Strict Nonblockingness of Reduced Shuffle-Exchange Networks

F. K. Hwang
Department of Applied Mathematics, National Chiaotung University, Hsinchu, Taiwan, Republic of China

Binwu Zhang
Department of Mathematics and Physics, Hohai University, Changzhou Branch, Changzhou, Jiangsu, People's Republic of China

The shuffle-exchange network is one of the most well-studied multistage interconnection networks. Whether a ($2n - 1$)-stage shuffle-exchange network is rearrangeable has been a challenging conjecture for some 30 years, and only recently a proof was claimed. In this article, we use the analysis method developed for EGSN networks to show that the shuffle-exchange network can be strictly nonblocking by deleting some inputs and outputs. **NETWORKS, Vol. 45(1), 4–8 2005**

Keywords: shuffle-exchange network; strictly nonblocking; EGSN network

1. INTRODUCTION

An s-stage d-nary shuffle-exchange network with $N = d^n$ inputs (outputs), denoted by SE(N, d, s), consists of s stages each having N/d $d \times d$ switches (crossbars), and the connection between two adjacent stages is of the *shuffle type:* if the input ports and output ports of stage i, $1 \leq i \leq s$, are each labeled by the d-nary n-sequence (u_1, u_2, . . . , u_n), then the output port (u_1, u_2, . . . , u_n) of stage i is connected to the input port (u_2, . . . , u_n, u_1) of stage $i + 1$ for $1 \leq i \leq s - 1$. We assume $s \geq n$ throughout this article. Figure 1 illustrates SE(27, 3, 4).

A network is called *rearrangeable* if given any set of (input, output) pairs, where the inputs and outputs are all distinct, there exists an equinumerous set of link–disjoint paths each connecting an (input, output) pair. A network is *strictly nonblocking* if given any sequence of additions and deletions of (input, output) pairs, the last added pair can always be connected by a path link–disjoint from all existing paths regardless how previous pairs have been connected.

The shuffle-exchange network has been extensively studied in both the computer network and the telecommunication network literature. In particular, SE(N, 2, n) is the Omega network [4], one of the most popular self-routing network. The conjecture that SE(N, 2, $2n - 1$) is rearrangeable [1] has fascinated many switching network theorists with a proof recently claimed by Cam [2].

A d-nary network has the nice property of using identical switches as components, which is good for manufacturing. However, it is well known [3] that a d-nary network cannot be strictly nonblocking. Hence, SE(N, d, s) is not strictly nonblocking for any s. This article is motivated by the aim to obtain a strictly nonblocking network that preserves the basic d-nary structure. Clearly, this is possible only if we reduce the traffic load of the network by reducing its number of inputs and outputs. One way is to leave some inputs (outputs) unused on each input (output) switch, while another is to leave some input and output switches totally unused. Let SE(N, d, s : r, v), $v \leq d$, denote a reduced SE(N, d, s), which keeps only v inputs (outputs) per input (output) switch and r $v \times d$ input switches (r $d \times v$ output switches). In this article we study the strict nonblockingness of SE(N, d, s : r, v). We do so by applying the sufficient condition for a strictly nonblocking EGSN network (defined in Section 2) to the special case SE(N, d, s : r, v).

2. THE EGSN NETWORK

Richards and Hwang [5] gave a sufficient condition for strict nonblockingness of a class of multistage networks called *extended generalized shuffle-exchange networks,* or EGSN. The main characteristic of this class is that the linking between two adjacent stages is of the extended

Received June 2003; accepted September 2004
Correspondence to: B. Zhang; e-mail: bwzhang71@163.com
This research was done when the author was visiting the Center of Mathematical Sciences, Zhejiang Univ., Hangzhou, P.R. China
Contract grant sponsor: National Natural Science Foundation of China; contract grant number: 10271110
DOI 10.1002/net.20043
Published online in Wiley InterScience (www.interscience.wiley.com).

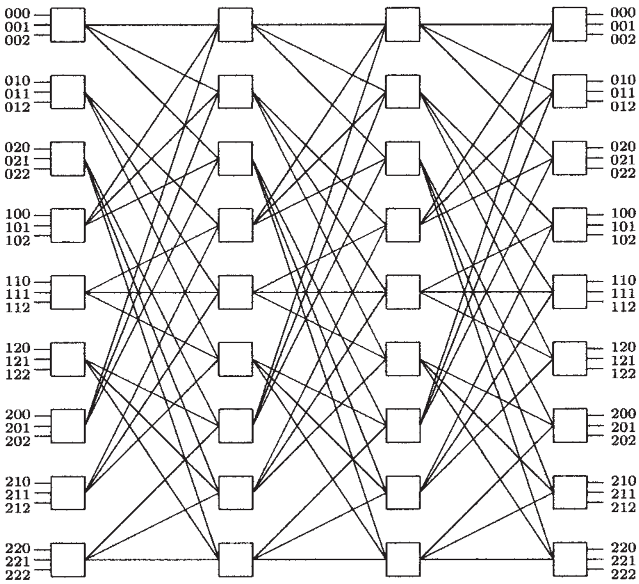

FIG. 1. SE(27, 3, 4).

generalized shuffle type defined as follows: let stage i consist of r_i $n_i \times m_i$ stage-$(i + 1)$ switches. The "shuffle" type means the first stage-i switches are linked to the first m_i stage-$(i + 1)$ switches, the second stage-i switches to the next m_i stage-$(i + 1)$ switches, and so on. "Generalized" means m_i does not have to divide r_{i+1}, and the r_{i+1} switches are treated cyclically in the shuffle-linking, that is, the first switch follows the last; "extended" means $n_i \times m_i$ can differ from stage to stage. Note that SE(N, d, s : r, v) clearly is an extended generalized shuffle-exchange network.

Define $N_{i,j} = \Pi_{k=i}^{j} n_k$, $M_{i,j} = \Pi_{k=i}^{j} m_k$ for $i \leq j$. Let N denote the number of inputs and M the number of outputs. We quote several results (Theorem 4.2a, Equation (7.3a) and Theorem 10.1) from [5].

Lemma 2.1. *The number P of paths (not necessarily disjoint) from an input x to an output y is $M_{1,s}/M$ (or $N_{1,s}/N$) if it is an integer.*

For a given pair (x, y) of inputs and outputs, an *intersecting connection* is a path from $x' \neq x$ to $y' \neq y$, which shares a link with a (x, y)-path.

Lemma 2.2. *Suppose $N_{1,k-1}$ divides N or vice versa, and $M_{k+1,s}$ divides M or vice versa. Then $w = N_{1,k-1} + M_{k+1,s} - 2$ is an upper bound on the total number of intersecting connections.*

(The original statement in [5] did not contain the "vice versa" part, but it is obvious.) An intersecting connection enters an (x, y) path at some stage and exits at a later stage. The number of intersecting connections that can enter or exit a given stage is restricted. Richards and Hwang [5] gave an assignment of the w intersecting connections to the entry stages and the exit stages to maximize the number B of blocked (x, y) paths.

Let t be the largest j such that $N_{1,j} < N$, and let u be smallest i such that $M_{i,s} < M$. Richards and Hwang [5] gave:

Lemma 2.3.

$$B = \sum_{i=1}^{s} [\min\{N_{1,i} - 1, w\} - \min\{N_{1,i-1} - 1, w\}] \lceil M_{i+1,s}/M \rceil$$

$$+ \sum_{j=1}^{s} [\min\{M_{j,s} - 1, w\} - \min\{M_{j+1,s} - 1, w\}] \lceil N_{1,j-1}/N \rceil$$

$$+ w \left[\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \right]$$

The network is nonblocking if $P > B$.

Remark. In Lemma 2.3, $N_{1,0} - 1$ and $M_{s+1,s} - 1$ are set to 0 as these terms are nonnegative.

For SE(N, d, s : r, v), $n_i = m_i = d$ except $n_1 = m_s = v$ and $n_2 = m_{s-1} = r/(d^{n-2}) \equiv q$. Hence $N_{1,1} = v$ and $N_{1,i} = d^{i-2}qv$ for $i \geq 2$, while $M_{s,s} = v$ and $M_{j,s} = d^{s-j-1}qv$ for $j \leq s - 1$. Further, $N = M = d^{n-2}qv$. By Lemma 2.1, $P = d^{s-n}$. In Lemma 2.2, if we take $k = \lceil s/2 \rceil$, then

$$w = \begin{cases} 2v - 2 & \text{for } s = 3, \\ v + qv - 2 & \text{for } s = 4, \\ d^{\lceil s/2 \rceil - 3}qv + d^{\lceil (s+1)/2 \rceil - 3}qv - 2 & \text{for } s \geq 5 \end{cases}$$

is an upper bound on the total number of intersecting connections.

It is easily checked that $t = n - 1$, $u = s - n + 2$, $N_{1,i} - 1 > w$ if and only if $i \geq \lceil (s + 1)/2 \rceil$, and $M_{i,s} - 1 > w$ if and only if $i \leq \lfloor (s + 1)/2 \rfloor$. Let

$$B = B_1 + B_2 + B_3,$$

where B_i is the ith term of B in Lemma 2.3. Then

$$B_1 = (N_{1,1} - 1) \left\lceil \frac{M_{2,s}}{M} \right\rceil + \sum_{i=2}^{\lceil (s-1)/2 \rceil} (N_{1,i} - N_{1,i-1}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil$$

$$+ (w - N_{1,\lceil (s-1)/2 \rceil} + 1)\left\lceil \frac{M_{\lceil (s+3)/2 \rceil, s}}{M} \right\rceil$$

$$= \begin{cases} (v-1)d^{2-n} + w - v + 1 & \text{for } s = 3, \\ (v-1)d^{3-n} + (qv - v)\lceil d^{2-n} \rceil \\ \quad + (w - qv + 1) & \text{for } s = 4, \\ (v-1)d^{s-n-1} + (qv - v)\lceil d^{s-n-i} \rceil + \sum_{i=3}^{\lceil (s-1)/2 \rceil} \\ \quad (d-1)d^{i-3}qv\lceil d^{s-n-1} \rceil + (d^{\lceil s/2 \rceil - 3}qv - 1) \\ \quad \times \lceil d^{\lfloor (s-2n-1)/2 \rfloor} \rceil & \text{for } s \geq 5 \end{cases}$$

$B_2 = B_1$ by symmetry.

$$B_3 = w\left[\sum_{k=n}^{s-n} \lceil d^{k-n} \rceil (\lceil d^{s-k-n} \rceil - \lceil d^{s-k-n-1} \rceil) - \lceil d^{s-2n} \rceil \right]$$

$$= \begin{cases} w[(d-1)(s-2n)d^{s-2n-1} - d^{s-2n}] & \text{for } s - n \geq n + 1 \\ -w & \text{for } s - n \leq n \end{cases}$$

3. SE(N, d, s : r, v)

There do not exist many tools to analyze whether a network is strictly nonblocking. Lemma 2.3, messy as it is, provides a general framework to check the strict nonblockingness of a multistage network. We will take full advantage of it in applying it to SE(N, d, s : r, v). More specifically, we determine the parameters r and v in an SE(N, d, s : r, v) such that $P > B$, where B is given in Lemma 2.3. To avoid trivial cases, we assume $s \geq 3$ and $s > n$ ($s = n$ corresponds to the Omega network for which $P = 1$, and is clearly not strictly nonblocking for all v and r except in some degenerate cases).

Using the formulas for B_1, B_2, and B_3 given in the last section, we separate the computation of B into several cases:

Case I. $n + 2 \leq s \leq 2n$.

For $n = 2$ and $s = 4$

$$B_1 = (v-1)d + (qv - v) + w - qv + 1 = v(d-1) + w - d + 1$$

$$B_3 = -w$$

Hence,

$$B = 2B_1 + B_3 = 2v(d-1) + 2 + w - 2d = 2vd + qv - v - 2d$$

and $P > B$ implies

$$d^2 > 2vd + qv - v - 2d$$

For $s \geq 5$, we have

$$B_1 = vd^{s-n-2}(d-1) - d^{s-n-1} + qvd^{s-n-2} + \sum_{i=3}^{s-n} (d-1)d^{i-3}qv\lceil d^{s-n-i} \rceil + \sum_{i=s-n+1}^{\lceil (s-1)/2 \rceil} (d-1)d^{i-3}qv$$

$$+ (d^{\lceil s/2 \rceil - 3}qv - 1) = vd^{s-n-2}(d-1) - d^{s-n-1} + qvd^{s-n-2} + (s-n-2)(d-1)\lceil d^{s-n-3} \rceil qv$$

$$+ (d^{\lceil (s-5)/2 \rceil} - d^{s-n-2})qv + (d^{\lceil s/2 \rceil - 3}qv - 1)$$

$$= vd^{s-n-2}(d-1) - d^{s-n-1} + (s-n-2)(d-1)\lceil d^{s-n-3} \rceil qv + d^{\lceil (s-5)/2 \rceil}qv + d^{\lceil s/2 \rceil - 3}qv - 1$$

$$B_3 = -w = -d^{\lceil (s-5)/2 \rceil}qv - d^{\lceil s/2 \rceil - 3}qv + 2$$

$$B = 2B_1 + B_3 = 2vd^{s-n-2}(d-1) + 2(s-n-2) \times (d-1)\lceil d^{s-n-3} \rceil qv - 2d^{s-n-1} + d^{\lceil (s-5)/2 \rceil}qv + d^{\lceil s/2 \rceil - 3}qv$$

thus, $B < P$ implies

$$v < \frac{d^{s-n} + 2d^{s-n-1}}{2d^{s-n-2}(d-1) + 2(s-n-2)(d-1)\lceil d^{s-n-3} \rceil q + d^{\lceil (s-5/2) \rceil}q + d^{\lceil s/2 \rceil - 3}q} \tag{2.1}$$

Note that in (2.1),

$$d^{s-n} + 2d^{s-n-1} \leq d^{\lceil (s-5)/2 \rceil}q + d^{\lceil s/2 \rceil - 3}q$$

$$\leq 2d^{s-n-2}(d-1) + 2(s-n-2)(d-1)\lceil d^{s-n-3} \rceil q + d^{\lceil (s-5)/2 \rceil}q + d^{\lceil s/2 \rceil - 3}q$$

Thus, the right-hand side of (2.1) is less than 1 unless $s = 2n - 4$, $n = 5$ and $q = 1$. But $n = 5$ and $s = 2n - 4 = 6$ is a case not satisfying the condition $s \geq n + 2$. Hence, (2.1) does not apply for $s \leq 2n - 4$. This does not imply that the network is necessarily blocking, but merely that the sufficient condition is not strong enough to cover these cases.

For $2n - 3 \leq s \leq 2n$, $P > B$ can have a positive solution v for any q. In particular, for $q = d$, (2.1) becomes

$$v < \frac{d^2 + 2d}{2(s-n-1)(d-1) + d^{\lceil (2n-s+1)/2 \rceil} + d^{\lceil (2n-s)/2 \rceil}} \tag{2.2}$$

The right-hand side of (2.2) is less than 1 for $s \le 2n - 3$. For $2n - 2 \le s \le 2n$, (2.2) can be further simplified to

$$v < \begin{cases} \dfrac{d^2 + 2d}{2(n-1)d - 2(n-2)} & \text{for } s = 2n - 1 \\ \dfrac{d^2 + 2d}{(2n-1)d - (2n-3)} & \text{for } s = 2n \end{cases}$$

Example 1. SE(27, 3, 5 : 6, 1) is strictly nonblocking since the right-hand side of (2.2) equals $(3^2 + 2 \times 3)/(2(3 - 1) + 2 + 2) = 15/8 > 1$.

Case ii. $s = n + 1$.

For $n = 2$ and $s = 3$

$$B_1 = (v - 1) + w - v + 1 = w$$

$$B_3 = -w$$

Hence,

$$B = 2B_1 + B_3 = w = 2v - 2$$

and $P > B$ implies

$$d > 2v - 2$$

For $n = 3$ and $s = 4$

$$B_1 = (v - 1) + qv - v + w - qv + 1 = w$$

$$B_3 = -w$$

Hence,

$$B = 2B_1 + B_3 = w = v + qv - 2$$

and $P > B$ implies

$$d > v + qv - 2$$

For $n \ge 4$, we have

$$B_1 = qv - 1 + \sum_{i=3}^{\lceil (s-1)/2 \rceil} (d-1)d^{i-3}qv + d^{\lceil s/2 \rceil - 3}qv - 1$$

$$= qv - 1 + [d^{\lceil (s-5)/2 \rceil}qv - qv] + d^{\lceil s/2 \rceil - 3}qv - 1$$

$$= w$$

B_3 is the same as in Case i. Thus,

$$B = 2B_1 + B_3 = w = d^{\lceil (s-5)/2 \rceil}qv + d^{\lceil s/2 \rceil - 3}qv - 2 \ge dqv = P$$

unless $n = 4$ or $n = 5$ and $qv = 1$.

Case iii. $s \ge 2n + 1$

$$B_1 = v(d-1)d^{s-n-2} + qvd^{s-n-2} - d^{s-n-1} + \sum_{i=3}^{\lceil (s-1)/2 \rceil} \times (d-1)d^{i-3}qvd^{s-n-i} + (d^{\lceil s/2 \rceil - 3}qv - 1)\lceil d^{\lfloor (s-2n-1)/2 \rfloor} \rceil$$

$$= v(d-1)d^{s-n-2} + qvd^{s-n-2} - d^{s-n-1} + \lceil (s-5)/2 \rceil \times (d-1)d^{s-n-3}qv + (d^{\lceil s/2 \rceil - 3}qv - 1)\lceil d^{\lfloor (s-2n-1)/2 \rfloor} \rceil$$

$$\ge qv - 1$$

$$B_3 = (d^{\lceil (s-5)/2 \rceil}qv + d^{\lceil s/2 \rceil - 3}qv - 2)[(s-2n) \times (d-1)d^{s-2n-1} - d^{s-2n}]$$

For $s \ge 2n + 4$

$$B_3 \ge (d^{\lceil (s-5)/2 \rceil}qv + d^{\lceil s/2 \rceil - 3}qv - 2)d^{s-2n}$$

and it is easily verified that

$$B = 2B_1 + B_3 \ge 2(qv - 1) + B_3 > d^{\lceil (s-5)/2 \rceil + s - 2n} \ge d^{s-n} = P$$

unless $s = 5$ and $qv = 1$ (in that case $B = 0 < P$).

For $s = 2n + 3$

$$B = (2v(d-1)d^{n+1} + 2d^{m+1}qv - 2d^{m+2} + 2(n-1) \times (d-1)d^nqv + 2(d^{n-1}qv - 1)d + 2(d^{n-1}qv - 1) \times (3(d-1)d^2 - d^3) = 2vd^{n+2} + 4d^{n+2}qv - 2vd^{n+1} + 2(n-4)d^{n+1}qv - 2(n-2)d^nqv - 2d^{n+2} - 4d^3 + 6d^2 - d < 2vd^n[d^2(1+2q) + (n-4)dq - d - (n-2)q] - 2d^{n+2}$$

$P > B$ implies

$$v < \frac{d^3 + 2d^2}{2(d^2(1+2q) + (n-4)dq - d - (n-2)q)}$$

For $s = 2n + 2$

$$B = 2v(d-1)d^n - 2d^{n+1} + 2d^nqv + 2(n-1) \times (d-1)d^{n-1}qv + 2(d^{n-2}qv - 1)$$

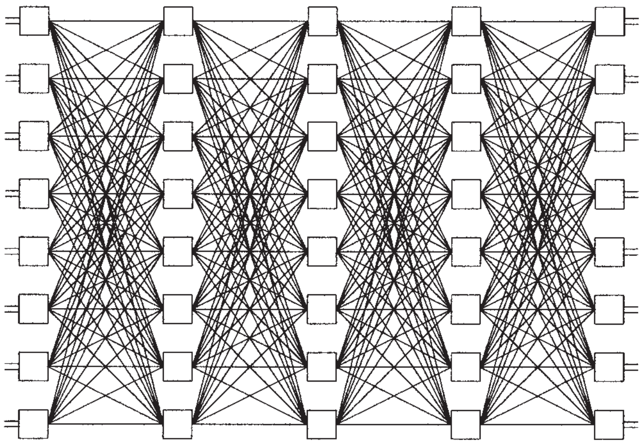

FIG. 2. SE(64, 8, 5 : 8, 2).

$$+(d^{n-1}qv + d^{n-2}qv - 2)(2(d-1)d - d^2)$$

$$= vd^{n+1}(2+q) - 2d^{n+1} - 2vd^n + (2n-1)d^n qv$$

$$-2nd^{n-1}qv + 2d^{n-2}qv - (2d^2 - 4d + 2)$$

$$< vd^{n-2}[(2+q)d^3 + ((2n-1)q - 2)d^2 - 2ndq + 2q] - 2d^{n+1}$$

$P > B$ implies

$$v < \frac{d^4 + 2d^3}{(2+q)d^3 + ((2n-1)q - 2)d^2 - 2ndq + 2q}$$

For $s = 2n + 1$

$$B = 2v(d-1)d^{n-1} - 2d^n + 2d^{n-1}qv + 2(n-2) \times (d-1)d^{n-2}qv + 2(d^{n-2}qv - 1)(1 + (d-1) - d)$$
$$\leq vd^{n-2}2(d^2 + (n-1)qd - d - (n-2)q) - 2d^n$$

$P > B$ implies

$$v < \frac{d^3 + 2d^2}{2(d^2 + (n-1)qd - d - (n-2)q)} \quad (2.3)$$

For $q = d$, (2.3) can be further simplified to

$$v < \frac{d^2 + 2d}{2(nd - (n-1))}.$$

For example, consider the case $d = q = 8$, $n = 2$, $s = 5$, $v = 2$. Because $(d^2 + 2d)/[2(nd - (n-1)] = (64 + 16)/2(16 - 1) > 2 = v$, the network in Figure 2 is strictly nonblocking.

Note that for the three cases $q = d$ leads to a solution of v, is largest for $s = 2n - 1$, then followed by $s = 2n$ and $s = 2n + 1$, contrary to our intuition that the range should increase in s. This again might be a reflection of the analysis, not the reality. These networks have size vr, which is about $d/2n$, and have about $2nd^2$ crosspoints, while a $d/2n \times d/2n$ crossbar has $(rd/2n)^2$ crosspoints. The ratio is $(4n^2)/r = (4n^2)/(d^{n-1})$. For example, for $n = 3$ and $d = 8$, the ratio is 9/16.

4. CONCLUSIONS

The Omega network was the first self-routing network proposed. Its multistage extension, the shuffle-exchange network, has the advantage of having the same linking pattern over all stages when compared to other similar networks. To prove the rearrangeability of the $(2n - 1)$-stage shuffle-exchange network has been a big challenge for a long time. In this article we show that the $(2n - 1)$-stage shuffle-exchange network can also be strictly nonblocking if enough input (output) switches, or inputs (outputs) per switch, are inoperative. We discuss the interaction between the two types of inoperativeness.

The method we used is the method developed for the EGSN network. The strictly nonblocking condition can be met only for $2n - 3 \leq s \leq 2n + 3$. Further, if $q = d$, that is, no input or output switches are inoperative, then the condition can be met only for $2n - 1 \leq s \leq 2n + 1$. When the network is connecting processors with memory devices, then $v = 1$ for $d = q$. It is easily verified that $d \geq 2n$ is a sufficient condition for strictly nonblocking.

Acknowledgment

The authors would like to thank Prof. Douglas R. Shier, the editor and the anonymous referees for their careful reading of the article and their many helpful comments.

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