in l_{xy} \backslash A_{xy}$ and $v \in A_{xy} \backslash l_{xy}$ are not adjacent.*

PROOF. If u and v are adjacent, then the edge uv is contained in two maximal cliques, say l_{uv} and A_{uv}. Since the singular line $l_{xy} \cap A_{xy}$ ($\subseteq \Gamma_1(u) \cap \Gamma_1(v)$) is a clique, it is contained in at least one of l_{uv} and A_{uv}. Hence the edge xy is in at least three maximal cliques, a contradiction. □

LEMMA 2.3. *Let l_{xy} and A_{xy} be the two maximal cliques containing the edge xy, and let m be a maximal clique different from l_{xy} and A_{xy} with $x \in m$. Then $|m \cap l_{xy}| = 1$ iff $|m \cap A_{xy}| > 1$, i.e. $m \cap A_{xy}$ is a singular line.*

PROOF. If $|m \cap l_{xy}| = 1$, then $m \cap l_{xy} = \{x\}$ and $\partial(y, m) = 1$. Then there exists $u \in \Gamma_1(y) \cap m$ ($u \neq x$) by assumption (A.3). Since $u \in \Gamma_1(x) \cap \Gamma_1(y)$ and $u \notin l_{xy}$, we have $u \in A_{xy}$. It follows that $x, u \in m \cap A_{xy}$, and hence $m \cap A_{xy}$ is a singular line.

Conversely, if $|m \cap A_{xy}| > 1$, let $w \in m \cap A_{xy}$, $w \neq x$. Clearly, $w \notin l_{xy}$; otherwise the edge xw will be contained in three maximal cliques. Similarly, $y \notin m$. Suppose, to the contrary, that $|m \cap l_{xy}| > 1$. Let $v \in m \cap l_{xy}$, $v \neq x$. Then $v \in l_{xy} \backslash A_{xy}$ and $w \in A_{xy} \backslash l_{xy}$ are adjacent (contained in m), a contradiction to Lemma 2.2. □

COROLLARY 2.3.1. *If l_1, l_2 and l_3 are any three distinct maximal cliques with $l_1 \cap l_2 = l_2 \cap l_3 = \{x\}$, then $l_1 \cap l_3 = \{x\}$.*

PROOF. If $|l_1 \cap l_3| > 1$, say $x, y \in l_1 \cap l_3$, then l_1 and l_3 are the two maximal cliques containing the edge xy. By Lemma 2.3, $|l_2 \cap l_1| = 1$ iff $|l_2 \cap l_3| > 1$, a contradiction. □

By Lemma 2.3 and Corollary 2.3.1, for any given $x \in V(\Gamma)$, the set of all maximal cliques containing x can be partitioned into two families in the following way. Let l_x and A_x be two maximal cliques containing the point x, with $l_x \cap A_x$ being a singular line. Let

$$\Sigma_1(x) = \{m \mid m \text{ is a maximal clique containing } x \text{ and } |m \cap l_x| = 1\},$$

$$\Sigma_2(x) = \{m \mid m \text{ is a maximal clique containing } x \text{ and } |m \cap A_x| = 1\}.$$

It is worth mentioning here that this partition is independent of the choice of l_x and A_x. Some properties of $\Sigma_1(x)$ and $\Sigma_2(x)$ are summarized in the following lemma.

LEMMA 2.4. (i) *$l_1 \cap l_2 = \{x\}$ for any $l_1, l_2 \in \Sigma_1(x)$;*
(ii) *$A_1 \cap A_2 = \{x\}$ for any $A_1, A_2 \in \Sigma_2(x)$;*
(iii) *if $l \in \Sigma_1(x)$ and $A \in \Sigma_2(x)$ then $l \cap A$ is a singular line; and*
(iv) *$\{l \backslash \{x\} \mid l \in \Sigma_1(x)\}$ and $\{A \backslash \{x\} \mid A \in \Sigma_2(x)\}$ are two partitions of $\Gamma_1(x)$.*

LEMMA 2.5. *Let l_1 and l_2 be two maximal cliques with $l_1 \cap l_2 = \{x\}$, and let m be a maximal clique intersecting l_1 in a single vertex different from x. Then $|m \cap l_2| \leq 1$.*

PROOF. Let $m \cap l_1 = \{y\}$. Suppose, to the contrary, that $m \cap l_2$ is a singular line. Clearly, $x \notin m \cap l_2$; otherwise $x, y \in m \cap l_1$, which contradicts $|m \cap l_1| = 1$. Similarly, $y \notin m \cap l_2$; otherwise $x, y \in l_1 \cap l_2$, which contradicts $|l_1 \cap l_2| = 1$. Then $y \in m \backslash l_2$ and $x \in l_2 \backslash m$ are adjacent (contained in l_1), a contradiction to Lemma 2.2. □

LEMMA 2.6. (i) *If $x, y \in V(\Gamma)$ with $\partial(x, y) = 2$, then there are exactly $q + 1$ cliques in $\Sigma_k(x)$ at distance 1 from y, $k = 1, 2$.*

(ii) *Every singular line of Γ has a size of $\alpha + 1$.*

(iii) $|\Gamma_1(x) \cap m| = 0$ *or* $\alpha + 1$, *for any vertex x and maximal clique m not containing x.*

PROOF. Let x and y be two vertices with $\partial(x, y) = 2$. Assume that $l_1, l_2, \ldots, l_s \in \Sigma_1(x)$ (resp. $\in \Sigma_2(x)$) are at distance 1 from y, and let $m_1, m_2, \ldots, m_t \in \Sigma_p(y)$ ($p = 1$ or 2) be at distance 1 from x such that $|l_i \cap m_j| = 1$ for some i, j ($1 \leqslant i \leqslant s; 1 \leqslant j \leqslant t$). By Lemma 2.5, each l_i and each m_j have at most one point in common.

For a given m_j ($1 \leqslant j \leqslant t$), let $z \in \Gamma_1(x) \cap m_j$. One of the two maximal cliques containing x and z, say A_{xz}, intersects m_j in a singular line, and $A_{xz} \cap m_j \subseteq \Gamma_1(x) \cap m_j$; hence, by (A.2),

$$|\Gamma_1(x) \cap m_j| \geqslant |A_{xz} \cap m_j| \geqslant \alpha + 1. \tag{3}$$

Clearly, $\{l_i \cap m_j \mid 1 \leqslant i \leqslant s; 1 \leqslant j \leqslant t\} \subseteq \Gamma_1(x) \cap \Gamma_1(y)$. Observe that $\{\Gamma_1(y) \cap l_i \mid 1 \leqslant i \leqslant s\}$ and $\{\Gamma_1(x) \cap m_j \mid 1 \leqslant j \leqslant t\}$ form two partitions of $\Gamma_1(x) \cap \Gamma_1(y)$ by Lemma 2.4(iv). Thus each vertex of $\Gamma_1(x) \cap \Gamma_1(y)$ lies in exactly one l_i and exactly one m_j. Hence $\Gamma_1(x) \cap \Gamma_1(y) = \{l_i \cap m_j \mid 1 \leqslant i \leqslant s; 1 \leqslant j \leqslant t\}$. Then

$$\begin{aligned}(q + 1)(\alpha + 1) &= |\Gamma_1(x) \cap \Gamma_1(y)| \qquad \text{(i.e. } c_2 \text{ in (1b))}\\ &= \sum_{j=1}^{t} |\Gamma_1(x) \cap m_j|\\ &\geqslant t(\alpha + 1) \qquad \text{(by (3))}.\end{aligned} \tag{4}$$

Hence $t \leqslant q + 1$. Similarly, $s \leqslant (q + 1)$. Furthermore,

$$\begin{aligned}(q + 1)q &\leqslant (q + 1)(\alpha + 1) \qquad \text{(by (A.2))}\\ &= |\Gamma_1(x) \cap \Gamma_1(y)|\\ &= \sum_{i=1}^{s} \sum_{j=1}^{t} |l_i \cap m_j|\\ &\leqslant st.\end{aligned} \tag{5}$$

It follows that $(q + 1)^2 \geqslant st \geqslant (q + 1)q$. Hence either $s = t = q + 1$, or $\{s, t\} = \{q, q + 1\}$. The possibility of $\{s, t\} = \{q, q + 1\}$ can be excluded in the following way.

Suppose, to the contrary, that $\{s, t\} = \{q, q + 1\}$. Then the equalities in (5) actually hold. Hence $\alpha + 1 = q$, and each l_i and each m_j must intersect. The subgraph induced on $\Gamma_1(x) \cap \Gamma_1(y)$ is therefore an $(\alpha + 1) \times (\alpha + 2)$-grid, which implies that the sizes of singular lines are $\alpha + 1$ and $\alpha + 2$ (both occur). Let x be a vertex contained in a singular line of size $\alpha + 2$, say $l \cap A$ where $l \in \Sigma_1(x)$ and $A \in \Sigma_2(x)$. Let $m \in \Sigma_1(x), m \neq l$ and $w \in m \cap A$, $w \neq x$. Clearly, $|\Gamma_1(w) \cap l| = |l \cap A| = \alpha + 2$. By Lemma 2.1(ii), $|A \cup l| = |A \cup m| = a_1 + 2$ in (1c), and hence $|l \backslash A| = |m \backslash A|$. Moreover, since every vertex $u \in l \backslash A$ is at distance 2 from w and $|\Gamma_1(w) \cap l| = \alpha + 2$, the subgraph induced on $\Gamma_1(u) \cap \Gamma_1(w)$ is an $(\alpha + 1) \times (\alpha + 2)$-grid. Hence $|\Gamma_1(u) \cap m| = \alpha + 1$. Count the set $\{(u, v) \mid u \in l \backslash A, v \in m \backslash A$ are adjacent$\}$ in two ways: $|l \backslash A|\, \alpha = \sum_{u \in l \backslash A} |\Gamma_1(u) \cap m - \{x\}| = \sum_{v \in m \backslash A} |\Gamma_1(v) \cap l - \{x\}| \geqslant |m \backslash A|\, \alpha = |l \backslash A|\, \alpha$. Hence $|\Gamma_1(v) \cap l| = \alpha + 1$ for all $v \in m \backslash A$. It follows that all singular lines through x contained in l except $l \cap A$ are of size $\alpha + 1$. Similarly, $l \cap A$ is the unique singular line of size $\alpha + 2$ through x contained in A. Hence $|l \cap A| = \alpha + 2$, and $|m \cap A| = \alpha + 1$. Let $A' \in \Sigma_2(x)$, $A' \neq A$. As shown above, $|l \cap A'| = |m \cap A'| = \alpha + 1$, and hence $|l \backslash A'| = |m \backslash A'| + 1$, a contradiction to $|l \cup A'| = |m \cup A'|$ (i.e. $a_1 + 2$).

Since $s = t = q + 1$, (i) follows. With this result, (ii) and (iii) follow from (3) and (4) immediately. □

COROLLARY 2.6.1. *Let l and m be two maximal cliques with $|l \cap m| = 1$. Then $|l| = |m|$.*

PROOF. Let $\{z\} = l \cap m$. By Lemma 2.6(iii), $|\Gamma_1(x) \cap m| = |\Gamma_1(y) \cap l| = \alpha + 1$ for all $x \in l, y \in m$ $(x, y \neq z)$. Counting the set $\{(x, y) \mid x \in l \backslash \{z\}, y \in m \backslash \{z\}$ are adjacent$\}$ in two ways show that $(|l| - 1)\alpha = (|m| - 1)\alpha$, and hence $|l| = |m|$. □

We now show that there are exactly two sizes for maximal cliques and hence the set of all maximal cliques of Γ can be partitioned into two families according to their sizes.

COROLLARY 2.6.2. *A maximal clique is of size $\beta + 1$ or $\alpha\left[{d \atop 1}\right] + 1$.*

PROOF. Let x and y be two adjacent vertices, and let $l_{xy} \in \Sigma_1(x)$ and $A_{xy} \in \Sigma_2(x)$ be the two maximal cliques containing x and y. By Lemma 2.4, we have $\{l \cap A_{xy} \backslash \{x\} \mid l \in \Sigma_1(x)\}$ forms a partition of $A_{xy} \backslash \{x\}$. So α divides $|A_{xy}| - 1$ by Lemma 2.6(ii). Similarly, α divides $|l_{xy}| - 1$. We may assume that $|A| = r\alpha + 1$ for all $A \in \Sigma_2(x)$, and $|l| = s\alpha + 1$ for all $l \in \Sigma_1(x)$ by Corollary 2.6.1 for some integers r and s. Note that $|\Sigma_1(x)| = (|A_{xy}| - 1)/\alpha = r$ and $|\Sigma_2(x)| = s$. Since $l_{xy} \cup A_{xy} = (\Gamma_1(x) \cap \Gamma_1(y)) \cup \{x, y\}$,

$$a_1 + 2 = \beta + 1 + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - 1\right) = (s\alpha + 1) + (r\alpha + 1) - (\alpha + 1);$$

hence

$$\beta = \alpha\left(r + s - \begin{bmatrix} d \\ 1 \end{bmatrix}\right). \tag{6}$$

Since $\Gamma_1(x) = \bigcup_{l \in \Sigma_1(x)} (l \backslash \{x\})$, $b_0 = \left[{d \atop 1}\right]\beta = \sum_{l \in \Sigma_1(x)} |l \backslash \{x\}| = rs\alpha$ and hence

$$\beta = rs\alpha \Big/ \begin{bmatrix} d \\ 1 \end{bmatrix}. \tag{7}$$

Comparing (6) and (7), we have

$$r + s - \begin{bmatrix} d \\ 1 \end{bmatrix} = rs \Big/ \begin{bmatrix} d \\ 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} d \\ 1 \end{bmatrix}^2 - (r + s)\begin{bmatrix} d \\ 1 \end{bmatrix} + rs = 0, \qquad \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - r\right)\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - s\right) = 0.$$

Hence either $r = \left[{d \atop 1}\right]$ or $s = \left[{d \atop 1}\right]$. We may assume that $r = \left[{d \atop 1}\right]$, i.e. $|A_{xy}| = \alpha\left[{d \atop 1}\right] + 1$; then $|l_{xy}| = s\alpha + 1 = \beta + 1$ by (6). □

At this point, two semilinear incidence structures can be derived from Γ by Lemma 2.1, Corollaries 2.6.1 and 2.6.2, and (A.1), without using the Bose–Laskar argument. Indeed, Corollary 2.6.2 and assumption (A.1) show that there are two different sizes for maximal cliques. Moreover, the connectedness of Γ and Corollary 2.6.1 show that the set of all maximal cliques of Γ can be partitioned into two families Σ_1 and Σ_2 according to their sizes, such that both the incidence structures $(V(\Gamma), \Sigma_1, \in)$ and $(V(\Gamma), \Sigma_2, \in)$ are semilinear. For the rest of this paper, we may assume that $\Sigma_1(x) \subseteq \Sigma_1$ and $\Sigma_2(x) \subseteq \Sigma_2$ for any vertex x, and that $|m| = \beta + 1$ if $m \in \Sigma_1$ and

$|m| = \alpha[{d \atop 1}] + 1$ if $m \in \Sigma_2$. It follows that every vertex of Γ is in $[{d \atop 1}]$ cliques of Σ_1 and in β/α cliques of Σ_2. Some properties of $(V(\Gamma), \Sigma_1, \in)$ and $(V(\Gamma), \Sigma_2, \in)$ will be summarized in Propositions 2.12 and 2.13, and Corollary 2.13.1, after we determine the geometric structures induced on each maximal clique in Theorem 2.7 to Corollary 2.11.1.

THEOREM 2.7. *$\alpha + 1 = q + 1$ or q.*

PROOF. Let x and y be two vertices with $\partial(x, y) = 2$, and let $A_y \in \Sigma_2(y)$ be at distance 1 from x. Then $|\Gamma_1(x) \cap A_y| = \alpha + 1$. Every vertex $z \in \Gamma_1(x) \cap A_y$ determines a unique maximal clique $m_{yx} \in \Sigma_1(y)$ at distance 1 from x, and distinct vertices of $\Gamma_1(x) \cap A_y$ determine distinct maximal cliques of $\Sigma_1(y)$ at distance 1 from x. Thus, by Lemma 2.6(i) and assumption (A.2), $q + 1 \geq \alpha + 1 \geq q$, and hence $\alpha + 1 = q + 1$ or q. □

Let $x, y \in V(\Gamma)$ with $\partial(x, y) = 2$. As shown in Lemma 2.6(i), let $l_1, \ldots, l_{q+1} \in \Sigma_k(x)$ $(k = 1, 2)$ be the maximal cliques at distance 1 from y, and let $m_1, \ldots, m_{q+1} \in \Sigma_k(y)$ be those at distance 1 from x. Then $|l_i \cap m_j| \leq 1$ $(1 \leq i, j \leq q + 1)$.

COROLLARY 2.8. *Let x, y and $l_1, \ldots, l_{q+1}, m_1, \ldots, m_{q+1}$ be as above. If $\alpha + 1 = q + 1$, then l_i intersects m_j for all $i, j = 1, 2, \ldots, q + 1$. If $\alpha + 1 = q$, then l_i intersects m_j iff $i \neq j$ $(1 \leq i, j \leq q + 1)$, up to a relabelling.*

PROOF. By Lemma 2.6(iii), $|\Gamma_1(x) \cap m_j| = \alpha + 1 \geq q$ and $|\Gamma_1(y) \cap l_i| = \alpha + 1 \geq q$ $(1 \leq i, j \leq q + 1)$. Then

$$\begin{aligned} |\{l_i \cap m_j | 1 \leq i, j \leq q + 1\}| &= |\Gamma_1(x) \cap \Gamma_1(y)| \\ &= (q + 1)(\alpha + 1) \qquad \text{(i.e. } c_2 \text{ in (1b))} \\ &= \begin{cases} (q + 1)^2 & \text{if } \alpha + 1 = q + 1, \\ (q + 1)q & \text{if } \alpha + 1 = q, \end{cases} \end{aligned}$$

as required. □

We shall show in the following that more common properties of the graphs $J_q(n, d)$ and $H_q(n, d)$ hold in Γ.

LEMMA 2.9. *Let x be a vertex and m be a maximal clique with $\partial(x, m) = 2$. Then*
(i) $|\Gamma_2(x) \cap m| = \alpha(q + 1) + 1$;
(ii) $|\Gamma_1(x) \cap \Gamma_1(w_1) \cap \Gamma_1(w_2)| = \alpha + 1$ *for all distinct vertices $w_1, w_2 \in \Gamma_2(x) \cap m$, and $\Gamma_1(x) \cap \Gamma_1(w_1) \cap \Gamma_1(w_2)$ is contained in a unique maximal clique.*

PROOF. First assume that $m \in \Sigma_1$, and let $y \in \Gamma_2(x) \cap m$. Then $m \in \Sigma_1(y)$, and $\partial(x, y) = 2$. By Lemma 2.6(i), there are exactly $q + 1$ maximal cliques $A_1, A_2, \ldots, A_{q+1} \in \Sigma_2(y)$ at distance 1 from x. Since $A_i \cap m \subseteq \Gamma_2(x) \cap m$ for each $i = 1, \ldots, q + 1$, we have

$$\begin{aligned} |\Gamma_2(x) \cap m| &\geq \left| \bigcup_{i=1}^{q+1} (A_i \cap m \backslash \{y\}) \right| + 1 \\ &= \sum_{i=1}^{q+1} |A_i \cap m \backslash \{y\}| + 1 \\ &= \alpha(q + 1) + 1. \end{aligned} \tag{8}$$

Let $\mathcal{M}$ be the set of $q+1$ cliques in $\Sigma_1(y)$ at distance 1 from x. Clearly, $m \in \Sigma_1(y)\backslash\mathcal{M}$. Observe that the above argument is valid for all cliques h in $\Sigma_1(y)\backslash\mathcal{M}$, and thus we have $|\Gamma_2(x) \cap h| \geq \alpha(q+1)+1$ and $|h\backslash\Gamma_2(x)| \leq \beta+1-(\alpha(q+1)+1)$. By Lemma 2.4(iv), we see that $\{h\backslash\Gamma_2(x) \mid h \in \Sigma_1(y)\backslash\mathcal{M}\}$ forms a partition of $\Gamma_3(x) \cap \Gamma_1(y)$. Then

$$\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - (q+1)\right)(\beta - \alpha(q+1)) = |\Gamma_3(x) \cap \Gamma_1(y)| \qquad \text{(i.e. } b_2 \text{ in (1a))}$$

$$= \sum_{h \in \Sigma_1(y)\backslash\mathcal{M}} |h\backslash\Gamma_2(x)|$$

$$\leq \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - (q+1)\right)(\beta - \alpha(q+1)).$$

Thus $|h\backslash\Gamma_2(x)| = \beta - \alpha(q+1)$, and hence $|\Gamma_2(x) \cap h| = \alpha(q+1)+1$ for all $h \in \Sigma_1(y)\backslash\mathcal{M}$. So $|\Gamma_2(x) \cap m| = \alpha(q+1)+1$. Similar arguments work for the case of $m \in \Sigma_2$, and hence (i) follows.

To prove (ii), assume that $m \in \Sigma_k$ $(k=1,2)$. Let $w_1, w_2 \in \Gamma_2(x) \cap m$, and let $A_1, \ldots, A_{q+1} \in \Sigma_j(w_1)$ $(j \neq k)$ be at distance 1 from x. Then $|\Gamma_1(x) \cap A_i| = \alpha+1$ for each $i=1,\ldots,q+1$, by Lemma 2.6(iii). As shown in (i), $|\Gamma_2(x) \cap m| = \alpha(q+1)+1 = \sum_{i=1}^{q+1} |(A_i \cap m\backslash\{w_1\})|+1$, and hence w_2 must be in one of $A_1, \ldots, A_{q+1}$, say A_1. Since

$$(\Gamma_1(w_1) \cap \Gamma_1(w_2)) \cup \{w_1, w_2\} = m \cup A_1$$

and

$$\Gamma_1(x) \cap m = \varnothing, \qquad \Gamma_1(x) \cap \Gamma_1(w_1) \cap \Gamma_1(w_2) = \Gamma_1(x) \cap A_1,$$

as required. □

We now determine the geometric structures induced on $T = \Gamma_2(x) \cap m$ for a pair (x, m) with x a vertex and m a maximal clique with $\partial(x, m) = 2$. Consider the incidence structure $\Pi_T = (T, L(T), \in)$, where $L(T) = \{A \cap m \mid A$ is a maximal clique with $\partial(x, A) = 1, |A \cap T| \geq 2\}$. Note that A and m must be in different families.

LEMMA 2.10. *Π_T is a projective plane of order q if $\alpha+1 = q+1$, or an affine plane of order q if $\alpha+1 = q$.*

PROOF. As shown in Lemma 2.9(ii), any two points of T are on a unique maximal clique A with $\partial(x, A) = 1$. Hence T is a linear space with $\alpha(q+1)+1$ points by Lemma 2.9(i) and lines of size $\alpha+1$ by Lemma 2.6(ii). Hence Π is a 2-$(\alpha(q+1)+1, \alpha+1, 1)$ design, and the lemma follows immediately from Theorem 2.7. □

We can remark further on the structures over Π_T as follows.

LEMMA 2.11. *If u, v and w are points of a maximal clique A not contained in any singular line, then there exists a vertex x with $\partial(x, A) = 2$ such that u, v and w belong to the projective (affine) plane $\Gamma_2(x) \cap A$.*

PROOF. In addition to A, let l_{uv} (resp. l_{uw}) be the other maximal clique containing the edge uv (resp. uw). Hence $l_{uv}, l_{uw} \in \Sigma_i$ and $A \in \Sigma_j$ $(i \neq j$ in $\{1, 2\})$. Let $y_1 \in l_{uv}\backslash A$. In addition to u, y_1 is adjacent to α vertices of $l_{uw}\backslash A$ by Lemma 2.6(iii). Let $y_2 \in \Gamma_1(y_1) \cap (l_{uw}\backslash A)$, and let m be the clique in Σ_i containing y_1 and y_2. By Lemma 2.2, $\partial(m, A) = 1$.

Let Y be the set of vertices in m at distance 1 from A, and let Z be the set of vertices in A at distance 1 from m. Counting the set $\{(y, z) \mid y \in Y \text{ and } z \in Z \text{ are adjacent}\}$ in two ways shows that $|Y|(\alpha+1) = |Z|(\alpha+1)$. So $|Y| = |Z|$. Observe that $|Y| \leq |m|$, $|Z| \leq |A|$ and $|m| \neq |A|$. If $|m| > |A|$ then there is a point $p \in m \setminus Y$ such that $Z \subseteq \Gamma_2(p) \cap A$; otherwise, there is a point $p' \in A \setminus Z$ such that $Y \subseteq \Gamma_2(p') \cap m$. So $|Y| = |Z| \leq \alpha(q+1)+1$. Hence there is a point $x \in m \setminus Y$ such that $\partial(x, A) = 2$ and $u, v, w \in \Gamma_2(x) \cap A$. □

REMARK. Indeed, $|Y| = |Z| = \alpha(q+1)+1$ in the above proof, by a counting argument similar to the one used in [10, Lemma 3.3]. This will be used in the proof of Proposition 2.13(ii) when the argument of [10, Proposition 3.4] is applied.

By Lemmas 2.10 and 2.11, the following corollary follows from a theorem of Veblen and Young [17] and a theorem of Buekenhout [5].

COROLLARY 2.11.1. *q is a prime power and every maximal clique together with the singular lines that it contains is either a projective space of order q if $\alpha+1 = q+1$, or an affine space of order q if $\alpha+1 = q$.*

From Lemma 2.6(iii) and Corollary 2.6.2, the following proposition holds.

PROPOSITION 2.12. (1) *The incidence structure $(V(\Gamma), \Sigma_1, \in)$ has the following properties:*
(i) *every line has $\beta+1$ points;*
(ii) *every point is on $[{d \atop 1}]$ lines;*
(iii) *for any line $l \in \Sigma_1$ and any point $x \notin l$, $|\Gamma_1(x) \cap l| = 0, \alpha+1$.*
(2) *The incidence structure $(V(\Gamma), \Sigma_2, \in)$ has the following properties:*
(i) *every line has $\alpha[{d \atop 1}]+1$ points;*
(ii) *every point is on β/α lines;*
(iii) *for any line $l \in \Sigma_2$ and any point $x \notin l$, $|\Gamma_1(x) \cap l| = 0, \alpha+1$.*

As shown above, maximal cliques in different families Σ_1 and Σ_2 share the same geometric structures but for their sizes. The roles that Σ_1 and Σ_2 play in $(V(\Gamma), \Sigma_1, \in)$ are interchanged in $(V(\Gamma), \Sigma_2, \in)$. For the rest of this section, Σ_k (resp. $\Sigma_j, j \neq k$) is called the *line* set (resp. the *assembly* set) of the incidence structure $(V(\Gamma), \Sigma_k, \in)$, where $1 \leq k, j \leq 2$.

Let us recall the definition of the axiom of parallelism, which holds for $(V(\Gamma), \Sigma_k, \in)$ in the case of $\alpha+1 = q$. For a semilinear incidence structure $(\mathscr{P}, \mathscr{L}, \in)$, two lines $m, l \in \mathscr{L}$ with $\partial(m, l) = 1$ are called *parallel* if $\partial(x, m) = 1$ for all $x \in l$ and $\partial(y, l) = 1$ for all $y \in m$. An incidence structure is said to satisfy the *axiom of parallelism* if for any point x and line m with $\partial(x, m) = 1$, there is a unique line l through x parallel to m.

PROPOSITION 2.13. (i) *In the case of $\alpha+1 = q+1$, $(V(\Gamma), \Sigma_k, \in)$ $(k = 1, 2)$ satisfies both Pasch's axiom and the dual of Pasch's axiom.*
(ii) *In the case of $\alpha+1 = q$, $(V(\Gamma), \Sigma_k, \in)$ $(k = 1, 2)$ satisfies the dual of Pasch's axiom and the axiom of parallelism.*

PROOF. First we prove that $(V(\Gamma), \Sigma_k, \in)$ satisfies the dual of Pasch's axiom in both cases. If x and y are two vertices of a line m, and u and v are two vertices not in m adjacent to both x and y, then u and v must be in the assembly containing x and y. Thus u and v are adjacent, and hence they are in a common line.

To prove that $(V(\Gamma), \Sigma_k, \in)$ satisfies Pasch's axiom in the case of $\alpha+1=q+1$, suppose that line m_i $(i=1,2)$ intersects the two lines l_1 and l_2 in vertices $x_{i,1}$ and $x_{i,2}$, respectively, distinct from $x=l_1\cap l_2$. If $x_{1,1}$ is adjacent to $x_{2,2}$, then the lines m_i and l_i meet the assembly on x and $x_{1,1}$ in the four lines of a projective plane containing $x_{i,j}$ for all i, j. Inside the plane one can find a point of intersection of m_1 and m_2. If $x_{1,1}$ and $x_{2,2}$ are not adjacent, then m_1 and m_2 intersect, by Corollary 2.8. Hence (i) follows.

To prove the rest of (ii), we first show that parallelism can be defined among lines. Let m and l be two lines with $\partial(m,l)=1$, and let $x\in l$ and $y\in m$ be adjacent. Since $\Gamma_1(x)\cap m$ and $\Gamma_1(y)\cap l$ are contained in the assembly on x and y, each $y'\in\Gamma_1(x)\cap m$ and each $x'\in\Gamma_1(y)\cap l$ are adjacent. Hence $\partial(u,m)=1$ for all $u\in l$ iff $\partial(v,l)=1$ for all $v\in m$. Now (ii) can be proved by an argument similar to the one used in [10, Proposition 3.4]. □

Let x be a point and m be a line with $\partial(x,m)=1$ and let $l_1,\ldots,l_{\alpha+1}$ be the lines of x intersecting m. For the case of $\alpha+1=q$, let l_0 be the unique line of x parallel to m. Proposition 2.13 leads to the following corollary, also considered by Wilbrink and Brouwer in [18].

COROLLARY 2.13.1. (i) *For the case of $\alpha+1=q+1$, if m' is a line not through x intersecting two lines of $l_1,\ldots,l_{q+1}$, then m' intersects m and all lines of $l_1,\ldots,l_{q+1}$.*

(ii) *For the case of $\alpha+1=q$, if m' is a line not through x intersecting l_0 and one line of $l_1,\ldots,l_q$, then m' intersects m and $q-1$ lines of $l_1,\ldots,l_q$.*

PROOF. (i) is an immediate consequence of Pasch's axiom. To prove (ii), let $m\cap l_i=\{y_i\}$, $i=1,\ldots,q$. Suppose that m' intersects l_0 and l_j for some j, and $m'\cap l_0=\{z\}$. If z is adjacent to y_j, then the lines m, m', l_0 and l_j meet the assembly on x and y_j in the four lines of an affine plane containing x, z and y_j. Since l_0 and m' are two lines of z, and l_0 is parallel to m, m' must intersect m and $q-1$ lines of $l_1,\ldots,l_q$ in the affine plane. If z and y_j are not adjacent, then $\partial(z,y_j)=2$. Since l_0 is parallel to m and $\partial(z,m)=\partial(y_j,l_0)=\partial(y_j,m')=1$, m' must intersect m, by Corollary 2.8. Let $m'\cap m=\{w\}$. If $w=y_i$ for some $i\neq j$ $(1\leqslant i,j\leqslant q)$, then $\partial(w,x)=1$ and m' intersects $q-1$ lines of $l_1,\ldots,l_q$ inside the affine plane containing x, z and w of the assembly on x and w; otherwise, $\partial(w,x)=2$, and the result thus follows from Corollary 2.8. □

3. PROOF OF THE MAIN THEOREM

We have established two semilinear spaces from Γ in Section 2. We now show in this section that these incidence structures are projective incidence structures if $\alpha+1=q+1$, or attenuated spaces if $\alpha+1=q$.

By Theorem 2.7, $\alpha+1=q+1$, or q. We first show that if $\alpha+1=q+1$, then the above incidence structures $(V(\Gamma), \Sigma_k, \in)$ $(k=1,2)$ mentioned in Proposition 2.12 are projective incidence structures. By assumption (A.1) with $d\geqslant 3$ and $\alpha+1=q+1$, it is easy to see that $(V(\Gamma), \Sigma_k, \in)$ $(k=1,2)$ satisfy the hypothesis of a theorem of Ray-Chaudhuri and Sprague in [11] (see also [6, Theorem 4.6]). Hence their collinearity graphs Γ are isomorphic to $J_q(n,d)$, where $\beta+1=\left[{n-d+1 \atop 1}\right]$ for some integer $n\geqslant 2d+1$ by (A.1). This gives assertion (1) of the Main Theorem. Note that the constraint $n\geqslant 3d$ for $J_q(n,d)$ in [13] is partially improved to $n\geqslant 2d+1$.

We now turn to the case of $\alpha+1=q$. An argument similar to that used in [7, Proposition 2.3] shows that the 2-spaces of $(V(\Gamma), \Sigma_1, \in)$ (resp. $(V(\Gamma), \Sigma_2, \in)$), obtained by the construction given in [10, Section 4], are the $(2,q,g)$- (resp. $(2,q,h)$-) attenuated spaces, where $\beta+1=q^g$ and $\alpha\left[{d \atop 1}\right]+1=q^h$, respectively. The second assertion of the Main Theorem follows from the arguments indicated in [10, Section 5] and of course [14] also applies. Hence their collinearity graphs Γ are isomorphic to $H_q(n, d)$ with $\beta + 1 = q^n$ for some integer $n \geqslant d + 1$ by (A.1). Note again that the constraint $n \geqslant 2d$ for $H_q(n, d)$ in [10] is partially improved to $n \geqslant d + 1$.

REMARK. As pointed out by one of the referees, the incidence structures $(V(\Gamma), \Sigma_k, \in)$ $(k = 1, 2)$ mentioned in Proposition 2.12 satisfy the hypothesis of [6, Proposition 3.2]. It follows that two intersecting lines of $(V(\Gamma), \Sigma_1, \in)$ (resp. $(V(\Gamma), \Sigma_2, \in)$) are contained in a geodesically closed subgraph of Γ isomorphic to $H_q(n, 2)$ (resp. $H_q(d, 2)$). Although the arguments used in [6, 7] work for the cases of $\alpha + 1 \geqslant 2$, the hypothesis $\alpha + 1 \geqslant \max\{5, q\}$ is assumed in the main theorem since Theorem A(ii) is used.

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