Sums of orthogonal projections ☆

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A B S T R A C T

In this paper, we consider the problem of characterizing Hilbert space operators which are expressible as a sum of (finitely many) orthogonal projections. We obtain a special operator matrix representation and some necessary/sufficient conditions for an infinite-dimensional operator to be expressible as a sum of projections. We prove that a positive operator with essential norm strictly greater than one is always a sum of projections, and if an injective operator of the form $I + K$, where K is compact, is a sum of projections, then either trace $K_+ =$ trace $K_- = \infty$ or K is of trace class with trace K a nonnegative integer. We also consider sums of those projections which have a fixed rank. The closure of the set of sums of projections is also characterized.

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Which bounded linear operator on a complex Hilbert space can be expressed as the sum of finitely many orthogonal projections? (An *orthogonal projection* is an operator P

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with $P^2 = P = P^*$.) This is the problem we are going to address in this paper. If the underlying space is finite dimensional, then a characterization of such operators was obtained before by Fillmore [6]: a finite-dimensional operator is the sum of projections if and only if it is positive, it has an integer trace and the trace is greater than or equal to its rank. In this paper, we consider this problem for operators on an infinite-dimensional separable space. It turns out that in this situation the necessary/sufficient conditions we obtained for sums of projections are, after some appropriate interpretation, not too much different from the finite-dimensional ones. Although we haven't been able to give a complete characterization, we can reduce the whole problem to the consideration of operators of the form identity + compact.

The organization of this paper is as follows. In Section 1 below, we start by giving a special operator matrix representation for sums of projections (Theorem 1.2). This is used to give a more conceptual proof of the above result of Fillmore (Corollary 1.4). The main result of Section 2 is Theorem 2.2. It says that every positive operator with essential norm strictly greater than one is the sum of projections. This essentially reduces our problem to operators of the form identity + compact. We then concentrate on this latter class in Section 3 and derive some necessary conditions for such operators to be sums of projections. It culminates in Theorem 3.3 (or Corollary 3.4) in which we show that if an injective operator of the form $I + K$, where K is compact, is a sum of projections, then the traces of the positive and negative parts K_+ and K_- of K are either both infinity or both finite with the difference trace $K_+ -$ trace $K_-(=$ trace $K)$ a nonnegative integer. We end this section by giving an example showing that the converse of this is in general false. Then, in Section 4, we consider a variation of the projection-sum problem. They involve the characterizations of sums of projections which have some fixed (finite or infinite) rank. Finally, Section 5 gives the characterization of the closure, in the norm topology, of the set of sums of projections.

We remark that some methods from [3] on convex combinations of projections are used in the present paper.

Recall that an operator T is *positive* (resp., *strictly positive*), denoted by $T \geq 0$ (resp., $T > 0$), if $\langle Tx, x \rangle \geq 0$ (resp., $\langle Tx, x \rangle > 0$) for any (resp., any nonzero) vector x. For Hermitian operators A and B, $A \geq B$ (resp., $A > B$) means that $A - B \geq 0$ (resp., $A - B > 0$), and similarly for $A \leq B$ (resp., $A < B$). For an operator T on H and $1 \leq n \leq \infty$, $T^{(n)}$ denotes the operator $\underbrace{T \oplus \cdots \oplus T}_{n}$ on $H^{(n)} = \underbrace{H \oplus \cdots \oplus H}_{n}$. For a (closed) subspace K of H, P_K denotes the (orthogonal) projection from H onto K. We use I_H to denote the identity operator on H. This will be abbreviated to I if the underlying space needs not be emphasized. For a positive integer m, 0_m and I_m denote the zero and identity operators on $\mathbb{C}^m$, respectively. For an infinite-dimensional separable space H, let $\mathcal{B}(H)$ be the algebra of all operators on H, $\mathcal{K}(H)$ the ideal of compact operators on H, $\mathcal{B}(H)/\mathcal{K}(H)$ the Calkin algebra, and $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ the quotient map. In the following, we will need the Fredholm theory of operators. For this, the reader may consult [4, Chapter XI].

1. Operator matrix representation

We start by showing that in considering sums of projections we may as well assume that the operator is strictly positive.

Lemma 1.1. *An operator of the form $T \oplus 0$ is a sum of n projections if and only if T itself is a sum of n projections.*

Proof. If $T \oplus 0 = \sum_{j=1}^{n} P_j$ is the sum of the n projections

$$P_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \quad j = 1, \dots, n,$$

then $\sum_j A_j = T$ and $\sum_j D_j = 0$. Since all the D_j's are positive, the latter equality implies that $D_j = 0$ for all j. Hence $B_j = 0$ and $C_j = 0$ and therefore $T = \sum_j A_j$ is the sum of the n projections A_j. $\quad\square$

The main result of this section is a characterization of sums of projections in terms of a certain operator matrix representation.

Theorem 1.2. *Let T be a strictly positive operator. Then T is a sum of n projections if and only if, for the zero operator 0 on some space, $T \oplus 0$ is unitarily equivalent to an operator matrix of the form*

$$\begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix},$$

where $I_1, \dots, I_n$ denote the identity operators on some spaces. (Some of these operators will be absent if their underlying spaces are of zero dimension.)

Proof. If $T = \sum_{j=1}^{n} P_j$ is a sum of projections on H, then, letting $H_j = \operatorname{Range} P_j$ and $V_j : H_j \to H$ be the inclusion map for each j, we have $V_j^* V_j = I_j$, the identity operator on H_j, and $V_j V_j^* = P_j$. Let

$$W = \begin{bmatrix} V_1^* \\ \vdots \\ V_n^* \end{bmatrix} T^{-1/2} : \operatorname{Range} T^{1/2} \to \sum_{j=1}^{n} \oplus H_j.$$

Since

$$\left\| W\left(T^{1/2}x\right) \right\|^2 = \left\| \begin{bmatrix} V_1^* \\ \vdots \\ V_n^* \end{bmatrix} x \right\|^2 = \left\langle \begin{bmatrix} V_1 \cdots V_n \end{bmatrix} \begin{bmatrix} V_1^* \\ \vdots \\ V_n^* \end{bmatrix} x, x \right\rangle = \left\langle \sum_j P_j x, x \right\rangle$$

$$= \langle Tx, x \rangle = \left\| T^{1/2} x \right\|^2$$

for any x in H, we can extend W to an isometry, which is also denoted by W, from $\overline{\operatorname{Range} T^{1/2}} = H$ to $\sum_j \oplus H_j$. Note that

$$WTW^* = \begin{bmatrix} V_1^* \\ \vdots \\ V_n^* \end{bmatrix} [V_1 \cdots V_n] = \begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix}.$$

Hence this latter operator matrix is unitarily equivalent to $T \oplus 0$, where 0 denotes the zero operator on $\ker W^* (= \{\sum_j \oplus x_j \in \sum_j \oplus H_j \colon \sum_j x_j = 0\})$.

Conversely, assume that $T \oplus 0$ is unitarily equivalent to

$$B = \begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix}.$$

Let $C = [C_{ij}]_{i,j=1}^n$ be the positive square root of the positive operator B and $D_i = [C_{i1} \cdots C_{in}]^*$ for $i = 1, \ldots, n$. Then $B = C^2 = \sum_{i=1}^n D_i D_i^*$ and $D_i^* D_i = I_i$ for all i. Note that $D_i D_i^*$ is Hermitian and $(D_i D_i^*)^2 = D_i(D_i^* D_i)D_i^* = D_i I_i D_i^* = D_i D_i^*$. Hence $B = \sum_i D_i D_i^*$ is the sum of the projections $D_i D_i^*$. Lemma 1.1 then implies that T is a sum of projections. □

For finite-dimensional operators, the preceding theorem can be stated more precisely.

Corollary 1.3. *Let T be a strictly positive k-by-k matrix. Then T is a sum of n projections with ranks $r_1, \ldots, r_n$ if and only if $T \oplus 0_m$ is unitarily equivalent to*

$$\begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix},$$

where $m = \sum_{j=1}^n r_j - k \geq 0$, and I_j is the r_j-by-r_j identity matrix, $j = 1, \ldots, n$.

This is essentially shown in the proof of Theorem 1.2. In particular, for the sufficiency part, we need the equality $\operatorname{rank} D_j D_j^* = \operatorname{rank} D_j^* D_j$.

The characterization of sums of projections among finite-rank operators can be obtained easily, the finite-dimensional case of which is due to Fillmore [6].

Corollary 1.4. *An operator T is the sum of finite-rank projections if and only if $T \geq 0$ is of finite rank, $\operatorname{trace} T$ is an integer and $\operatorname{trace} T \geq \operatorname{rank} T$. In this case, T is the sum of $\operatorname{trace} T$ many rank-one projections.*

Proof. Since every finite-rank operator is the direct sum of a finite-dimensional operator and a zero operator, we may assume that T itself acts on a, say, n-dimensional space. To

prove the nontrivial sufficiency part, we may, as in the proof of [6, Theorem 1], subtract some rank-one projections from T and thus assume that trace $T = \operatorname{rank} T = n$. By [7, Corollary 2], T is unitarily equivalent to an n-by-n matrix with diagonal entries all equal to 1. Theorem 1.2 then implies that T is a sum of n rank-one projections. □

The condition in Theorem 1.2 can also be expressed in terms of the isotropic subspaces. Recall that a (closed) subspace K of H is *isotropic* for an operator T on H if $\langle Tx, x \rangle = 0$ for all x in K. This is the same as saying that $P_K T P_K = 0$.

Proposition 1.5. *The following conditions are equivalent for an operator T on a Hilbert space H:*

(a) *H is the direct orthogonal sum of n isotropic subspaces of T;*
(b) *there is a unitary operator U on H with $\sum_{j=0}^{n-1} U^{-j} T U^j = 0$;*
(c) *there is a unitary operator W on H with $W^n = I$ and $\sum_{j=0}^{n-1} W^{-j} T W^j = 0$.*

Proof. (a) $\Rightarrow$ (b). Under (a), there are projections $P_1, \ldots, P_n$ such that $P_k T P_k = 0$ for each k, $P_k P_l = 0$ for $k \neq l$, and $\sum_k P_k = I$. Let $\omega = \exp(2\pi i/n)$, an nth primitive root of unity, and $U = \sum_{k=1}^{n} \omega^k P_k$. Then

$$U^* = \sum_k \bar{\omega}^k P_k = \sum_k \omega^{-k} P_k = U^{-1}$$

and

$$\sum_{j=0}^{n-1} U^{-j} T U^j = \sum_{j=0}^{n-1} \sum_{k,l=1}^{n} \omega^{(l-k)j} P_k T P_l$$

$$= \sum_{k,l=1}^{n} \left(\sum_{j=0}^{n-1} \omega^{(l-k)j} \right) P_k T P_l = \sum_{k=1}^{n} P_k T P_k = 0$$

since $\sum_{j=0}^{n-1} \omega^{(l-k)j} = 0$ for $k \neq l$. These show that U is the required unitary operator.

Note that since $U^n = \sum_k \omega^{nk} P_k = \sum_k P_k = I$, the preceding arguments also prove (a) $\Rightarrow$ (c).

(b) $\Rightarrow$ (c). Let U be the asserted operator. Then

$$T - U^{-n} T U^n = \sum_{j=0}^{n-1} U^{-j} T U^j - \sum_{j=1}^{n} U^{-j} T U^j = 0 - U^{-1} \left(\sum_{j=0}^{n-1} U^{-j} T U^j \right) U = 0.$$

Hence T commutes with U^n. Let $f(e^{i\theta}) = e^{i\theta/n}$ for $\theta \in [0, 2\pi)$, and let $W = f(U^n)^{-1} U$. Then W is unitary,

$$W^n = f(U^n)^{-n} U^n = U^{-n} U^n = I,$$

and

$$\sum_{j=0}^{n-1} W^{-j}TW^j = \sum_{j=0}^{n-1} U^{-j}f(U^n)^j Tf(U^n)^{-j}U^j = \sum_{j=0}^{n-1} U^{-j}TU^j = 0$$

since T commutes with $f(U^n)^j$ for all j.

(c) $\Rightarrow$ (a). Let W be as asserted. Since $W^n = I$, the spectrum of W consists of eigenvalues from the set $\{\omega^k : 1 \le k \le n\}$. Hence, by the spectral theorem, $W = \sum_{k=1}^{n} \omega^k P_k$ for some projections P_k with $\sum_k P_k = I$. Since $WP_k = \omega^k P_k$ for each k, we have

$$0 = \frac{1}{n} P_k \left(\sum_{j=0}^{n-1} W^{-j}TW^j \right) P_k = \frac{1}{n} \sum_{j=0}^{n-1} (\bar{\omega}^{jk} P_k) T (\omega^{jk} P_k)$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} P_k T P_k = P_k T P_k.$$

This proves (a). $\square$

In the preceding proposition, the equivalence of (a) and (b) for $n = 2$ yields that T is unitarily equivalent to $-T$ if and only if T is unitarily equivalent to an operator matrix of the form $\begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix}$. Here the two diagonal zero operators may not be of the same size.

Theorem 1.6. *Let T be an operator on a Hilbert space H. Consider the following conditions*:

(a) *There are projections $P_1, \ldots, P_n$ on H such that $P_k T P_k = P_k$ for each k and $\sum_k P_k = I$.*

(b) *There is a unitary operator U on H such that $(1/n)\sum_{j=0}^{n-1} U^{-j}TU^j = I$.*

(c) *There is a unitary operator W on H such that $W^n = I$ and $(1/n)\sum_{j=0}^{n-1} W^{-j}TW^j = I$.*

(d) *T is a sum of n idempotent operators.*

Then (a), (b) and (c) are equivalent, which imply (d).

Furthermore, if T is positive, then the equivalent (a), (b) and (c) imply

(d′) *T is a sum of n projections.*

Proof. The equivalence of (a), (b) and (c) follows from Proposition 1.5 by letting A there be $T - I$ and noting that projections summed to I have their ranges orthogonal to each other.

(a) $\Rightarrow$ (d). Let the P_k's be as asserted. Then $T = \sum_k TP_k$ with $(TP_k)^2 = T(P_k T P_k) = TP_k$ for each k.

(a) $\Rightarrow$ (d$'$) for $T \geq 0$. In this case, we have $T = \sum_k T^{1/2} P_k T^{1/2}$ with

$$\left(T^{1/2} P_k T^{1/2}\right)^2 = T^{1/2}(P_k T P_k)T^{1/2}$$
$$= T^{1/2} P_k T^{1/2} = \left(T^{1/2} P_k T^{1/2}\right)^*. \qquad \square$$

Note that, in the preceding theorem, (a) is equivalent to the condition in Theorem 1.2 without the zero summand and thus (d) or (d$'$) does not imply (a), (b) and (c) (witness $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$). Also note that in Proposition 1.5 (resp., Theorem 1.6) condition (a) is described by $3n+1$ operator equations: $P_k^2 = P_k = P_k^*$ and $P_k A P_k = 0$ (resp., $P_k T P_k = P_k$) for $k = 1, \ldots, n$, and $\sum_k P_k = I$, (b) is described by 2 equations: $U^* = U^{-1}$ and $\sum_j U^{-j} A U^j = 0$ (resp., $\sum_j U^{-j} T U^j = nI$), and (c) by 3 equations: $U^* = U^{-1}$, $U^n = I$ and $\sum_j U^{-j} A U^j = 0$ (resp., $\sum_j U^{-j} T U^j = nI$).

The next corollary is useful in Section 2.

Corollary 1.7. *If $T = T_1 \oplus \cdots \oplus T_n$ on $H^{(n)}$, where the T_j's are positive operators satisfying $T_1 + \cdots + T_n = nI$, then T is a sum of n projections.*

Proof. Let

$$U = \begin{bmatrix} 0 & I & & & 0 \\ & 0 & I & & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & I \\ I & & \cdots & & 0 \end{bmatrix} \quad \text{on } H^{(n)}.$$

Then U is unitary and satisfies $(1/n)\sum_{j=0}^{n-1} U^{-j} T U^j = I$. Hence T is a sum of projections by Theorem 1.6. $\square$

2. Essential norm

For the remaining part of this paper, we only consider operators on infinite-dimensional separable spaces unless otherwise specified. We start with the following necessary conditions for sums of projections. They greatly facilitate our search for the exact characterization.

Proposition 2.1. *Let T be a sum of projections.*

(a) *If $\|T\| < 1$, then $T = 0$.*
(b) *If $\|T\|_e < 1$, then T is of finite rank.*
(c) *If $\|T\| \leq 1$, then T is a projection.*

Here $\|T\|_e$ denotes the *essential norm* of T: $\|T\|_e = \inf\{\|T + K\|: K \text{ compact}\}$. We remark that in the above situation if $\|T\|_e \leq 1$, then T is the sum of a projection and

a compact operator. This can either be proved as in (b) by using (c) or follow from Theorem 3.1 below.

Proof of Proposition 2.1. Let $T = \sum_{j=1}^{n} P_j$, where the P_j's are projections.

(a) Since $P_j \le T$ for all j, we have $\|P_j\| \le \|T\| < 1$. It follows that $P_j = 0$ for all j and hence $T = 0$.

(b) Consider $\pi(T) = \sum_j \pi(P_j)$ in the Calkin algebra, where $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is the quotient map, and represent $\pi(T)$ and $\pi(P_j)$'s as operators on some Hilbert space. Since $\|\pi(T)\| = \|T\|_e < 1$, (a) implies that $\pi(T) = 0$ or T is compact. Now $0 \le P_j \le T$ implies that P_j is also compact for every j (cf. [6, p. 146]). Hence P_j must be of finite rank. The same is then true for T.

(c) If $\|T\| \le 1$, then $0 \le P_1 + P_2 \le T \le I$. It was proved in [6, p. 151] that sums of two projections are exactly those which are unitarily equivalent to an operator of the form $A \oplus (2I - A) \oplus 0 \oplus 2I$, where $0 \le A \le I$. From this, we infer that $P_1 + P_2$ is actually itself a projection. Repeating this argument with other projections in the sum $T = \sum_j P_j$ yields that T is a projection. $\square$

In view of Proposition 2.1 (b) and Corollary 1.4, to characterize sums of projections we need only consider positive operators with essential norm at least one. The case when the essential norm is strictly greater than one is taken care of by the next theorem, which is the main result of this section. It was announced previously in [13, Theorem 4.12].

Theorem 2.2. *Any positive operator with essential norm strictly greater than one is the sum of finitely many projections.*

This will be proved via the following lemma.

Lemma 2.3. *If $n \ge 2$ and $0 \le T \le (1+(1/n))I$ on a Hilbert space H, then $T \oplus (1+(1/n))I$ on $H \oplus H$ is a sum of projections.*

Proof. Let $\lambda = 1 + (1/n)$. Since $T \oplus \lambda I$ is unitarily equivalent to the sum of the two operators

$$T \oplus T^{(\infty)} \oplus (\lambda I - T)^{(\infty)} \oplus (\lambda I)^{(\infty)} \oplus 0^{(\infty)}$$

and

$$0 \oplus (\lambda I - T)^{(\infty)} \oplus T^{(\infty)} \oplus 0^{(\infty)} \oplus (\lambda I)^{(\infty)}$$

on $H \oplus H^{(\infty)} \oplus H^{(\infty)} \oplus H^{(\infty)} \oplus H^{(\infty)}$ and since these latter two operators are both unitarily equivalent to $(T \oplus (\lambda I - T) \oplus \lambda I \oplus 0)^{(\infty)}$, to prove our assertion we need only check that $T \oplus (\lambda I - T) \oplus \lambda I$ is a sum of projections. This is indeed the case since

$T \oplus (\lambda I - T) \oplus \lambda I$ is unitarily equivalent to $T \oplus (\lambda I - T) \oplus (\lambda I)^{(n-1)}$ and the latter is a sum of projections by Corollary 1.7, completing the proof. □

Proof of Theorem 2.2. First assume that $0 \leq T \leq 2I$. As $\|T\|_e > 1$, there is an integer $n \geq 2$ such that $\|T\|_e > 1 + (1/n) \equiv \lambda$. We decompose T as $T_1 \oplus T_2 \oplus T_3$, where $0 \leq T_1 \leq \lambda I$ and $T_2, T_3 \geq \lambda I$, the latter two acting on infinite-dimensional spaces. Then $T = (T_1 \oplus \lambda I \oplus (T_3 - \lambda I)) + (0 \oplus (T_2 - \lambda I) \oplus \lambda I)$. Since $0 \leq T_1 \oplus (T_3 - \lambda I) \leq \lambda I$ and $0 \leq T_2 - \lambda I \leq \lambda I$, Lemma 2.3 implies that these latter two operators are sums of projections. Hence the same is true for T.

In general, for $\|T\| > 2$ (and $\|T\|_e > 1$), consider the function $f : [0, \|T\|] \to [0, 2]$ defined by

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 2, \\ 2 + t - \lceil t \rceil & \text{otherwise,} \end{cases}$$

where $\lceil t \rceil$ denotes the smallest integer larger than or equal to t. Then $0 \leq f(t) \leq 2$ for all t and $f(t) > 1$ for $t > 1$. Hence $0 \leq f(T) \leq 2I$ and $\|f(T)\|_e > 1$. Therefore, by the first paragraph of this proof, $f(T)$ is a sum of projections. On the other hand, since $t - f(t)$ is a nonnegative integer for all t, $T - f(T)$ is itself a sum of projections. We conclude that the same is true for $T = f(T) + (T - f(T))$. □

Corollary 2.4. *A scalar operator λI (on an infinite-dimensional space) is a sum of projections if and only if either $\lambda = 0$ or $\lambda \geq 1$.*

This is an easy consequence of Proposition 2.1 (a) and Theorem 2.2.

In recent years, there have been some works by Bownik et al. [1], Casazza et al. [2] and Kruglyak et al. [11,12] on characterizing, for each fixed $N \geq 1$ and $n \geq 1$, the scalars λ for which λI_N is the sum of n many N-by-N projections. The ones in [1,2] are in terms of the frame theory. In particular, [2] solves the case of λI_N as the sum of n fixed-rank projections, which is related to our results in Section 4 below.

3. Identity + compact

In light of the results in Sections 1 and 2, for the characterization of sums of projections we may restrict ourselves to operators which are strictly positive and have essential norm equal to one. For such operators, we can strengthen Theorem 1.2 to the following main theorem of this section.

Theorem 3.1. *Let T be a strictly positive operator on the infinite-dimensional separable space H with $\|T\|_e = 1$. Then T is a sum of n projections if and only if, for some nonnegative integer m, $T \oplus 0_m$ is unitarily equivalent to an operator X of the form*

$$\begin{bmatrix} I_1 & & * \\ & \ddots & \\ * & & I_n \end{bmatrix}, \tag{1}$$

where the off-diagonal entries are all compact operators. In particular, in this case $T = I + K$, where K is a compact Hermitian operator with $K > -I$. Furthermore, if K is of trace class, then $m = \operatorname{trace} K$ and all off-diagonal entries of X are trace-class operators.

Proof. Assume first that T is a sum of n projections. Then Theorem 1.2 gives the unitary equivalence of $T \oplus 0$ (for 0 on some space M) and an operator X of the form (1). Hence, $\|X\|_e = \|T \oplus 0\|_e = \|T\|_e = 1$. Represent $\pi(X)$, where $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is the quotient map, as an operator on a Hilbert space. Then $\pi(X)$ can also be considered to be of the form (1) (with a possibly smaller n). Since $0 \leq \pi(X) \leq I$, we infer that $\pi(X) = I_1 \oplus \cdots \oplus I_n$, and hence the off-diagonal entries of X are all compact. In particular, this implies that $T = I + K$ for some compact Hermitian K with $K > -I$ and $0 = 0_m$ on the m-dimensional M, $0 \leq m < \infty$. The converse of our assertion follows from Theorem 1.2.

Furthermore, if K is of trace class, then so is $(T \oplus 0) - I$ or $X - I$. It follows that the off-diagonal entries of X are also of trace class and

$$0 = \operatorname{trace}(X - I) = \operatorname{trace}\big((T \oplus 0) - I\big) = \operatorname{trace}\big(K \oplus (-I_m)\big) = \operatorname{trace} K - m.$$

Hence $m = \operatorname{trace} K$, completing the proof. $\square$

Note that if T is a sum of projections (on an infinite-dimensional separable Hilbert space) with $\|T\|_e = 1$, then, from Theorem 3.1, we deduce that T is the sum of a projection and a compact operator. It would be interesting to know which operators in this latter class are sums of projections.

Corollary 3.2. *Let $T = T_1 \oplus I$ on $H_1 \oplus H_2$, where $\dim H_1 < \infty$. Then the following are equivalent*:

(a) *T is a sum of projections,*
(b) *T_1 is a sum of projections, and*
(c) *$T_1 \geq 0$, $\operatorname{trace} T_1$ is a nonnegative integer and $\operatorname{trace} T_1 \geq \operatorname{rank} T_1$.*

Proof. The implication (c) $\Rightarrow$ (b) follows from Corollary 1.4. The implication (b) $\Rightarrow$ (a) is obvious. Thus we only need to prove (a) $\Rightarrow$ (c). Under (a), we may even assume, by Lemma 1.1, that T is strictly positive. Since $\|T\|_e = 1$, Theorem 3.1 implies that $\operatorname{trace}(T - I)$ is a nonnegative integer, say, m. Therefore,

$$\operatorname{trace} T_1 = \operatorname{trace}(T_1 - I) + \dim H_1 = \operatorname{trace}(T - I) + \dim H_1 = m + \dim H_1.$$

Thus $\operatorname{trace} T_1$ is a nonnegative integer and $\operatorname{trace} T_1 \geq \operatorname{rank} T_1$. This proves (c). $\square$

For operators of the form identity + compact, the next theorem gives necessary conditions for them to be sums of projections.

Theorem 3.3. *If $T = I + K$ is a sum of projections, where K is a compact Hermitian operator with $K > -I$, then $\operatorname{trace} K_+ \geq \operatorname{trace} K_-$ (possibly infinity on either side). Moreover, under these conditions, the following hold*:

(a) *K is of trace class if and only if K_+ or K_- is of trace class.*
(b) *K is of finite rank if and only if K_+ or K_- is of finite rank.*

Corollary 3.4. *If $T = I + K$ is a sum of projections, where K is an infinite-rank compact Hermitian operator with $K > -I$, then*

(a) *both K_+ and K_- have infinite rank, and*
(b) *either $\operatorname{trace} K_+ = \operatorname{trace} K_- = \infty$ or K is of trace class with $\operatorname{trace} K$ a nonnegative integer.*

Recall that the *positive* and *negative parts* of an operator A are by definition $A_+ = (|A| + A)/2$ and $A_- = (|A| - A)/2$, respectively, where $|A| = (A^*A)^{1/2}$.

Proof of Theorem 3.3. Assume that T is a sum of n projections. Then, for some nonnegative integer m, $T \oplus 0_m$ is unitarily equivalent to an operator X of the form (1) by Theorem 3.1. If $A = X - I$, then A_+ (resp., A_-) is unitarily equivalent to $K_+ \oplus 0_m$ (resp., $K_- \oplus I_m$). Hence $\operatorname{trace} A_+ = \operatorname{trace} K_+$ and $\operatorname{trace} A_- = \operatorname{trace} K_- + m$. If $\{e_j\}_j$ is the orthonormal basis for which $\langle Ae_j, e_j \rangle = 0$ for all j, then

$$\operatorname{trace} A_+ = \sum_j \langle A_+ e_j, e_j \rangle$$

$$= \sum_j \langle Ae_j, e_j \rangle + \sum_j \langle A_- e_j, e_j \rangle$$

$$= \sum_j \langle A_- e_j, e_j \rangle = \operatorname{trace} A_-.$$

It follows that $\operatorname{trace} K_+ = \operatorname{trace} K_- + m \geq \operatorname{trace} K_-$ as asserted.

(a) is an easy consequence of the above. We now prove (b). Since A is unitarily equivalent to $A' \equiv K \oplus (-I_m) = (K_+ \oplus 0_m) - (K_- \oplus I_m)$, by Proposition 1.5 there is a unitary operator U such that $\sum_{j=0}^{n-1} U^{-j} A' U^j = 0$. We thus obtain $\sum_j U^{-j}(K_+ \oplus 0_m)U^j = \sum_j U^{-j}(K_- \oplus I_m)U^j$. Hence

$$\operatorname{rank} K_+ \leq \operatorname{rank}\left(\sum_j U^{-j}(K_+ \oplus 0_m)U^j\right) = \operatorname{rank}\left(\sum_j U^{-j}(K_- \oplus I_m)U^j\right)$$

$$\leq \sum_j \operatorname{rank}\left(U^{-j}(K_- \oplus I_m)U^j\right) = n \cdot \operatorname{rank}(K_- \oplus I_m) = n(\operatorname{rank} K_- + m)$$

and, similarly, $\operatorname{rank} K_- \leq n \cdot \operatorname{rank} K_+ - m$. (b) follows immediately. □

The preceding theorem gives some necessary conditions on the trace and rank in order that an operator of the form identity + compact be a sum of projections. Are these conditions sufficient? Unfortunately, the next example shows that the answer turns out to be "NO".

Example 3.5. Let K be the diagonal operator $\mathrm{diag}(a_1, a_2, \ldots) \oplus \mathrm{diag}(-b_1, -b_2, \ldots)$ on $l^2 \oplus l^2$, where $a_k = 1/2^k$ for $k \geq 1$ and the b_k's satisfy $b_1 \geq b_2 \geq \cdots > 0$ and $\sum_{k=1}^{l^2} b_k = \sum_{k=1}^{l} a_k$ for each $l \geq 1$. Then K is of trace class with trace $K = 0$. We show that the strictly positive $T \equiv I + K$ is not a sum of projections. Indeed, if T is a sum of n projections, then Theorem 3.1 and Proposition 1.5 imply that there is a unitary operator U such that $\sum_{j=0}^{n-1} U^{-j} K U^j = 0$. Let $e_k = (0, \ldots, 0, \underset{k\text{th}}{1}, 0, \ldots) \oplus 0$ in $l^2 \oplus l^2$ for $1 \leq k \leq n$, $L = \bigvee_{k=1}^{n} \{e_k\}$ and $M = \bigvee_{j=0}^{n-1} U^{-j} L$. Express K as $K_+ - K_-$. Note that, for each j, $0 \leq j \leq n-1$, $P_M(U^{-j} K_+ U^j) P_M$ is positive compact with $a_1, \ldots, a_n$ as its n largest eigenvalues. Hence trace $P_M(U^{-j} K_+ U^j) P_M \geq \sum_{k=1}^{n} a_k$. On the other hand, $P_M(U^{-j} K_- U^j) P_M$ is also positive compact with no more than $n^2 - n$ many (strictly) positive eigenvalues. Thus we have trace $P_M(U^{-j} K_- U^j) P_M < \sum_{k=1}^{n^2} b_k$. These two together lead to

$$0 = \mathrm{trace}\, P_M \left(\sum_{j=0}^{n-1} U^{-j} K U^j \right) P_M$$

$$= \sum_{j=0}^{n-1} \mathrm{trace}\, P_M \left(U^{-j} K_+ U^j \right) P_M - \sum_{j=0}^{n-1} \mathrm{trace}\, P_M \left(U^{-j} K_- U^j \right) P_M$$

$$> n \sum_{k=1}^{n} a_k - n \sum_{k=1}^{n^2} b_k = 0,$$

a contradiction. Hence T cannot be a sum of projections.

We conclude this section with more information on sums of projections of the form identity + compact.

Theorem 3.6. Let $I + K = \sum_{j=1}^{n} P_j$ on a Hilbert space H, where K is compact and P_j's are projections. Then

(a) $P_j K P_j \geq 0$ for all j,
(b) $P_j P_k$ is compact for all $j \neq k$, and
(c) there are projections $Q_1, \ldots, Q_n$ with orthogonal ranges such that $\sum_j Q_j = I$ and $P_j - Q_j$ is compact for all j.

Proof. Since

$$0 \leq (P_j P_k)(P_j P_k)^* = P_j P_k P_j \leq P_j(I + K - P_j)P_j = P_j K P_j$$

for any $j \neq k$, we infer that $P_j K P_j \geq 0$ and $(P_j P_k)(P_j P_k)^*$ is compact. Through the polar decomposition of $(P_j P_k)^*$, we obtain the compactness of $(P_j P_k)^* = P_k P_j$. These prove (a) and (b).

To prove (c), let $Q_1 = P_1$ and $Q_2 = f((I - P_1)P_2(I - P_1))$, where f is the function on $[0, 1]$ given by

$$f(t) = \begin{cases} 1 & \text{if } 1/2 \leq t \leq 1, \\ 0 & \text{if } 0 \leq t < 1/2. \end{cases}$$

If $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is the quotient map from $\mathcal{B}(H)$ to $\mathcal{B}(H)/\mathcal{K}(H)$, then

$$\begin{aligned} \pi(Q_2) &= \pi\big(f\big((I - P_1)P_2(I - P_1)\big)\big) \\ &= f\big(\pi\big((I - P_1)P_2(I - P_1)\big)\big) = f\big(\pi(P_2)\big) \\ &= \pi\big(f(P_2)\big) = \pi(P_2), \end{aligned}$$

where $\pi((I - P_1)P_2(I - P_1)) = \pi(P_2)$ holds because

$$(I - P_1)P_2(I - P_1) - P_2 = -P_1 P_2 - P_2 P_1 + P_1 P_2 P_1$$

is compact by (b). This shows that $K' \equiv P_2 - Q_2$ is compact. Since $Q_1 + Q_2$ is a projection, $(Q_1 + Q_2) + P_3 + \cdots + P_n$ is a sum of $n - 1$ projections and is equal to $\sum_{j=1}^{n} P_j - K' = I + K - K'$ with $K - K'$ compact. Hence we may repeat the above arguments inductively to obtain projections $Q_3, \ldots, Q_n$ with orthogonal ranges such that $Q \equiv \sum_{j=1}^{n} Q_j$ is of the form identity + compact and $P_j - Q_j$ is compact for all j. In particular, Q is a projection with nullity $Q < \infty$. Replacing Q_n by $Q'_n \equiv Q_n + P_{\ker Q}$ yields the projections $Q_1, \ldots, Q_{n-1}, Q'_n$, which satisfy all our requirements. □

In the remaining part of this paper, we consider some variations of the projection-sum problem. There are two different types. One involves sums of projections with some additional properties. For example, we may require that the projections have some fixed (finite or infinite) rank. Results of this nature are given in Section 4 below. Another type concerns operators which can be approximated by sums of projections in the norm topology. These will be in Section 5.

4. Fixed-rank projections

We start with a characterization of sums of infinite-rank projections.

Proposition 4.1. *An operator T is a sum of n infinite-rank projections if and only if it has infinite rank and is a sum of n projections.*

Proof. The necessity is trivial. To prove the sufficiency, assume that $T = \sum_{j=1}^{n} P_j$ is a sum of n projections, where P_j is of finite rank for $j = 1, \ldots, m$ ($1 \leq m < n$) and

of infinite rank otherwise. Let K be the finite-dimensional subspace $\bigvee_{j=1}^m((\text{Range } P_j) \vee P_{m+1}(\text{Range } P_j))$. Then K is invariant for $P_1, \ldots, P_{m+1}$. Let $P_j = Q_j \oplus 0$, $j = 1, \ldots, m$, and $P_{m+1} = Q_{m+1} \oplus R$ on the decomposition $K \oplus K^\perp$. Then $T' \equiv \sum_{j=1}^{m+1} P_j = (\sum_{j=1}^{m+1} Q_j) \oplus R$. Since the infinite-rank R is unitarily equivalent to $I^{(m+1)} \oplus 0$, where the identity operator I acts on an infinite-dimensional space, T' is unitarily equivalent to the sum of the operators

$$Q_j \oplus (0 \oplus \cdots \oplus \underset{j\text{th}}{I} \oplus \cdots \oplus 0) \oplus 0, \quad j = 1, \ldots, m+1,$$

each of which is an infinite-rank projection. It follows that $T = T' + \sum_{j=m+2}^n P_j$ is a sum of n infinite-rank projections. $\quad\square$

For a characterization of sums of rank-k projections for a fixed $k \geq 1$, we need the following lemma.

Lemma 4.2. *If $T = P_1 + P_2$ on a Hilbert space H, where P_1 and P_2 are projections with* $\text{rank } P_1 \leq \text{rank } P_2 < \infty$*, then there are two projections Q_1 and Q_2 with* $\text{rank } Q_1$ *equal to* $\text{rank } Q_2$ *or* $\text{rank } Q_2 - 1$ *such that $T = Q_1 + Q_2$.*

Proof. Considering PTP instead of T, where P is the projection with range equal to $(\text{Range } P_1) \vee (\text{Range } P_2)$, we may assume that H is finite dimensional. Let $n = \dim H$ and $n_j = \text{rank } P_j$, $j = 1, 2$. Then

$$\dim\big((\text{Range } P_1)^\perp \cap (\text{Range } P_2)\big) = \dim(\text{Range } P_1)^\perp + \dim \text{Range } P_2$$
$$- \dim\big((\text{Range } P_1)^\perp \vee (\text{Range } P_2)\big)$$
$$\geq (n - n_1) + n_2 - n = n_2 - n_1 \geq 0.$$

Let K be a subspace of $(\text{Range } P_1)^\perp \cap (\text{Range } P_2)$ with dimension $\lfloor (n_2 - n_1)/2 \rfloor$, the largest integer less than or equal to $(n_2 - n_1)/2$, and let $P_3 = P_K$. Then $P_3 P_1 = 0$ (since $K \subseteq (\text{Range } P_1)^\perp$) and $P_3 \leq P_2$ (since $K \subseteq \text{Range } P_2$). If $Q_1 = P_1 + P_3$ and $Q_2 = P_2 - P_3$, then Q_1 and Q_2 are projections, $Q_1 + Q_2 = P_1 + P_2 = T$,

$$\text{rank } Q_1 = \text{rank } P_1 + \text{rank } P_3 = n_1 + \lfloor (n_2 - n_1)/2 \rfloor$$

and

$$\text{rank } Q_2 = \text{rank } P_2 - \text{rank } P_3 = n_2 - \lfloor (n_2 - n_1)/2 \rfloor.$$

Hence $\text{rank } Q_1$ equals $\text{rank } Q_2$ or $\text{rank } Q_2 - 1$ depending on whether $n_2 - n_1$ is even or odd. This completes the proof. $\quad\square$

Repeated applications of Lemma 4.2 yield the following.

Proposition 4.3. *If T is a sum of n finite-rank projections, then T can be rewritten as a sum of projections $P_1, \ldots, P_n$ satisfying*

$$\operatorname{rank} P_1 \leq \operatorname{rank} P_2 \leq \cdots \leq \operatorname{rank} P_n \leq \operatorname{rank} P_1 + 1 < \infty.$$

Corollary 4.4. *Let n and k be two positive integers. Then T is the sum of n rank-k projections if and only if T is the sum of n projections and $\operatorname{trace} T = nk$.*

Proof. The necessity is trivial. To prove the sufficiency, let $T = \sum_{j=1}^{n} P_j$ be a sum of n projections and $\operatorname{trace} T = nk$. By Proposition 4.3, we may assume that $k_j \equiv \operatorname{rank} P_j$, $1 \leq j \leq n$, satisfy $k_1 \leq \cdots \leq k_n \leq k_1 + 1$. Since

$$nk_1 \leq \sum_{j=1}^{n} k_j = \operatorname{trace} T = nk \leq nk_n,$$

we obtain $k_1 \leq k \leq k_n \leq k_1 + 1$. Hence k equals k_1 or $k_1 + 1$. In either case, we have $k_j = k$ for all j. This completes the proof. □

We now proceed to a more detailed analysis of the projections which appear in a sum by considering both the rank and nullity of a projection. We start with the following two lemmas.

Lemma 4.5. *Let P and Q be two projections on an infinite-dimensional Hilbert space. If either $\operatorname{rank} P$ is infinite and $\operatorname{rank} Q$ is finite or $\operatorname{nullity} P$ is infinite and $\operatorname{nullity} Q$ is finite, then there are two projections R_1 and R_2 whose ranks and nullities are all infinity such that $P + Q = R_1 + R_2$.*

Proof. First assume that $\operatorname{rank} P = \infty$ and $\operatorname{rank} Q < \infty$. Then $(\operatorname{Range} P) \cap (\operatorname{Range} Q)^{\perp}$ must be infinite dimensional. We can express it as the direct sum $H_1 \oplus H_2$ of two infinite-dimensional subspaces H_1 and H_2. If $R = P_{H_1}$, then $RP = PR = R$ and $RQ = QR = 0$. Thus $R_1 = P - R$ and $R_2 = Q + R$ are two projections with infinite ranks and nullities such that $P + Q = R_1 + R_2$.

On the other hand, if $\operatorname{nullity} P = \infty$ and $\operatorname{nullity} Q < \infty$, then apply the above to $I - P$ and $I - Q$ to obtain projections R_1' and R_2' with infinite ranks and nullities such that $(I - P) + (I - Q) = R_1' + R_2'$. Then $P + Q = (I - R_1') + (I - R_2')$ is the required sum. □

Lemma 4.6. *Let T be the sum of n projections on an infinite-dimensional Hilbert space. If both T and $nI - T$ have infinite ranks, then there are projections $P_1, \ldots, P_n$, all of which have infinite ranks and nullities, such that $T = P_1 + \cdots + P_n$.*

Proof. Assume that $T = \sum_{j=1}^{n} Q_j$, where the Q_j's are projections. As $\operatorname{rank} T = \operatorname{rank}(nI - T) = \infty$, there are some j and k $(1 \leq j, k \leq n)$, not necessarily distinct,

such that $\operatorname{rank} Q_j = \operatorname{rank}(I - Q_k) = \operatorname{nullity} Q_k = \infty$. Applying Lemma 4.5 repeatedly, we may replace $Q_1, \ldots, Q_n$, one pair at a time, by projections $P_1, \ldots, P_n$ so that all of the latter have infinite ranks and nullities and satisfy $T = \sum_{j=1}^{n} P_j$. □

We can now combine together Propositions 4.1 and 4.3 and Lemma 4.6 into the following theorem.

Theorem 4.7. *If T is the sum of n projections, then there are projections $P_1, \ldots, P_n$ such that $T = P_1 + \cdots + P_n$ with the following additional conditions:*

(a) $\operatorname{rank} P_1 \leq \operatorname{rank} P_2 \leq \cdots \leq \operatorname{rank} P_n = \operatorname{rank} P_1 + 1$, *if T has finite rank and trace T is a positive integer not divisible by n,*

(b) $\operatorname{nullity} P_1 \leq \operatorname{nullity} P_2 \leq \cdots \leq \operatorname{nullity} P_n = \operatorname{nullity} P_1 + 1$, *if $nI - T$ has finite rank and $\operatorname{trace}(nI - T)$ is a positive integer not divisible by n,*

(c) *the P_j's are unitarily equivalent to each other, if otherwise.*

Proof. (a) follows from Proposition 4.3 and (b) from applying (a) to $nI - T$. To prove (c), note that if $\operatorname{rank} T = \operatorname{rank}(nI - T) = \infty$, then the assertion follows from Lemma 4.6. On the other hand, if $\operatorname{rank} T < \infty$ and $\operatorname{trace} T$ is divisible by n, then Proposition 4.3 implies that $T = \sum_{j=1}^{n} P_j$ for some projections P_j with equal finite ranks, whose unitary equivalences then follow immediately. Similarly, if $\operatorname{rank}(nI - T) < \infty$ and $\operatorname{trace}(nI - T)$ is divisible by n, then apply the above to $nI - T$ to obtain $nI - T = \sum_{j=1}^{n} Q_j$ with unitarily equivalent projections Q_j. Letting $P_j = I - Q_j$ for $1 \leq j \leq n$, we thus have $T = \sum_{j=1}^{n} P_j$ with unitarily equivalent P_j's as required. These cover all the remaining cases of (a) and (b). □

5. Closure of sums of projections

In this final section, we turn to problems on approximating operators by sums of projections. Note that in the finite-dimensional case, the set of sums of (resp., sums of two) projections is itself closed. These are easy consequences of the results of Fillmore [6]. We now turn to sums of two projections on infinite-dimensional spaces. Recall that an operator is such a sum if and only if it is unitarily equivalent to an operator of the form $0 \oplus I \oplus 2I \oplus A \oplus (2I - A)$, where $0 < A < I$ (cf. [6, p. 151]). The next result gives a characterization of operators (on an infinite-dimensional space) which can be approximated by such operators in norm.

Proposition 5.1. *For an infinite-dimensional separable Hilbert space H, let $\mathcal{S} = \{S \in \mathcal{B}(H): S$ is a sum of two projections$\}$. Then T is in the norm closure of $\mathcal{S}$ if and only if it is unitarily equivalent to an operator of the form $0 \oplus I \oplus 2I \oplus A \oplus (2I - B)$, where $0 < A, B < I$, $\sigma(A) = \sigma(B)$ and A and B have the same multiplicity for each of their common isolated eigenvalues.*

Proof. Let $T = A \oplus (2I - B)$, where A and B satisfy the asserted properties. For any positive integer n, consider a partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, with $t_{j+1} - t_j < 1/n$ for all j, $1 \le j \le n$, such that each t_j is not an eigenvalue of either A or B. This is possible since, on a separable space, A and B can have only countably many eigenvalues. Define a step function $f : [0, 1] \to [0, 1]$ by $f(t) = t_j$ for t in $[t_j, t_{j+1})$, $0 \le j \le n$, and $f(1) = 1$. Then $0 \le f(A), f(B) \le I$, $\|f(A) - A\|, \|f(B) - B\| < 1/n$, $\sigma(f(A)) = \sigma(f(B)) \subseteq \{t_0, \ldots, t_n\}$, and $f(A)$ and $f(B)$ have the same multiplicity at each t_j. Hence $\|T - f(A) \oplus (2I - f(B))\| < 1/n$ and $U^* f(A) U = f(B)$ for some unitary operator U. It follows that $f(A) \oplus (2I - f(B))$ is unitarily equivalent to $f(A) \oplus (2I - f(A))$ while the latter is the sum of the two projections

$$P_{\pm} = \begin{bmatrix} I & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} C & \pm(C - C^2)^{1/2} \\ \pm(C - C^2)^{1/2} & I - C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix},$$

where $C = f(A)/2$. This shows that T is in the asserted closure. We remark that this can also be proved by appealing to [8, Theorem 1] on the approximate unitary equivalence of normal operators and [6, p. 151] on the characterization of sums of two projections.

Conversely, assume that $\{T_n\}$ is a sequence of sums of two projections which converges to an operator T in norm. Then $C_n \equiv (T_n - I)_+$ (resp., $D_n \equiv (T_n - I)_-$) converges to $C \equiv (T - I)_+$ (resp., $D \equiv (T - I)_-$) in norm. Since $0 \le T \le 2I$, we have $0 \le C, D \le I$. We first show that $\sigma(C) \cup \{0, 1\} = \sigma(D) \cup \{0, 1\}$. Indeed, from the structure of sums of two projections, we have $\sigma(C_n) \cup \{0, 1\} = \sigma(D_n) \cup \{0, 1\}$ for all n. Since the function which maps an operator to its spectrum is continuous when restricted to the normal ones (cf. [9, Problem 105]), we obtain, as n approaches infinity, $\sigma(C) \cup \{0, 1\} = \sigma(D) \cup \{0, 1\}$ as asserted. Next, let λ, $0 < \lambda < 1$, be any isolated eigenvalue of C. We will show that λ, as an (isolated) eigenvalue of D, has the same multiplicity as that for C. Let $0 < \varepsilon_1 < \varepsilon_2 < \min\{\lambda, 1 - \lambda\}$ be such that $\sigma(C) \subseteq \Omega \equiv [0, \lambda - \varepsilon_2) \cup (\lambda - \varepsilon_1, \lambda + \varepsilon_1) \cup (\lambda + \varepsilon_2, 1]$, and let $f : [0, 1] \to [0, 1]$ be a continuous function such that

$$f(t) = \begin{cases} 1 & \text{if } t \in (\lambda - \varepsilon_1, \lambda + \varepsilon_1), \\ 0 & \text{if } t \in [0, \lambda - \varepsilon_2) \cup (\lambda + \varepsilon_2, 1]. \end{cases}$$

Since $\sigma(C_n)$ converges to $\sigma(C)$ as n approaches infinity, there exists an N such that $\sigma(C_n) \subseteq \Omega$ for all $n \ge N$. From $C_n \to C$ in norm, we obtain $f(C_n) \to f(C)$ in norm (cf. [9, Problem 126]). Since $P_n \equiv f(C_n)$ and $P \equiv f(C)$ are projections, we infer that $\operatorname{rank} P_n = \operatorname{rank} P$ for all large n (cf. [9, Problem 57]). Similarly, we have $\operatorname{rank} Q_n = \operatorname{rank} Q$ for all large n, where $Q_n = f(D_n)$ and $Q = f(D)$ are projections. Since $\operatorname{rank} P_n = \operatorname{rank} Q_n$ for all n by the structure of sums of two projections, we obtain $\operatorname{rank} P = \operatorname{rank} Q$ or $\dim\{x: Cx = \lambda x\} = \dim\{y: Dy = \lambda y\}$. The assertions in the statement of our proposition then follow immediately. $\square$

Our final result is a characterization of the norm closure of the set of sums of projections. We start with the following lemma.

Lemma 5.2. *If A and B are positive finite-rank operators on a Hilbert space H, then* $\text{trace}(A - B) \leq (\text{rank } A)\|A - B\|$ *and* $|\text{trace } A - \text{trace } B| \leq \max\{\text{rank } A, \text{rank } B\}\|A - B\|$ *hold.*

Proof. We may assume that $n \equiv \text{rank } A > 0$. Let P be a projection on H with $\text{rank } P = n$ and $A = PAP$. Then

$$\|A - B\| \geq \|P(A - B)P\| \geq \frac{1}{n}\text{trace}\big(P(A - B)P\big) = \frac{1}{n}\big(\text{trace}(PAP) - \text{trace}(PBP)\big)$$

$$\geq \frac{1}{n}(\text{trace } A - \text{trace } B)$$

and our assertions follow. □

Note that the inequalities in the preceding lemma are sharp. This is seen by the n-by-n strictly positive matrices A and B satisfying $A = B + tI_n$ for some $t > 0$, in which case the equalities

$$\text{trace}(A - B) = nt = (\text{rank } A)\|A - B\| = \max\{\text{rank } A, \text{rank } B\}\|A - B\|$$

hold.

Consider, for each positive integer k, the closure of the set $\mathcal{S}_k$ consisting of sums of k rank-one projections on a space H. As proven in Corollary 1.4, each $\mathcal{S}_k$ is equal to the set $\{T \in \mathcal{B}(H): T \geq 0, \text{rank } T \leq \text{trace } T = k\}$ and the set $\mathcal{S}_0$ of sums of finite-rank projections equals $\bigcup_{k=1}^{\infty} \mathcal{S}_k$. The next proposition says that all the $\mathcal{S}_k$'s, $k \geq 1$, are norm-closed.

Proposition 5.3. *For each positive integer k, the set $\mathcal{S}_k = \{T \in \mathcal{B}(H): T \text{ is a sum of } k \text{ rank-one projections}\}$ is norm-closed.*

Proof. Let $T = \lim_n T_n$ in norm, where $T_n \geq 0$ and $\text{rank } T_n \leq \text{trace } T_n = k$ for each n. Obviously, we have $T \geq 0$. We now show that T also satisfies $\text{rank } T \leq \text{trace } T = k$. Indeed, assume first that $\text{rank } T > k$. Then there is a projection P with $k < \text{rank } P < \infty$ and $\varepsilon P \leq T$ for some $\varepsilon > 0$. As $\lim_n T_n = T$ in norm, there is an integer n_0 such that $\|T_{n_0} - T\| < \varepsilon/2$. Therefore, $T_{n_0} - T + (\varepsilon/2)I > 0$ and hence

$$PT_{n_0}P - \frac{\varepsilon}{2}P = P\left(T_{n_0} - T + \frac{\varepsilon}{2}I\right)P + P(T - \varepsilon P)P \geq 0.$$

This implies that $\text{rank } PT_{n_0}P \geq \text{rank } P > k$, which contradicts $\text{rank } PT_{n_0}P \leq \text{rank } T_{n_0} \leq k$. Thus we must have $\text{rank } T \leq k$. Finally, Lemma 5.2 yields that

$$|\text{trace } T - k| = |\text{trace}(T - T_n)| \leq \max\{\text{rank } T, \text{rank } T_n\}\|T - T_n\| \leq k\|T - T_n\|$$

for each n. Letting n approach infinity, we obtain trace $T = k$. This shows that T is in $\mathcal{S}_k$ and thus $\mathcal{S}_k$ is norm-closed as asserted. $\quad\square$

Next we show the norm-closedness of $\mathcal{S}_0 = \bigcup_{k=1}^{\infty} \mathcal{S}_k$. Note that the sets $\mathcal{S}_k$, $k \geq 1$, are separated by the norm. Indeed, if $A, B \geq 0$ are such that rank $A \leq$ trace $A = k$ and rank $B \leq$ trace $B = l$ with $k < l$, then

$$\|A - B\| \geq \frac{l - k}{\operatorname{rank} B} \geq \frac{l - k}{l} = 1 - \frac{k}{l}$$

by Lemma 5.2. This shows that $\operatorname{dist}(\mathcal{S}_k, \mathcal{S}_l) \geq 1 - \min\{k/l, l/k\}$ for any $k, l \geq 1$.

Proposition 5.4. *The set $\mathcal{S}_0 = \{T \in \mathcal{B}(H)$: T is a sum of finite-rank projections$\}$ is norm-closed.*

Proof. Let $T = \lim_n T_n$ in norm, where each T_n, $n \geq 1$, is a sum of finite-rank projections. If n_0 is an integer such that $\|T_n - T_{n_0}\| \leq 1/2$ for all $n > n_0$, then, by Lemma 5.2,

$$
\begin{aligned}
\operatorname{trace}(T_n - 2T_{n_0}) &= 2\operatorname{trace}(T_n - T_{n_0}) - \operatorname{trace} T_n \\
&\leq 2(\operatorname{rank} T_n)\|T_n - T_{n_0}\| - \operatorname{trace} T_n \\
&\leq \operatorname{rank} T_n - \operatorname{trace} T_n \leq 0,
\end{aligned}
$$

that is, trace $T_n \leq 2$ trace T_{n_0} for all large n. It follows that $\{\operatorname{trace} T_n\}_{n=1}^{\infty}$ is a bounded sequence of positive integers. Passing to a subsequence if necessary, we may assume that trace T_n has a constant value, say, k for all n. Hence T_n is a sum of k rank-one projections by Corollary 1.4 for all n. Proposition 5.3 then implies that T itself is a sum of k rank-one projections. This proves the norm-closedness of $\mathcal{S}_0$. $\quad\square$

The following is a result of related interest.

Proposition 5.5. *For each positive integer k, the set $\{T \in \mathcal{B}(H)$: T is a sum of k finite-rank projections$\}$ is norm-closed.*

Proof. Let $T = \lim_n T_n$ in norm, where, for each n, $T_n = \sum_{j=1}^{k} P_j^{(n)}$ is a sum of k finite-rank projections $P_j^{(n)}$. By Proposition 5.4, T is of finite rank. Let Q be the finite-rank projection on H with range equal to $(\operatorname{Range} T) \vee (\operatorname{Range} T^*)$. Then $QT = TQ = T$, $\lim_n QT_nQ = QTQ = T$ and $\lim_n (I - Q)T_n(I - Q) = (I - Q)T(I - Q) = 0$. Passing to a subsequence, we may assume that, for each j, $QP_j^{(n)}Q$ converges, as n approaches infinity, to a finite-rank positive operator R_j in norm. Thus $T = \sum_{j=1}^{k} R_j$. It remains to show that R_j is a projection for all j. Note that for any two projections P and Q, we have

$$
\begin{aligned}
\left\|QPQ - (QPQ)^2\right\| &= \left\|QP(I - Q)PQ\right\| \\
&\leq \left\|P(I - Q)\right\| = \left\|(I - Q)P(I - Q)\right\|^{1/2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\| R_j - R_j^2 \right\| &= \lim_n \left\| Q P_j^{(n)} Q - \left(Q P_j^{(n)} Q \right)^2 \right\| \\
&\le \liminf_n \left\| (I-Q) P_j^{(n)} (I-Q) \right\|^{1/2} \\
&\le \lim_n \left\| (I-Q) T_n (I-Q) \right\|^{1/2} \\
&= \left\| (I-Q) T (I-Q) \right\|^{1/2} = 0.
\end{aligned}
$$

This proves that $T = \sum_{j=1}^{k} R_j$ is the sum of k finite-rank projections. $\square$

Finally, we can characterize the norm closure of sums of projections.

Theorem 5.6. *The norm closure of the set of sums of projections consists of all positive operators which either have essential norm greater than or equal to one or are sums of finite-rank projections.*

Proof. If T is a positive operator with $\|T\|_e \ge 1$, then, for every $n \ge 1$, $T + (1/n)I$ is a positive operator with $\|T + (1/n)I\|_e > 1$ and hence is a sum of projections by Theorem 2.2. Hence T, as a norm limit of the sequence $\{T + (1/n)I\}$, is in the asserted closure.

Conversely, let $T = \lim_n T_n$ in norm, where each T_n is a sum of projections. Assume that $\|T\|_e < 1$. Since $\|T_n\|_e$ converges to $\|T\|_e$, we may assume that $\|T_n\|_e < 1$ for all n. Proposition 2.1 (b) implies that T_n is of finite rank. By Proposition 5.4, T is a sum of finite-rank projections, completing the proof. $\square$

To conclude this paper, we remark that on an infinite-dimensional space, the closure of the set of sums of projections in the weak operator topology (WOT) consists of all positive operators. Indeed, if $T \ge 0$, then, letting $n \ge \|T\|$, we have $0 \le (1/n)T \le I$. [9, Problem 224] implies that there are projections P_j such that $P_j \to (1/n)T$ in the WOT. Hence $nP_j \to T$ in the WOT, which shows that T is in the WOT-closure of sums of projections. The same proof also shows that for any $n \ge 1$, the WOT-closure of sums of n projections equals the set $\{T : 0 \le T \le nI\}$.

In recent years, there have been works on expressing an operator as the sum of a sequence of projections in the strong operator topology: $T = \sum_{n=1}^{\infty} P_n$ in SOT. For these and their applications to frame theory, the readers can consult [5,10] and the references therein.

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