



On a piecewise-smooth map arising in ecology



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ABSTRACT

In this paper, we study a two-dimensional piecewise smooth map arising in ecology. Such map, containing two parameters d and β , is derived from a model describing how masting of a mature forest happens and synchronizes. Here d is the energy depletion quantity and β is the coupling strength. Our main results are the following. First, we obtain a “weak” Sharkovskii ordering for the map on its nondiagonal invariant region for a certain set of parameters. In particular, we show that its Sharkovskii ordering is the natural number (resp., the positive even number) for $\beta > 1$ (resp., $0 < \beta < 1$). Second, we obtain a region of parameter space for which its corresponding global dynamics can be completely characterized.

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1. Introduction

Many dynamical systems contain terms that are non-smooth functions of their arguments. Important examples are electrical circuits having switches, mechanical devices in which components make impact on each other, problems with friction, sliding or squealing [4]. Even one-dimensional piecewise-smooth maps are known to have surprisingly rich dynamics, including periodic orbits with high period and the period-adding bifurcations [9,10]. In this paper, we are to investigate a certain properties and the global dynamics of a piecewise-smooth map arising in ecology. Specifically, we are concerned with a two dimensional piecewise-smooth map $F_{d,\beta}$ with two positive parameters d and β of the form:

$$\begin{aligned} F_{d,\beta}(x, y) &= (x + 1 - (d[y]_+^\beta + 1)[x]_+, y + 1 - (d[x]_+^\beta + 1)[y]_+) \\ &=: (F_1(x, y), F_2(x, y)), \\ \text{where } 1 - d \leq x, y \leq 1, d, \beta > 0, \text{ and } [x]_+ &= x \text{ if } x \geq 0; [x]_+ = 0 \text{ if } x < 0. \end{aligned} \quad (1.1)$$

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We then define its associated lower dimensional map $f_{d,\beta}$ to be of the form:

$$f_{d,\beta}(x) = \begin{cases} x + 1 =: f_1(x), & \text{if } x \leq 0, \\ -dx^{\beta+1} + 1 =: g_{d,\beta}(x), & \text{if } x > 0. \end{cases} \quad (1.2)$$

It should be remarked that the diagonal set S , $S = \{(x, y): x = y, 1 - d \leq x, y \leq 1\}$, is invariant under the map $F_{d,\beta}$. In fact, $f_{d,\beta}(x)$ is obtained from $F_{d,\beta}$ by restricting its dynamics on S . Hence, we shall call $f_{d,\beta}$ the synchronous map of the system. Let $I_d = [1 - d, 1]$ and $D_d = I_d \times I_d$. Then all initial iterates of $f_{d,\beta}$ and $F_{d,\beta}$ enter the invariant region I_d and D_d , respectively, at finite time. We shall, henceforth, treat the domains of $f_{d,\beta}$ and $F_{d,\beta}$ to be I_d and D_d , respectively. The ecological meaning of the parameters d and β are to be explained in Section 2.

2. Forest model

To give some ideas as what model gives rise to (1.1) and (1.2), we shall begin with a brief introduction of the coupled trees model considered by Isagi et al. [5], and Satake and Iwasa [11]. Let $Y(k)$ be the amount of normalized energy reserve at the beginning of year k . Here $Y(k)$ is normalized in a way that 0 is the critical energy level for a tree to reproduce and that its energy level lies in between 1 and $-d + 1$, where d is the depletion coefficient. Due to the photosynthesis, each tree will gain a normalized amount of energy each year. Moreover, if the energy level of the tree is below critical at the year k , then all its energy is preserved to the following year. On the other hand, if its energy exceeds the critical level, then it will set flower and grow seeds. As a result, its energy is decreased after a reproductive year. The quantity d is a measurement to such energy depletion. Consequently, the motion of its energy reserve yearly for an individual tree reads as follows.

$$Y(k+1) = \begin{cases} Y(k) + 1 & \text{if } Y(k) \leq 0, \\ -dY(k) + 1 & \text{if } Y(k) > 0. \end{cases} \quad (2.1)$$

If the resource depletion by fruit production is heavy, meaning a larger d , the reproductive activities tend to fluctuate between years with a large variance. In a mature forest, fruiting efficiency may depend on the flowering activity of the other trees in a forest. This is because the pollination efficiency changes with the number of plants flowering in a population. To model the pollen limitation of the reproduction, d in (2.1) is replaced by $dP_i(k)$, where $P_i(k)$ is a factor smaller than or equal to 1, and indicates outcross pollen availability for the i -th tree. Then the normalized energy reserve of the i -th tree at year $k + 1$ is

$$Y_i(k+1) = \begin{cases} Y_i(k) + 1 & \text{if } Y_i(k) \leq 0, \\ -dP_i(k)Y_i(k) + 1 & \text{if } Y_i(k) > 0, \end{cases} \quad (2.2a)$$

where

$$P_i(k) = \left(\frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq i}}^n [Y_j(k)]_+ \right)^\beta. \quad (2.2b)$$

Here n is the total number of trees in the forest and β is the coupling strength, which measures the efficiency of the spread of outcross pollen produced by other flowering activities. In fact, the rate of setting seeds and fruits is limited by its pollen availability, which depends on the coupling strength β . The coupling strength β determines the shape of the outcross pollen availability function $P_i(k)$ and controls the degree of dependence of fruit production on $P_i(k)$. If β is chosen to be closed to zero, then the fruit production is almost independent of the reproductive activities of the other trees in a forest. Small β means that a small

fraction of flowering is sufficient to achieve good fruiting success. This, in turn, indicates that the forest has either a high pollination efficiency or a high density of trees. On the other hand, a large β indicates a strong dependence of seed and fruit production on the reproductive activities of the other trees in the forest. Note that $P_i(k) = 1$ only if the other trees reproduce at full intensity. Model (2.2a), (2.2b) is a system of coupled map lattices. The dynamics of the model in (2.2a) and (2.2b) with $n = 2$ is depicted by the map defined in (1.1). Coupled map lattices are models for studying fundamental questions in spatially extended dynamical systems. They exhibit very rich phenomena (see e.g. [1,6–8] and the work cited therein), including a wide variety of both spatial and temporal periodic structure, intermittence, chaos as well as synchronization. The purpose of this work is to investigate a certain properties and the global dynamics of $F_{d,\beta}$. We obtain the following results. First, we derive a “weak” Sharkovskii ordering for the map on its nondiagonal invariant region. In particular, we show that on such region, its weak version of the Sharkovskii ordering is the natural number (resp., the positive even number) for $\beta > 1$ (resp., $0 < \beta < 1$). Second, we prove that if $d|\beta - 1| < 1$, then all initial states of $F_{d,\beta}$ converge to the diagonal. We further prove the above mentioned assertion also holds true for $d \leq 1$ and $\beta > 0$.

3. Weak Sharkovskii ordering

It is well-known that for a continuous one dimensional map f , the ordering of the periods of its periodic points follows the so called “Sharkovskii” order (see e.g., [3]). In this section, we shall investigate a weak version of the “Sharkovskii” order for the two dimensional map $F_{d,\beta}$ on its nondiagonal region $D_d - S$. To this end, we first set up the following notations. Let $D_d = R_1 \cup R_2 \cup R_3 \cup R_4$, where the subscript k means that the corresponding region R_k is in the k -th quadrant. All the regions R_k , $k = 1, 2, 3, 4$, are closed. Hence, they are not disjoint. We further divide R_1 into four closed regions $i_{1,0}$, $i_{2,0}$, $i_{3,0}$ and $i_{4,0}$ so that $F_{d,\beta}(i_{k,0}) =: i_{k,1} \subset R_k$. Clearly, those regions $i_{k,0}$, $k = 1, 2, 3, 4$, are separated by two boundary curves γ_1 and γ_2 . Here

$$\gamma_1: -dy^\beta x + 1 = 0 \quad \text{and} \quad \gamma_2: -dx^\beta y + 1 = 0. \tag{3.1}$$

For $\beta = 1$, $\gamma_1 = \gamma_2$, and, hence, $i_{2,0} \cup i_{4,0} = \gamma_1 \cup \gamma_2$. For $\beta > 1$ and $0 < \beta < 1$, the corresponding $i_{4,0}$ lie below and above the diagonal, respectively, as illustrated in Fig. 3.1. Let

$$\ell_{\theta,0} \text{ be the line segments in } D_d \text{ passing through the point } (1, 1) =: B \text{ with slope } \tan \theta. \tag{3.2}$$

Define

$$\mathbf{0}_0 := (1, (1/d)^{1/\beta}) =: (\bar{0}_0, \tilde{0}_0) \tag{3.3a}$$

and

$$\mathbf{0}_n := (\bar{0}_n, \tilde{0}_n) := F_{d,\beta}(\mathbf{0}_{n-1}). \tag{3.3b}$$

Proposition 3.1. *Let*

$$h_1(x, y) = \begin{cases} (x^\beta y - y^\beta x)/(x - y), & y \neq x, \\ (\beta - 1)y^\beta, & y = x, \end{cases} \tag{3.4a}$$

and for $1 < \beta < 2$, we set

$$h_1(x, y) = 1/d, \tag{3.4b}$$

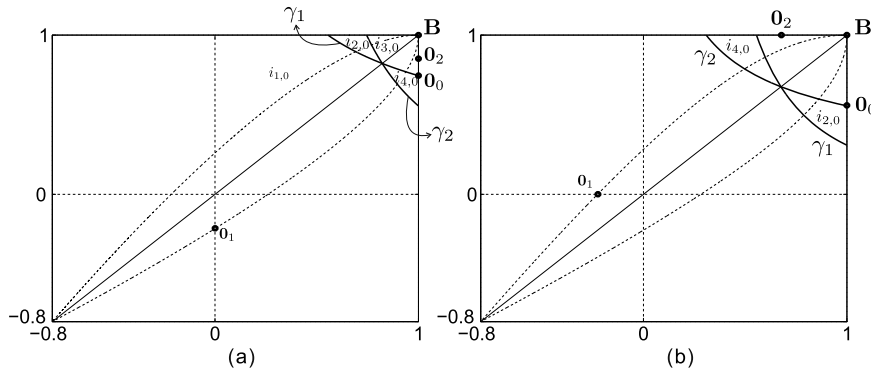


Fig. 3.1. (a) $\beta = \frac{3}{2}$, $d = 1.8$, (b) $\beta = 0.5$, $d = 1.8$. The regions $i_{k,0}$, $k = 1, 2, 3, 4$, for $\beta > 1$ and $0 < \beta < 1$ are illustrated in (a) and (b), respectively.

and for $0 < \beta < 1$, we let

$$h_1(x, y) = -1/d. \tag{3.4c}$$

Then Eq. (3.4b) (resp., (3.4c)) defines a curve $y = h_2(x)$ near $(x, y) = (x_s, x_s)$, where

$$x_s = (1/(d(\beta - 1)))^{1/\beta} \quad (\text{resp., } (1/(d(1 - \beta)))^{1/\beta}). \tag{3.4d}$$

Moreover, $h'_2(x) < 0$ and $h''_2(x) > 0$. In particular, $h_2(x)$ is tangent to the straight line $x + y = 2x_s$ at (x_s, x_s) .

Proof. Consider the case that $1 < \beta < 2$. Let $g(x, y) = h_1(x, y) - 1/d$. Then

$$g_x(x, y) = \begin{cases} \frac{(\beta-1)x^\beta y - \beta x^{\beta-1} y^2 + y^{\beta+1}}{(x-y)^2} := \frac{\bar{a}(x,y)}{(x-y)^2}, & x \neq y, \\ \frac{\beta(\beta-1)}{2} \cdot x_s^{\beta-1}, & x = y = x_s, \end{cases}$$

and

$$g_y(x, y) = \begin{cases} \frac{x^{\beta+1} - \beta y^{\beta-1} x^2 + (\beta-1)y^\beta x}{(x-y)^2} := \frac{\bar{b}(x,y)}{(x-y)^2}, & x \neq y, \\ \frac{\beta(\beta-1)}{2} \cdot x_s^{\beta-1}, & x = y = x_s. \end{cases}$$

For $\beta \neq 1$, we have, via the Implicit Function Theorem, that Eq. (3.4b) defines a curve $y = h_2(x)$ with $h'_2(x_s) = -g_x/g_y|_{x=x_s, y=y_s} = -1$. To compute the second derivative of h_2 , we need the second derivatives of g . After some laborious calculations, we have that

$$g_{xx}(x, y) = \begin{cases} \frac{(\beta-1)(\beta-2)x^\beta y - 2\beta(\beta-2)x^{\beta-1} y^2 - 2y^{\beta+1} + \beta(\beta-1)x^{\beta-2} y^3}{(x-y)^3} := \frac{\bar{c}(x,y)}{(x-y)^3}, & x \neq y, \\ \frac{\beta(\beta-1)(\beta-2)}{3} \cdot (x_s)^{\beta-2}, & x = y = x_s, \end{cases}$$

$$g_{yy}(x, y) = \begin{cases} \frac{-\beta(\beta-1)y^{\beta-2} x^3 + 2\beta(\beta-2)y^{\beta-1} x^2 - (\beta-1)(\beta-2)y^\beta x + 2x^{\beta+1}}{(x-y)^3} := \frac{\bar{d}(x,y)}{(x-y)^3}, & x \neq y, \\ \frac{\beta(\beta-1)(\beta-2)}{3} \cdot (x_s)^{\beta-2}, & x = y = x_s, \end{cases}$$

and

$$g_{yx} = g_{xy} = \begin{cases} \frac{(\beta-1)x^{\beta+1} - (\beta+1)x^\beta y + (\beta+1)y^\beta x - (\beta-1)y^{\beta+1}}{(x-y)^3} := \frac{\bar{e}(x,y)}{(x-y)^3}, & x \neq y, \\ \frac{\beta(\beta-1)(\beta+1)}{6} \cdot x_s^{\beta-2}, & x = y = x_s. \end{cases}$$

Note that $h_2''(x_s)$ satisfies the following equation

$$g_{xx} + g_{xy}h_2' + (g_{yx} + g_{yy}h_2')h_2' + g_y h_2'' = 0.$$

Hence,

$$h_2''(x) = (2g_x g_y g_{xy} - g_y^2 g_{xx} - g_x^2 g_{yy}) / g_y^3.$$

To complete the assertions of the proposition, we need to know the signs of \bar{a} , \bar{b} , \bar{c} , \bar{d} and \bar{e} . To this end, we compute partial derivatives of them. The resulting calculations are displayed in the following

$$\begin{aligned} \bar{a}_x &= \beta(\beta - 1)x^{\beta-2}y(x - y), \\ \bar{b}_y &= \beta(\beta - 1)xy^{\beta-2}(y - x), \\ \bar{c}_x &= \beta(\beta - 1)(\beta - 2)x^{\beta-3}y(x - y)^2, \\ \bar{d}_y &= -\beta(\beta - 1)(\beta - 2)xy^{\beta-3}(x - y)^2, \\ \bar{e}_x &= (\beta - 1)(\beta + 1)x^\beta - \beta(\beta + 1)x^{\beta-1}y + (\beta + 1)y^\beta, \\ \text{and } \bar{e}_{xx} &= \beta(\beta - 1)(\beta + 1)x^{\beta-1}(x - y). \end{aligned}$$

From \bar{a}_x and \bar{b}_y , we conclude that \bar{a} and \bar{b} have a unique absolute minimum at $x = y$. Consequently, $g_x(x, y)$ and $g_y(x, y)$ are positive for all x and y . Since $\bar{c}_x \leq 0$ and $\bar{c}(x, x) = 0$, we have that $\bar{c}(x, y) > 0$ if $x < y$ and $\bar{c}(x, y) < 0$ if $x \geq y$. Consequently, $g_{xx}(x, y) < 0$ for all x and y . Similarly, $g_{yy}(x, y) < 0$. Since $\bar{e}_{xx} < 0$ if $x < y$ and $\bar{e}_{xx} > 0$ if $x > y$, \bar{e}_x has a unique absolute minimum at $x = y$. Therefore, $\bar{e}_x(x, y) \geq \bar{e}_x(x, x) = 0$ and so, $\bar{e}(x, y) < 0$ if $x < y$ and $\bar{e}(x, y) > 0$ if $x > y$. This implies $g_{xy} > 0$ for all x, y . Combining the above calculations, we have that $h_2''(x) > 0$ for all x in its domain. \square

Lemma 3.1. *Assume that*

$$d|\beta - 1| > 1 \tag{3.5}$$

Then each of Eqs. (3.4b) and (3.4c) with $x = 1$ has exactly one solution \bar{y} for which $0 < \bar{y} < 1$.

Proof. We shall only illustrate the case for Eq. (3.4b). The solution to this case can be formulated as the intersection of two functions $f_1(y)$ and $f_2(y)$, where $f_1(y) = dy^\beta$ and $f_2(y) = (d + 1)y - 1$. Using the facts that $f_1(1) = f_2(1)$, $f_1'(1) > f_2'(1)$ and $f_1''(y) > 0$ on $(0, 1)$, we conclude that the assertion of the lemma holds as claimed. \square

Proposition 3.2. (i) *Let $\beta \geq 1$ and $1 < d \leq 2$. Then $\tilde{0}_2 > \tilde{0}_0$ if and only if*

$$(1 + d)^\beta < 2^\beta d \quad \text{or, equivalently,} \quad \beta < \frac{\ln d}{\ln(1 + d) - \ln 2} =: \Gamma_1(d). \tag{3.6}$$

Moreover, $\Gamma_1(d)$ is strictly decreasing on $d \in (1, \infty)$, $\lim_{d \rightarrow 1^+} \Gamma_1(d) = 2$ and $\lim_{d \rightarrow \infty} \Gamma_1(d) = 1$.

(ii) *Let A_n be the set of period n points of $F_{d,\beta}$. Suppose (3.6) is satisfied and*

$$(A_2 \cap i_{3,0}) - S = \emptyset. \tag{3.7}$$

Define $\Gamma_2(\beta, k) = \frac{\beta^\beta}{(k+1)^\beta(\beta-1)^{\beta+1}}$. Then

$$d \leq \Gamma_2(\beta, 1). \tag{3.8}$$

- (iii) If (3.6) and (3.7) are satisfied, then $F_{d,\beta}$ does not have a fixed point in $(i_{1,0} - S)$.
- (iv) Let (3.5) be satisfied and $d > \Gamma_2(\beta, 0)$. Then $F_{d,\beta}$ has a fixed point on $(i_{1,0} - S)$.

Proof. Note that $\tilde{0}_2 > \tilde{0}_0$ if and only if $(1 + d) < 2d^{\frac{1}{\beta}}$, or equivalently, $\beta < \Gamma_1(d)$. To see $\Gamma_1(d)$ is decreasing in d , we note that $\Gamma_1'(d) < 0$ provided that $\Gamma_3(d) := (d + 1)[\ln(d + 1) - \ln(2)] - d \ln d < 0$. Now, $\Gamma_3'(d) = \ln(\frac{d+1}{2d}) < 0$. Thus $\Gamma_3(d) < \Gamma_3(1) = 0$. Hence, $\Gamma_1(d)$ is a decreasing function of d . The remaining assertions for (i) are obvious, and, thus, omitted.

Let $(x, y) \in A_2 \cap i_{3,0}$. Then (x, y) satisfies the following equations with $m = 2$.

$$\begin{aligned} -dxy^\beta + m &= x, \\ -dx^\beta y + m &= y. \end{aligned} \tag{3.9}$$

Consequently, x satisfies the equation $h_3(x) = h_4(x)$ where $h_3(x) = \frac{x}{2-x}$ and $h_4(x) = \frac{1}{d}(\frac{1+dx^\beta}{2})^\beta$. Both functions are increasing and concave upward. For $1 < d \leq 2$, one can show that $f_{d,\beta}$ exists a unique period two point $p > 0$ and $f_{d,\beta}(p) < 0$. Note that such p satisfies the equation $-dx^{\beta+1} + 2 = x$ and that (3.6) implies that $h_4(1) < h_3(1)$. Since these two functions $h_3(x)$ and $h_4(x)$ only intersect at p , we must have that

$$\frac{2}{(2-p)^2} = h'_3(p) \geq h'_4(p) = \frac{\beta^2}{2^\beta} (p + dp^{\beta+1})^{\beta-1} = \frac{\beta^2 2^{\beta-1}}{2^\beta} = \frac{\beta^2}{2}. \tag{3.10}$$

Otherwise, $h_3(x)$ and $h_4(x)$ also intersect at a point x^* with $p < x^* < 1$. Moreover, the corresponding solution y^* to (3.9) is $\frac{2}{1+d(x^*)^\beta}$, which is less than 1, a contradiction to (3.7). To have the inequality in (3.10) held, we must have that $p \in [2 - \frac{2}{\beta}, 1)$. Since $-dx^{\beta+1} + 2 =: h_5(x)$ is a decreasing function, to ensure $p \in [2 - \frac{2}{\beta}, 1)$, $h_5(2 - \frac{2}{\beta})$ must stay on or above the diagonal. Hence, $-d(2 - \frac{2}{\beta})^{\beta+1} + 2 \geq 2 - \frac{2}{\beta}$, or equivalently, (3.8) holds.

We next turn our attention to the proof of the third assertion of the proposition. Using (3.9), we have that

$$d(xy^\beta - x^\beta y) = y - x, \quad 0 \leq x \leq 1. \tag{3.11a}$$

Clearly, the graph of (3.11a) is the union of diagonal segment $y = x$, $0 \leq x \leq 1$ and a curve defined by Eq. (3.4b). The solutions to Eq. (3.9) are determined by the intersection of curves (3.11a) and the curve

$$y = \frac{2}{1 + dx^\beta} =: h_6(x). \tag{3.11b}$$

Since $h_6(0) = 2$ and $h_6(1) = \frac{2}{1+d} < 1$, $h_6(x)$ intersects $y = x$, $0 \leq x \leq 1$, exactly at one point. It then follows from (3.7) that the curve decided by (3.4b) must completely lie above or below the curve described by $y = h_6(x)$. To see which one is the case, we first compute the intersection of the diagonal and the curve $y = h_2(x)$ as defined in Proposition 3.1. Some direct calculations yield that the x -coordinate of such intersection is

$$x_s = \left(\frac{1}{d(\beta - 1)} \right)^{\frac{1}{\beta}}, \tag{3.11c}$$

and that

$$h_6(x_s) - x_s \leq 0. \tag{3.11d}$$

Note that (3.11d) is equivalent to (3.8). Consequently, the curve $1 = dh_1(x, y)$ lies completely above or on the curve $y = h_6(x)$. The fixed point of $F_{d,\beta}$ in $i_{1,0}$ satisfies (3.9) with $m = 1$. Finding the solutions in

the above equations is equivalent to solving equations (3.4b) and $y = \frac{1}{1+dx^\beta} =: h_7(x)$. However, the curve $y = h_7(x)$ lies completely below $h_6(x)$. The assertion of the proposition (iii) now follows.

To conclude the proof of the proposition, we first note that $h_1(1, h_7(1)) < \frac{1}{d}$ for $0 < y < 1$. Consequently, $h_7(1) < \bar{y}$. Here \bar{y} is the unique solution of

$$h_1(1, y) = \frac{1}{d} \tag{3.12}$$

The existence and uniqueness of \bar{y} have been proved in Lemma 3.1. The assumption that $d > \Gamma_2(\beta, 0)$ yields $h_7(x_s) - x_s > 0$. This in turn implies that the intersection of $h_7(x)$ and the diagonal $y = x$ is above the point (x_s, x_s) . The assertion of proposition-(iv) now follows. \square

Proposition 3.3. *Assume that $\beta > 1$, $d > 2$ and that (3.5) is satisfied. Let $[d]$ be the least integer that is equal to or greater than d . Let $n = [d]$. Suppose, for $k = 0, 1, \dots, n - 2$, let the parameters d and β satisfy (3.5) and the following inequalities.*

$$\beta < \frac{\ln(d) - \ln(k)}{\ln(1+d) - \ln(k+1)} \tag{3.13}$$

and

$$d > \Gamma_2(\beta, k). \tag{3.14}$$

Then $F_{d,\beta}$ has a period $k + 1$ point (x, y) with $x \neq y$ satisfying (3.9) with $m = k + 1$. Furthermore, $F_{d,\beta}$ has a period ℓ point satisfying (3.9) with $m = \ell$, where $1 \leq \ell < k + 1$ on its nondiagonal region.

Proof. Following the same set up for proving Proposition 3.2-(iv), we have that the corresponding $h_7(x)$ equals to $\frac{k+1}{1+dx^\beta}$ and that (3.14) is equivalent to $h_7(x_s) - x_s > 0$. Moreover, (3.13) is amount to the condition that $h_7(1) < \bar{y}$, where \bar{y} is defined as in Lemma 3.1. Consequently, the first assertion of the proposition follows. Let $1 \leq \ell < k + 1$. Consider the equation

$$\begin{aligned} -dxy^\beta + \ell &= x, \\ -dx^\beta y + \ell &= y. \end{aligned}$$

Since the corresponding $h_7(x)$ for the above equation has the property that $h_7(1) = \frac{\ell}{1+d} < \frac{k+1}{1+d}$. Hence, the above equation has a solution (x, y) with $x \neq y$. \square

For $d > 1$ and $0 < \beta < 1$, the corresponding $\mathbf{0}_0$, the intersection of γ_2 and $\ell_{\frac{\pi}{2},0}$ is

$$\mathbf{0}_0 := (\bar{0}_0, \tilde{0}_0) := (1, 1/d).$$

Moreover, its corresponding $\mathbf{0}_1$ and $\mathbf{0}_2$ now locate above the diagonal, as shown in Fig. 3.1. To see the above, we note that $\ell_{\frac{\pi}{2},1} = F_{d,\beta}(\ell_{\frac{\pi}{2},0}) = (-dt^\beta + 1, -dt + 1)$, $0 \leq t \leq 1$. For $0 < \beta < 1$, the x -coordinate of $\ell_{\frac{\pi}{2},1}$ is less than or equal to its y -coordinate. Therefore $\ell_{\frac{\pi}{2},1}$ is above the diagonal. As a result, $\mathbf{0}_1$ whose y -coordinate is the zero is the image of $\mathbf{0}_0$ for which its y -coordinate has to be $\frac{1}{d}$. As a result, the corresponding condition (3.6) is

$$\bar{0}_2 > \tilde{0}_0, \quad \text{or, equivalently,} \quad \beta > 2 - (\ln(2d - 1))/\ln(d) := \Gamma_3(d). \tag{3.15}$$

We further note that the line $\ell_{\frac{\pi}{2},0}$ will keep flipping over the diagonal under the dynamics guided by $F_{d,\beta}$. This observation leads to the following propositions.

Proposition 3.4. *Let $1 < d \leq 2$ and $0 < \beta < 1$. Then the following hold.*

- (i) $\Gamma_3(d)$ is strictly increasing in $d \in (1, \infty)$. Moreover, $\lim_{d \rightarrow 1^+} \Gamma_3(d) = 0$ and $\lim_{d \rightarrow \infty} \Gamma_3(d) = 1$.
- (ii) Let

$$d > \frac{(2 - \beta)^\beta}{(1 - \beta)^{\beta+1}}. \tag{3.16}$$

If (3.5) and (3.16) hold, then $F_{d,\beta}$ exists a period two point on $(i_{1,0} - S)$.

Proof. To study $\Gamma_3(d)$, we compute its derivative. The resulting calculation yields that $\Gamma_3'(d) = \frac{-2d \ln d + (2d-1) \ln(2d-1)}{(\ln d)^2 d(2d-1)} =: \frac{\Gamma_4(d)}{(\ln d)^2 d(2d-1)}$ and $\Gamma_4'(d) = 2(\ln(2d-1) - \ln d) \geq 0$. Since $\Gamma_4'(1) \geq 0$, $\Gamma_4(d) \geq 0$ on $[1, \infty)$. Consequently, $\Gamma_3(d)$ is strictly increasing on $d \in (1, \infty)$. Some direct calculations give that $\Gamma_3(1) = 0$ and $\Gamma_3(\infty) = 1$. Due to symmetry of the map and the flipping natural of the map around the diagonal, we consider a period two point (x, y) in $i_{3,0}$ satisfying the following equation with $m = 1$.

$$\begin{aligned} -dxy^\beta + m &= y, \\ -dx^\beta y + m &= x. \end{aligned} \tag{3.17}$$

Following the similar approach in proving Proposition 3.2-(iv), we get that the corresponding x_s and $h_7(x)$ have, respectively, the form $x_s = (\frac{1}{d(1-\beta)})^{\frac{1}{\beta}}$, and $h_7(x) = \frac{1-x}{dx^\beta}$. Moreover, Eq. (3.4b) becomes (3.4c). It is clear that (3.16) is equivalent to $h_7(x_s) - x_s > 0$. Since $h_7(1) = 0 < \bar{y}$, where \bar{y} is the given as in Lemma 3.1 for the case that $0 < \beta < 1$. Hence, there exists a period two point in $i_{1,0} - S$. \square

Proposition 3.5. *Assume that $0 < \beta < 1$ and $d > 2$. Let (3.5) hold. Let $n = [d]$. Suppose, for $k = 1, 2, \dots, n - 2$, the following inequalities hold.*

$$d > \frac{(2 - \beta)^\beta}{k^\beta (1 - \beta)^{\beta+1}}, \tag{3.18}$$

and

$$\beta > \frac{2 \ln d - \ln(dk - k + 1)}{\ln d - \ln(k - 1)}. \tag{3.19}$$

Then $F_{d,\beta}$ has a period $2k$ point with $x \neq y$ satisfying (3.17) with $m = k$. Furthermore, $F_{d,\beta}$ has a period 2ℓ point satisfying (3.17) with $m = 2\ell$ for $1 \leq \ell < k + 1$ on its nondiagonal region.

We note that assumptions (3.18) and (3.19) are, respectively, amount to $h_7(x_s) > x_s$ and $h_7(1) < \bar{y}$. The proof of the proposition is similar to that of Proposition 3.2 and is thus skipped.

Remark 3.1. Propositions 3.4-(ii) and 3.5 amount to saying that for a certain parameters a “weak Sharkovskii ordering” of periodic points of $F_{d,\beta}$ for each of the cases that $\beta > 1$ and $0 < \beta < 1$ on its nondiagonal region is, respectively, the natural numbers and positive even numbers. It should also be noted that such region of parameters is nonempty.

4. Line-order preserving maps

The following concepts of monotonicity and line-order preserving play the important role of understanding the global dynamics of $F_{d,\beta}$.

Definition 4.1. Let D be a compact set in \mathbb{R}^2 . A smooth map $F : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is said to be monotonically preserving on D if γ is an increasing and smooth curve in D , then so is $F(\gamma)$. The map is line-order preserving on D if the following hold.

- (i) There exists a set $M = \{\ell_{\theta,0} : a \leq \theta \leq b, a, b \in \mathbb{R}\}$ of line segments such that $D = M$.
- (ii) Let $x_{\theta,k} \in D$ be the x -coordinate of the unique intersection of $y = k$ and $F(\ell_{\theta,0})$. Then for any k , $x_{\theta,k}$ is strictly monotonic in θ .

Lemma 4.1. *Let*

$$K_0 \text{ be the triangle whose vertices are } (0, 0), (1, 0) \text{ and } (1, 1) =: \mathbf{B}. \tag{4.1}$$

Then $F_{d,\beta}$ is monotonicity and line-order preserving on D_1 and K_0 , respectively.

Proof. Let γ be an increasing curve in D_1 . Without loss of generality, we assume that $\gamma(t) = \{(x(t), y(t)) : 0 \leq t \leq 1\}$. Let dx/dt and dy/dt have the same sign for all t . Since $F_{d,\beta}(x(t), y(t)) = (F_1(x(t), y(t)), F_2(x(t), y(t))) = (-d(y(t))^\beta x(t) + 1, -d(x(t))^\beta y(t) + 1)$, we have that

$$dF_1/dt = -d\beta(y(t))^{\beta-1}(dy/dt)x(t) - d(y(t))^\beta(dx/dt),$$

and

$$dF_2/dt = -d\beta(x(t))^{\beta-1}(dx/dt)y(t) - d(x(t))^\beta(dy/dt).$$

Hence, dF_1/dt and dF_2/dt have the same sign for all t . Consequently, $F_{d,\beta}(\gamma)$ is also an increasing curve. Thus, $F_{d,\beta}$ is monotonicity preserving on D_1 .

To see $F_{d,\beta}$ is line-order preserving on K_0 , we first note that $K_0 = \{\ell_{\theta,0} : \pi/4 \leq \theta \leq \pi/2\}$. Here $\ell_{\theta,0}$ are defined in (3.2). The equations of $\ell_{\theta,0}$ are

$$x = 1 + \cot \theta(y - 1), \quad 0 \leq y \leq 1. \tag{4.2}$$

Treating y as a parameter t , then $F_{d,\beta}(\ell_{\theta,0}) = (-dt^\beta(1 + \cot \theta(t - 1)) + 1, -d(1 + \cot \theta(t - 1))^\beta t + 1) =: \ell_{\theta,1}$. Here $0 \leq t \leq 1$ and the second component k of the subscript of $\ell_{\theta,k}$ is to be used as the iteration index under $F_{d,\beta}$. Since $F_{d,\beta}$ is monotonically preserving, $\ell_{\theta,1}$ is an increasing curve.

Let $x_{\theta,k}$ be the unique intersection of the line $y = k$ and $\ell_{\theta,1}$, as illustrated in Fig. 4.1. (4.3)

Clearly, $x_{\theta,k}$ satisfies the following equations

$$x_{\theta,k} = -dt^\beta(1 + \cot \theta(t - 1)) + 1, \tag{4.4a}$$

and

$$k = -d(1 + \cot \theta(t - 1))^\beta t + 1. \tag{4.4b}$$

Note that t is an implicit function of θ . Differentiating (4.4a) and (4.4b) with respect to θ , we have that

$$\partial x_{\theta,k} / \partial \theta = -dt^{\beta-1} [(dt/d\theta)(\beta + \beta t \cot \theta - \beta \cot \theta + t \cot \theta) - (\csc^2 \theta)t(t - 1)],$$

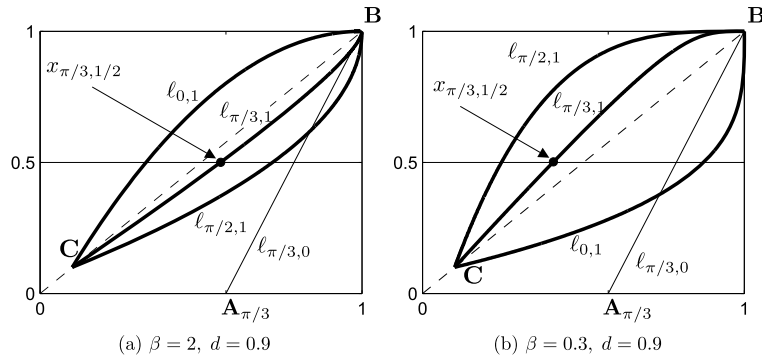


Fig. 4.1. (a) and (b) give pictorial images of A_θ , B , C , $\ell_{\theta,0}$, $\ell_{\theta,1}$ and $x_{\theta,k}$. For $\beta > 1$ (resp., $0 < \beta < 1$), $\ell_{\pi/3,1}$ stays below (resp., above) the diagonal. For $\beta = 1$, $\ell_{\theta,1}$ coincides with the diagonal for all θ .

and

$$dt/d\theta = ((-\csc^2 \theta)\beta t(1-t))/(\beta t \cot \theta + 1 + (t-1) \cot \theta).$$

Combining the above, we get that

$$\partial x_{\theta,k}/\partial \theta = d(\csc^2 \theta)t^\beta(1-t)(\beta^2 - 1)(1 + t \cot \theta - \cot \theta)/(\beta t \cot \theta + 1 + (t-1) \cot \theta),$$

which is nonnegative (resp., nonpositive) whenever $\beta \geq 1$ (resp., $0 < \beta \leq 1$) for all t and θ , $\pi/4 \leq \theta \leq \pi/2$. Moreover,

$$\text{for } \theta \neq \pi/2, t \neq 0, t \neq 1 \text{ and } \beta \neq 1, \partial x_{\theta,k}/\partial \theta > 0 \text{ (resp., } < 0) \text{ whenever } \beta > 1 \text{ (resp., } < 1). \tag{4.5}$$

Note that for $t = 0$ or $t = 1$ or $\beta = 1$, $x_{\theta,k}$ is either a point or does not exist for all θ , $\pi/4 \leq \theta \leq \pi/2$. Hence, $x_{\theta,k}$ is strictly monotonic in θ for any k . \square

Lemma 4.2. Let $\ell_{\theta,0}$, $0 \leq \theta \leq \pi/2$, be given as in (4.2). Let $\mathbf{A}_\theta = (1 - \cot \theta, 0)$ and $\mathbf{C} = (1 - d, 1 - d)$. Then the following hold true.

- (i) For all $d, \beta > 0$ and $0 \leq \theta \leq \pi/2$, $\ell_{\theta,1}$ are increasing curves with $F_{d,\beta}(\mathbf{A}_\theta) = \mathbf{B}$ and $F_{d,\beta}(\mathbf{B}) = \mathbf{C}$, as seen in Fig. 4.1.
- (ii) Let $d \leq 1$. Then $F_{d,\beta}(D_d)$ is symmetric with respect to $y = x$, for which its boundaries are $\ell_{0,1} \cup \ell_{\pi/2,1}$. Furthermore, if $\beta > 1$ (resp., $0 < \beta < 1$), $\ell_{\theta,1}$ moves from the upper left (resp., the lower right) to the lower right (resp., the upper left) as θ varies from $\pi/4$ to $\pi/2$. In both cases, $\ell_{\pi/4,0}$ is a fixed line, i.e., $\ell_{\pi/4,0} = \ell_{\pi/4,1}$. See Fig. 4.1 for the illustration.

Proof. The first assertion of the lemma follows from some direct calculations and the fact that $F_{d,\beta}$ is monotonicity preserving. Since the map $F_{d,\beta}$ is line-order preserving, $\ell_{0,1} \cup \ell_{\pi/2,1}$ are the boundaries of $F_{d,\beta}(D_1)$ with their relative positions being as claimed in the lemma. \square

5. Global dynamics of $F_{d,\beta}$

In this section, we shall investigate the global dynamics of $F_{d,\beta}$. We begin with the following results.

Theorem 5.1. Let $d > 1$. Assume that

$$d|\beta - 1| < 1. \tag{5.1}$$

Then $\lim_{n \rightarrow \infty} F_{d,\beta}^n(x_0, y_0) \in S$ for any $(x_0, y_0) \in D_d$, or equivalently, the coupled map lattices (2.2a) and (2.2b) with $n = 2$ is globally synchronized.

Proof. Assume that $\beta \geq 1$. Let $(x_0, y_0) \in R_1$. We may assume that $0 \leq x_0 \leq y_0 \leq 1$. Let $(x_1, y_1) = F_{d,\beta}(x_0, y_0)$. Then, for some $z_0 \in [x_0, y_0]$, $|x_1 - y_1| = |dx_0y_0(x_0^{\beta-1} - y_0^{\beta-1})| = |d(\beta - 1)x_0y_0z_0^{\beta-2}||x_0 - y_0| \leq |d(\beta - 1)x_0z_0^{\beta-1}||x_0 - y_0| \leq d|\beta - 1||x_0 - y_0|$. Consequently, the point (x_0, y_0) moves close to the diagonal S after the iteration under the map $F_{d,\beta}$. In fact, its contraction rate is smaller than or equal to $d|\beta - 1|$. For $(x_0, y_0) \in R_2 \cup R_3 \cup R_4$, it is easy to see that $|x_0 - y_0| \leq |x_1 - y_1|$. Let $(x_0, y_0) \in D_d$. Then (x_0, y_0) must enter R_1 infinitely many times. As a result, $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$.

For $0 < \beta < 1$ and $0 \leq x_0 \leq y_0 \leq 1$, we have that $|x_1 - y_1| = |dx_0^\beta y_0^\beta (x_0^{1-\beta} - y_0^{1-\beta})| = d|1 - \beta||x_0^\beta y_0^\beta z_0^{-\beta}||x_0 - y_0| \leq d|1 - \beta||x_0 - y_0|$. Here $z_0 \in [x_0, y_0]$. Similarly, we have that $\lim_{n \rightarrow \infty} F_{d,\beta}^n(x_0, y_0) \in S$ for any $(x_0, y_0) \in D_d$. \square

For $0 < d \leq 1$, the sufficient condition (5.1) can be further improved. In particular, we shall prove that the corresponding system is globally synchronized for any $\beta > 0$. To this end, we first prove that the corresponding smooth system is locally synchronized with the help of the Schwarzian derivative and a result of Singer [12]. The Schwarzian derivative was first introduced into the study of one dimensional systems by Singer [12] in 1978. It is a valuable tool for a number of reasons. For instance, it can be used to prove that a certain map has an entire interval on which the map is chaotic. In this paper, we shall use it to prove that the existence and the uniqueness of a globally stable periodic orbit for $f_{d,\beta}$ with $0 < d \leq 1$.

Definition 5.1. Let I be a compact interval and $f : I \rightarrow I$ be a piecewise monotone continuous map. This means that f is continuous and that f has a finite number of turning points, i.e., points in the interval of I where f has a local extremum. Such a map is called unimodal (resp., l -model) if $f(\partial I) \subset \partial I$ and if f has precisely one (resp., l) turning point (resp., points). Here ∂I is the boundary of I . We say that a periodic attractor is essential if it contains a turning point in its basin.

Definition 5.2. The Schwarzian derivative of a function f at x is

$$Sf(x) = f'''(x)/f'(x) - 3/2(f''(x)/f'(x))^2.$$

Theorem 5.2. (See [2].) Let $f \in C^3(I)$. Assume that f has negative Schwarzian derivative on I . Then

- (i) the immediate basin of any attracting periodic orbit contains either a turning point of f or a boundary point of the interval;
- (ii) each neutral periodic point is attracting.

Proposition 5.1. (i) Let $d \leq 1/((\beta + 1)((\beta + 1)/(\beta + 2))^\beta) =: d_{1,\beta}$. Then $f_{d,\beta}$ has a unique attracting fixed point at $p_{d,\beta}$. Moreover, $p_{d_{1,\beta},\beta} = (\beta + 1)/(\beta + 2)$.

(ii) If $x \in I_d$, then $\lim_{n \rightarrow \infty} f_{d,\beta}^n(x) = p_{d,\beta}$.

(iii) For $d_{1,\beta} < d \leq 1$, $f_{d,\beta}$ has a unique stable periodic two orbit $\{x_1, x_2\}$, which attracts all initial points except $p_{d,\beta}$. Consequently, $|f'_{d,\beta}(x_1)f'_{d,\beta}(x_2)| < 1$.

Proof. Note that $d_{1,\beta} < 1$. Hence, $f_{d,\beta}$ is a smooth map on I_d . Moreover, some direct calculations yield that $Sf_{d,\beta}(x) < 0$, $p_{d_{1,\beta},\beta} = (\beta + 1)/(\beta + 2)$ and that $0 > f'_{d,\beta}(p_{d,\beta}) > -1$ if and only if $d < d_{1,\beta}$. The first part the proof can then be completed by applying Theorem 5.2-(ii). Since $f_{d,\beta}(x)$ on I_d has no critical point, it then follows from Theorem 5.2-(i) that the immediate basin of contains a boundary of the interval I_d . Consequently, $p_{d,\beta}$ is a globally attracting fixed point of $f_{d,\beta}$ on I_d . This completes the proof of the second part. For $1 \geq d > d_{1,\beta}$, the fixed point becomes unstable. Hence, the graph of $f_{d,\beta}^2(x)$ must be under (resp.,

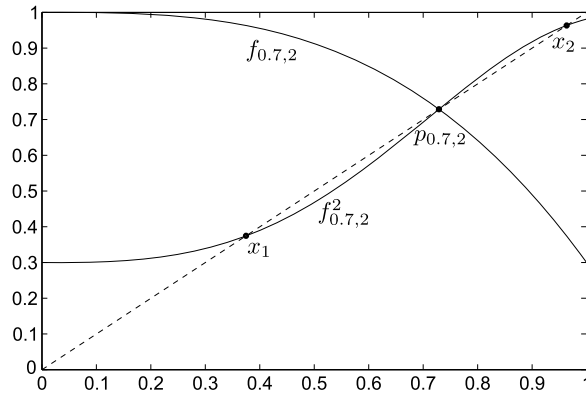


Fig. 5.1. The graphs of $f_{0.7,2}$ and $f_{0.7,2}^2$.

above) the diagonal whenever x is a little bit to the left (resp., right) of the fixed point $p_{d,\beta}$, as shown in Fig. 5.1. Otherwise, $p_{d,\beta}$ is stable. Since $f_{d,\beta}^2(0) = 1 - d$, the graph of $f_{d,\beta}^2$ must intersect the diagonal at points other than the fixed point. From the graph analysis, we see that at least one stable periodic point with period two must exist. Applying Theorem 5.2-(i), we conclude that the orbit of the period two points is unique. The assertions of the proposition-(iii) now follow from Theorem 5.2-(i). \square

Proposition 5.2. For $0 < d \leq 1$ and $\beta > 0$, $\lim_{n \rightarrow \infty} F_{d,\beta}^n(x_0, y_0) \in S$ provided that (x_0, y_0) is sufficiently close to B .

Proof. For $0 < d \leq 1$, the map $F_{d,\beta}$ is smooth. Depending on the range of d (see Proposition 5.1), we compute the derivative of $F_{d,\beta}$ at the fixed point $(p_{d,\beta}, p_{d,\beta})$ or the stable period two points $\{(x_1, x_1), (x_2, x_2)\}$. Then

$$F'_{d,\beta}(p_{d,\beta}, p_{d,\beta}) = -dp_{d,\beta}^\beta \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \quad \text{or} \quad (F_{d,\beta}^2)'(x_1, x_1) = d^2 x_1^\beta x_2^\beta \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}^2.$$

Thus, the largest eigenvalue λ in magnitude of $F'_{d,\beta}$ on $(p_{d,\beta}, p_{d,\beta})$ or (x_1, x_1) is $|\lambda| = |f'_{d,\beta}(p_{d,\beta})|$ or $|\lambda| = |f'_{d,\beta}(x_1)f'_{d,\beta}(x_2)|$, respectively. According to Proposition 5.1, we see that $|\lambda| < 1$. Hence, those periodic points on the diagonal is stable with respect to the map $F_{d,\beta}$. Using Proposition 5.1, we have that B is in the basin of the attraction for $F_{d,\beta}$. The assertion of the proposition now follows. \square

Theorem 5.3. Let $0 < d \leq 1$ and $\beta > 0$. Then $\lim_{n \rightarrow \infty} F_{d,\beta}^n(x_0, y_0) \in S$ for any initial condition $(x_0, y_0) \in D_d$.

Proof. Due to the symmetry of $F_{d,\beta}$ with respect to the diagonal $y = x$, it suffices to show that the assertion of the theorem holds true for $(x_0, y_0) \in K_0$. Assume that $\beta \geq 1$. Consider $F_{d,\beta}(D_d) \cap K_0$. It follows from Proposition 5.2 that there exists an ϵ such that if $|x_0 - y_0| < \sqrt{2}\epsilon$, then $\lim_{n \rightarrow \infty} F_{d,\beta}^n(x_0, y_0)$ lies on the diagonal. Let $\ell_{\theta_1,0}$ be the line segment joining the points \mathbf{B} and $x_{\pi/2,1-\epsilon}$, as illustrated in Fig. 5.2. Here $x_{\pi/2,1-\epsilon}$ is defined as in (4.3). Clearly $\theta_1 < \pi/2$. Let K_1 be the region whose boundaries are $\ell_{\theta_1,0}$, the x -axis and the diagonal. It then follows from Lemmas 4.1 and 4.2 that $F_{d,\beta}(K_1)$ contains a boundary $\ell_{\theta_1,1}$ which is strictly to the left of $\ell_{\pi/2,1}$ except at \mathbf{C} and \mathbf{B} . We may define inductively that $\ell_{\theta_n,0}$ are the line segments joining \mathbf{B} and $x_{\theta_{n-1},1-\epsilon}$ and K_n are the regions whose boundaries are $\ell_{\theta_n,0}$, the x -axis and the diagonal. Again, from Lemmas 4.1 and 4.2, one sees that $\lim_{n \rightarrow \infty} K_n = \{(x, x) : 1 - d \leq x \leq 1\}$. Note that $F_{d,\beta}(D_d) \cap K_0 \subset K_1 \cup \{(x, y) : y \leq x, 1 - \epsilon \leq y \leq 1, 1 - \epsilon \leq x \leq 1\} =: K_1 \cup \Gamma$. Moreover, K_1 and Γ both converge to the diagonal under the map $F_{d,\beta}$. Thus, for $\beta > 1$, $F_{d,\beta}^n(x, y)$ converges to the diagonal as n

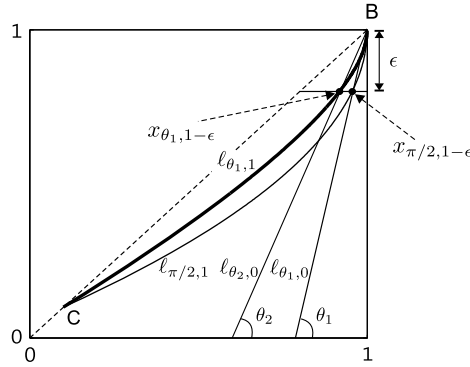


Fig. 5.2. The points $x_{\pi/2, 1-\epsilon}$ and $x_{\theta_1, 1-\epsilon}$. The graphs of $\ell_{\theta_1, 0}$, $\ell_{\pi/2, 1}$, $\ell_{\theta_1, 1}$ and $\ell_{\theta_2, 0}$ for $\beta > 1$.

approaches to infinity for any $(x, y) \in D_d$. The case for $0 < \beta < 1$ is similar. It is obvious that the theorem holds for $\beta = 1$. \square

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