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Pricing barrier stock options with discrete dividends by approximating analytical formulae

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Deriving accurate analytical formulas for pricing stock options with discrete dividend payouts is a hard problem even for the simplest vanilla options. This is because the falls in the stock price process due to discrete dividend payouts will significantly increase the mathematical difficulty in pricing the option. On the other hand, much literature uses other dividend settings to simplify the difficulty, but these settings may produce inconsistent pricing results. This paper derives accurate approximating formulae for pricing a popular path-dependent option, the barrier stock option, with discrete dividend payouts. The fall in stock price due to dividend payout at an exdividend date t is approximated by an accumulated price decrement due to a continuous dividend yield up to time t. Thus, the stock price process prior to time t and after time t can be separately modelled by two different lognormal-diffusive stock processes which help us to easily derive analytical pricing formulae. Numerical experiments suggest that our formulae provide more accurate and coherent pricing results than other approximation formulae. Our formulae are also robust under extreme cases, like the high volatility (of the stock price) case. Besides, our formulae also extend the applicability of the first-passage model (a type of structural credit risk model) to measure how the firm's payout influences its financial status and the credit qualities of other outstanding debts.

Keywords: Barrier option; Derivative pricing; Discrete dividend; First-passage model

JEL Classification: G1, G13

1. Introduction

Developing a feasible option pricing model that captures the phenomena of financial markets is an important issue in the financial field. Black and Scholes (1973) derive option pricing formulae for non-dividend-paying stocks. To deal with the dividend payout problem, Merton (1973) extends the Black-Scholes formulae by assuming that the stock pays a fixed continuous dividend yield. However, most dividend payments are paid discretely rather than continuously. Pricing the option on the stock that pays a fixed dividend discretely seems to be first investigated in Black (1975). In addition, Ehrhardt and Brigham (2009) also argue that most stocks pay stable dividends discretely to maintain the investors' confidence. Although this discrete-payment setting might be more realistic than the continuous one, it gives rise to significant mathematical difficulty in pricing the options. This is because the underlying stock price process becomes much more complicated due to the jumps caused by the discrete payments.

Pricing stock options with discrete dividend payouts has drawn a lot of attention in the literature. Frishling (2002) shows that the underlying stock price processes are usually modelled in the following three different ways. **Model 1** suggests that the stock price minus the present value of future dividends over the life of the option follows a lognormal diffusion process (see Roll 1977, Geske 1979). **Model 2** suggests that the stock price plus the forward values of the dividends paid from today up to the option's maturity follows a lognormal diffusion process (see Heath and Jarrow 1988, Musiela and Rutkowski 1997). **Model 3** suggests that the stock price falls by the amount of the dividend paid at the exdividend date, and follows the lognormal diffusion process between two adjacent exdividend dates. For pricing vanilla options, Frishling (2002) argues that these three models are incompatible with each other and generate very different prices. In addition, Frishling (2002), Bender and Vorst (2001), and Bos and Vandermark (2002) argue that only **Model 3** can reflect the reality and generate consistent option prices. Apart from the three aforementioned models, Chiras and Manaster (1978) suggest that the discrete dividends can be transformed into a fixed continuous dividend yield. The vanilla

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stock option can then be analytically solved by the Merton pricing formula (see Merton 1973). However, Dai and Lyuu (2009) show that the pricing results of their approach can deviate significantly from those generated by **Model 3**. On the other hand, pricing vanilla options under **Model 3** can be mathematically intractable since the downward jumps due to the dividend payouts cause the stock price process to no longer follow a lognormal diffusion process. Bender and Vorst (2001), Bos and Vandermark (2002), Vellekoop and Nieuwenhuis (2006), and Dai and Lyuu (2009) derive the approximated distribution of the stock price at the maturity date and derive approximating analytical pricing formulae. Besides, Dai (2009) constructs a numerical pricing approach, namely, the 'stair tree', that produces accurate pricing results by faithfully modelling the evolution of the stock price process with downward jumps.

Similarly, pricing barrier stock options with the discrete dividend payout with the aforementioned models other than **Model 3** can produce unreasonable pricing results (see Frishling 2002). A barrier option is a popular exotic option whose payoff depends on whether the price path of the underlying stock has reached a certain predetermined price level called a barrier. Reiner and Rubinstein (1991) derive an analytical pricing formula for the barrier option on the stock that pays a fixed continuous dividend yield. In their model, the stock price follows a lognormal diffusion process, and the joint density of the extreme stock price over the option life and the stock price at the option maturity date can therefore be derived by taking advantage of the reflection principle and Girsanov's theorem. Unfortunately, their approach can not be directly extended to price barrier stock options with discrete dividends. While Zvan *et al.* (2000) and Gaudenzi and Zanette (2009) develop numerical methods to price barrier options under **Model 3**, to our knowledge, no announced papers derive analytical pricing formulae for pricing barrier stock options with discrete dividend payouts.

The major contribution of this paper is to derive approximating analytical formulae under **Model 3**. The numerical experiments in section 5 suggest that our formulae produce accurate pricing results. Besides, our option pricing formulae also extend the applicability of the first-passage model—a credit risk model that simulates the evolution of the firm value and that triggers the default event once the firm value reaches the so-called 'default boundary'. Therefore, the firm's equity can be treated as a barrier call option on the firm value, and other outstanding debts can also be evaluated by taking advantage of our approach. Much of the literature puts restrictions on selling the firm's assets to finance the loan repayments or dividend payouts (see Geske 1977, Kim *et al.* 1993, Leland 1994, Longstaff and Schwartz 1995). However, the empirical studies in Billett *et al.* (2007) suggest that up to 64.5% of their debt issue samples have asset sale clauses; that is, most debts allow the issuing firm to sell its assets to finance the debt repayments. Eom *et al.* (2004) also argue that 29 out of 31 bonds in their sample have asset sale clauses. They claim that selling assets to finance the repayment of one bond would damage the values of other outstanding bonds. Therefore, evaluating the impact of the asset sale clause on the debt value is important. Lando (2004) argues that dealing with the asset sale clause can be mathematically intractable since the jumps in the firm's value due to discrete payouts make the firm's value process complicated. Indeed, by substituting the issuing firm's value process and the discrete debt repayments for the stock price process and the dividend payout in our pricing formulae, we obtain new formulae to evaluate the credit risk for the debts that have asset sale clauses. Our numerical experiments suggest that the pricing results of our formulae match the empirical finding in Linn and Stock (2005): When the junior debt matures prior to the senior unsecured debt, the security of the senior unsecured debt is threatened and the default spread (of senior debt) may increase. On the other hand, another model that limits the firm to maintaining a constant continuous payout ratio—which is widely adopted by much of the literature such as Kim *et al.* (1993) and Longstaff and Schwartz (1995)—might fail to capture this finding.

Our pricing formulae are derived based on a piecewise stock price process designed to approximate the stock price process under **Model 3**. The stock price $S(t)$ at time t given that no dividend is paid out during the time interval $[0, t]$ follows the lognormal diffusion process:

$$S(t) = S(0)e^{\mu t + \sigma W(t)}, \tag{1}$$

where $\mu \equiv r - 0.5\sigma^2$, r denotes the annual risk-free interest rate, σ denotes the volatility, and $W(t)$ denotes the standard Brownian motion. Under **Model 3**, the stock pays dividends c_1, $c_2, c_3, \ldots$ at exdividend dates $t_1, t_2, t_3, \ldots$, respectively, where $t_1 < t_2 < t_3 \ldots$. At the exdividend date t_i, the stock price falls by the amount c_i due to the dividend payout as suggested in Black (1975) and Zvan *et al.* (2000). For convenience, define the stock return at time t as $\frac{S(t)}{S(0)}$. The process of the stock return prior to the exdividend date t_1 can be expressed by the drifting Brownian motion: $\mu t + \sigma W(t)$ as described in equation (1). However, the stock price at any time t between the exdividend dates t_1 and t_2 is

$$S(t) = \left(S(0)e^{\mu t_1 + \sigma W(t_1)} - c_1\right) e^{\mu(t-t_1)+\sigma(W(t)-W(t_1))}, \tag{2}$$

and the stock return is no longer a drifting Brownian motion. To make the pricing problem tractable, the amount c_1 by which the stock price falls at the exdividend date t_1 is approximated by the accumulated price decrement caused by a continuous dividend yield q_1 paid from time 0 to time t_1. That is,

$$S(t_1) = S(0)e^{\mu t_1 + \sigma W(t_1)} - c_1 \equiv S(0)e^{(\mu - q_1)t_1 + \sigma W(t_1)}. \tag{3}$$

Thus, we construct another lognormal diffusion process with a continuous payout rate q_1 paid from time 0 to time t_1 to approximate the stock price process between the time interval $[t_1, t_2]$ in equation (2) as follows:

$$S(t) = S(t_1)e^{\mu(t-t_1)+\sigma(W(t)-W(t_1))} = S(0)e^{\mu t - q_1 t_1 + \sigma W(t)}, \tag{4}$$

where $t \in [t_1, t_2]$. Since q_1 in equation (3) can be approximately solved by the first-order Taylor expansion as an affine function of $W(t_1)$, the process of the stock return between the exdividend dates t_1 and t_2, $\mu t - q_1 t_1 + \sigma W(t)$, can be approximated by another drifting Brownian motion. Let the option maturity $T < t_2$ for simplicity. The joint distribution of the extreme stock price over the time interval $[0, t_1)$ ($[t_1, T]$) and the stock price at time t_1 (T) can be solved for by applying the reflection principle and Girsanov's theorem to the drifting Brownian motion $\mu t + \sigma W(t)$ (another drifting Brownian motion $\mu t - q_1 t_1 + \sigma W(t)$). The pricing formulae can then be derived by applying the risk-neutral valuation method to these two joint distributions. Our approach can be extended to the multiple-dividend case by repeating the aforementioned steps to derive the approximated stock return process between any two adjacent exdividend dates.

The remainder of this paper is organized as follows. Section 2 introduces the required financial and mathematical background knowledge. Section 3 derives mathematical properties that are useful for later deriving the pricing formulae. Our approximation pricing formulae are then derived in section 4. The experimental results given in section 5 verify the accuracy of our pricing formulae and demonstrate how our approach extends the applicability of the first-passage model. Section 6 concludes the paper.

2. Preliminaries

2.1. Barrier options and the first passage model

Assume that a barrier stock option with a strike price K is initiated at time 0 and matures at time T. The payoff of an up-and-out option at maturity is as follows:

$$\texttt{payoff} = \begin{cases} (\theta S(T) - \theta K)^+ & \text{if } S_{\max} < B \\ 0 & \text{if } S_{\max} \geq B \end{cases},$$

where $(A)^+$ denotes $\max(A, 0)$, $S_{\max}$ denotes the maximum underlying stock price between time 0 and time T, B denotes the barrier and θ equals 1 for call options and -1 for put options. Similarly, the payoff of a down-and-out option at maturity is as follows:

$$\texttt{payoff} = \begin{cases} (\theta S(T) - \theta K)^+ & \text{if } S_{\min} > B \\ 0 & \text{if } S_{\min} \leq B \end{cases},$$

where $S_{\min}$ denotes the minimum stock price between time 0 and time T. For simplicity, our paper will focus on an up-and-out call option and the extensions to other barrier options are straightforward.

The same mathematical settings can be used to model the first passage model by redefining the symbol B as the default boundary, T as the debt maturity, and K as the debt repayment due at maturity. The firm value process $S(t)$ is assumed to follow equations (1)–(4), where σ denotes the volatility of the firm value and c_i denotes the loan repayment or dividend payout at time t_i. The firm defaults once its value falls below the default boundary prior to the maturity date or it can not meet the debt obligation at the maturity date. Thus, the equity value can be evaluated as a down-and-out call option on the firm value and each debt issued by the firm can be priced by treating it as a contingent claim on the firm value.

2.2. Pricing barrier stock options without discrete dividends

The payoff of an up-and-out call depends on whether the underlying stock price process has ever risen above the barrier over the life of this option. The stock price process has risen above the barrier during the time interval $[0, \tau]$ if and only if the maximum stock price during the time interval $[0, \tau]$ is greater than the barrier. The following theorem, derived from the reflection principle and Girsanov's theorem (see Shreve 2007), can be applied to describe the joint density of the stock price at time τ and the maximum stock price during the time interval $[0, \tau]$.

THEOREM 2.1 *Let $\tilde{W}(t) = \alpha t + W(t)$ be a Brownian motion with a drift term αt and $\tilde{M}(\tau) = \max_{0\leq t\leq\tau} \tilde{W}(t)$ be its maximum value over a certain time interval $[0, \tau]$. The joint density function of $(\tilde{M}(\tau), \tilde{W}(\tau))$ is given by*

$$f_{\tilde{M}(\tau),\tilde{W}(\tau)}(m, w) = \begin{cases} \frac{2(2m-w)}{\tau\sqrt{2\pi\tau}} e^{\alpha w - \frac{1}{2}\alpha^2\tau - \frac{1}{2\tau}(2m-w)^2} & \text{if } m \geq w^+ \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

The set of points (m, w) that make density values non-zero, also known as the support of a density, is illustrated in figure 1(a).

Reiner and Rubinstein (1991) derive analytical formulae for barrier stock options without discrete dividends by Theorem 2.1. We derive some lemmas that can be used to derive their pricing formulae. These lemmas can also be applied to derive our barrier stock option pricing formulae with discrete dividend payouts. Define the stock return in equation (1), $\mu t + \sigma W(t)$ as $\sigma \hat{W}(t)$, where the drifted Brownian motion $\hat{W}(t)$ is defined as $\mu t/\sigma + W(t)$. Define the maximum value of the Brownian motion $\hat{M}(\tau)$ as $\max_{0\leq t\leq\tau} \hat{W}(t)$. Thus, the value of an up-and-out call option C can be derived as follows:

$$\begin{aligned} C &= e^{-rT} E\left\{ (S(T) - K)^+ 1_{\left\{\max\limits_{0\leq t\leq T} S(t) < B\right\}} \right\} \\ &= e^{-rT} E\left\{ (S(0)e^{\sigma\hat{W}(T)} - K) 1_{\{S(0)e^{\sigma\hat{W}(T)} \geq K, S(0)e^{\sigma\hat{M}(T)} < B\}} \right\} \\ &= e^{-rT} E\left\{ (S(0)e^{\sigma\hat{W}(T)} - K) 1_{\{\hat{W}(T)\geq k, \hat{M}(T) < b\}} \right\}, \end{aligned} \tag{6}$$

where k and b in equation (6) stand for $\frac{1}{\sigma} \ln \frac{K}{S(0)}$ and $\frac{1}{\sigma} \ln \frac{B}{S(0)}$, respectively. By substituting equation (5) into equation (6) with $\alpha = \sigma/\mu$, we have

$$C = \int_k^\infty \int_{-\infty}^b e^{-rT} \left(S(0)e^{\sigma w} - K\right) f_{\hat{M}(T),\hat{W}(T)}(m, w) dm dw \tag{7}$$

$$= \int_k^b \int_{w^+}^b e^{-rT} \left(S(0)e^{\sigma w} - K\right) \times \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} dm dw, \tag{8}$$

where the domain of integral in equation (7), i.e. $-\infty < m < b$ and $k \leq w < \infty$, is the support of the indicator function in equation (6) as illustrated in figure 1(b). The domain of integral in equation (8) is the intersection of the support of the joint density function $f_{\hat{M}(T),\hat{W}(T)}(m, w)$ and the support of indicator function $1_{\{\hat{W}(T)\geq k, \hat{M}(T) < b\}}$ as illustrated in figure 1(c).

In the double integral formula equation (8), since only $f_{\hat{M}(T),\hat{W}(T)}(m, w)$ contains the variable m, $\int_{w^+}^{b} f_{\hat{M}(T),\hat{W}(T)}(m, w) dm$ can be evaluated first by the following lemma†:

†Proofs of this lemma is available upon request.

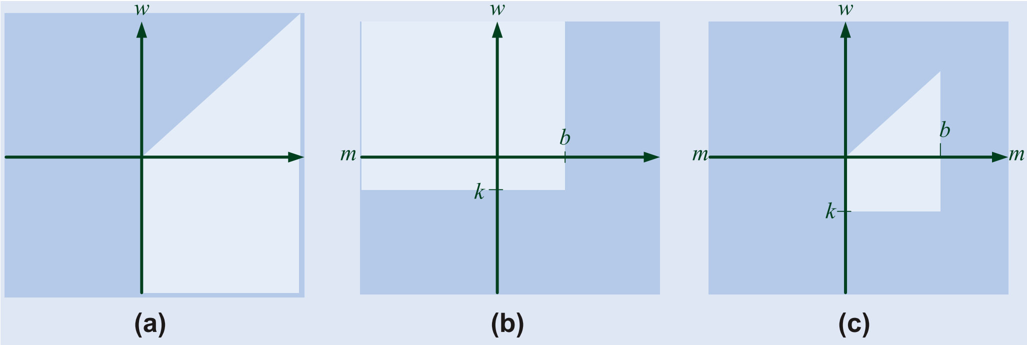

Figure 1. The Integral Domain of equation (8). Notes: The shaded area in Panel (a) denotes the support of the density function $f_{\tilde{M}(T),\tilde{W}(T)}$ in equation (5). The shaded area in Panel (b) denotes the in the money region of the up-and-out barrier call option in equation (7). It is also the support of the indicator function of equation (6). The shaded area in Panel (c) denotes the intersection of the shaded areas in Panel (a) and (b), which is the domain of integral in equation (8).

LEMMA 2.2

$$\int_{v^+}^{\beta} \frac{2(2u-v)}{\Delta\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{1}{2\Delta}(2u-v)^2} du = \frac{1}{\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{v^2}{2\Delta}} \left(1 - e^{\frac{2\beta(v-\beta)}{\Delta}}\right).$$

By applying Lemma 2.2, equation (8) can be rewritten as

$$\begin{aligned} C &= e^{-rT}\int_k^b \left(S(0)e^{\sigma w} - K\right) \\ &\quad \times \left(\int_{w^+}^b \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} dm\right) dw \\ &= e^{-rT}\int_k^b \left(-\frac{K}{\sqrt{2\pi T}} e^{-\frac{w^2}{2T} + \alpha w - \frac{T\alpha^2}{2}} \right. \\ &\quad + \frac{K}{\sqrt{2\pi T}} e^{-\frac{w^2}{2T} + \alpha w - \frac{T\alpha^2}{2} + \frac{2b(w-b)}{T}} \\ &\quad + \frac{S'(0)}{\sqrt{2\pi T}} e^{-\frac{w^2}{2T} + \alpha w + \sigma w - \frac{T\alpha^2}{2}} \\ &\quad \left. - \frac{S'(0)}{\sqrt{2\pi T}} e^{-\frac{w^2}{2T} + \alpha w + \sigma w - \frac{T\alpha^2}{2} + \frac{2b(w-b)}{T}} \right) dw. \end{aligned} \quad (9)$$

In equation (9), each term of the integrand is of the form $Le^{a_2w^2+a_1w+a_0}$ for some constants a_0, a_1, a_2, and L. The following identity can convert the integrals of the aforementioned form into the cumulative distribution function (CDF) of the standard normal distribution by completing the square:

$$\int_{-\infty}^{l} e^{a_2x^2+a_1x+a_0}dx = \sqrt{\frac{\pi}{-a_2}} e^{-\frac{a_1^2-4a_0a_2}{4a_2}} N\left(\frac{l-\mathbf{m}}{\mathbf{s}}\right), \quad (10)$$

where $a_2 < 0$ to ensure that the integral is finite, $\mathbf{m} = -\frac{a_1}{2a_2}$, $\mathbf{s} = \frac{1}{\sqrt{-2a_2}}$, and $N(\cdot)$ denotes the CDF of the standard normal distribution. Thus, the Reiner and Rubinstein (1991) pricing formula can be derived as a linear combination of tail probability values, which can be evaluated by the CDF of the standard normal distribution.

3. Derivations of useful mathematical properties

3.1. *Approximate the stock price process under* Model 3 *piecewisely with lognormal diffusion processes*

We derive a systematic approach for constructing a series of lognormal diffusion processes to piecewisely approximate the stock price process under **Model 3**. To be precise, we decompose the stock price process into several parts by exdividend dates. Each part of the stock price process is approximated by a lognormal diffusion process that makes the stock return process (for this part of the stock price process) a drifted Brownian motion. Therefore, Theorem 2.1 can be applied to derive the pricing formulae.

Note that the stock return process for the time interval $[0, t_1]$ is already a drifted Brownian motion, $\mu t + \sigma W(t)$, as illustrated in equation (1). So we do not need to derive the approximated process for this interval. On the other hand, the stock return process between the time interval $[t_1, t_2]$ (see equation (2)) is not a drifted Brownian motion due to the discrete dividend c_1 paid at time t_1. The stock price drop due to the dividend payout is approximated by the accumulated price decrement caused by a continuous dividend q_1 paid from time 0 to time t_1 as illustrated in equation (3), so the resulting stock price process after time t_1 can be expressed as equation (4). To make this modified price process a lognormal diffusion one, q_1 is approximately solved as a linear function of $W(t_1)$ from equation (3) as follows:

$$S(0)e^{\mu t_1+\sigma W(t_1)} - c_1 \approx S(0)e^{\mu t_1+\sigma W(t_1)}(1-q_1t_1) \quad (11)$$

$$\Rightarrow \quad q_1 \approx \frac{c_1e^{-\mu t_1}(1-\sigma W(t_1))}{t_1S(0)}, \quad (12)$$

where the first-order Taylor expansion $e^x \approx 1+x$ is used in equations (11) and (12). By substituting $k_1 \equiv \frac{c_1e^{-\mu t_1}}{S(0)} - 1$, $q_1 \approx \frac{(k_1-1)(1-\sigma W(t_1))}{t_1}$ into equation (4), the stock price at any time $t \in [t_1, t_2]$ can be approximated by the lognormal diffusion process expressed as follows:

$$S(t) \approx S(0)e^{(\mu t - k_1+1)+k_1\sigma W(t_1)+\sigma(W(t)-W(t_1))}. \quad (13)$$

Thus, the stock return for the time interval $[t_1, t_2]$ can be expressed as a drifted Brownian motion and Theorem 2.1 can be applied to solve the joint density of the stock price at time

t_2 and the maximum stock price for this time interval. The aforementioned procedure can be repeated to find the lognormal diffusion process that approximates the stock price process between the exdividend dates t_2 and t_3. Again, the discrete stock price jump due to the payout of the dividend c_2 is approximated by the accumulated price decrement caused by a continuous dividend yield q_2 paid from time t_1 to time t_2; that is,

$$\begin{aligned} S(t) &= \left(S(t_1)e^{\mu(t_2-t_1)+\sigma(W(t_2)-W(t_1))} - c_2\right) \\ &\quad \times e^{\mu(t-t_2)+\sigma(W(t)-W(t_2))} \\ &= S(0)e^{(\mu-q_1)t_1+\sigma W(t_1)} \\ &\quad \times e^{(\mu-q_2)(t_2-t_1)+\sigma(W(t_2)-W(t_1))}e^{\mu(t-t_2)+\sigma(W(t)-W(t_2))}. \end{aligned} \tag{14}$$

Note that q_2 can be approximately solved by the first-order Taylor expansion to obtain

$$q_2 \approx \frac{(k_2-1)\left[1-k_1\sigma W(t_1)-\sigma(W(t_2)-W(t_1))\right]}{t_2-t_1}, \tag{15}$$

where $k_2 \equiv \frac{c_2 e^{-\mu t_2+k_1-1}}{S(0)} - 1$. Therefore, the stock price at time $t \in [t_2, t_3]$ can be approximated by a lognormal diffusion process by substituting equation (15) into equation (14) to obtain

$$\begin{aligned} S(t) &\approx S(0) \\ &\quad \times e^{(\mu t-k_1-k_2+2)+k_1k_2\sigma W(t_1)+k_2\sigma(W(t_2)-W(t_1))+\sigma(W(t)-W(t_2))}. \end{aligned} \tag{16}$$

Note that the aforementioned procedure can be repeatedly applied to obtain lognormal diffusion processes for approximating the stock price processes in time intervals $[t_3, t_4]$, $[t_4, t_5], \ldots$, and so on. For simplicity, the following discussion will focus on the time interval $[0, t_3]$. The approximated stock price process $\hat{S}(t)$ used to derive the pricing formulae later is constructed by combining equations (1), (13) and (16) as follows:

$$\hat{S}(t) = \begin{cases} S(0)e^{\mu t+\sigma W(t)} & 0 \le t < t_1, \\ S(0)e^{(\mu t-k_1+1)+k_1\sigma W(t_1)+\sigma(W(t)-W(t_1))} & t_1 \le t < t_2, \\ S(0)e^{(\mu t-k_1-k_2+2)+k_1k_2\sigma W(t_1)+k_2\sigma(W(t_2)-W(t_1))+\sigma(W(t)-W(t_2))} & t_2 \le t < t_3. \end{cases} \tag{17}$$

To derive the pricing formulae with Theorem 2.1, the stock return for any time interval listed in equation (17) should be reexpressed in terms of a drifted Brownian motion. First, the stock price process $\hat{S}(t)$ for the first time interval $[0, t_1)$ can be reexpressed as

$$S(0)e^{\sigma\hat{W}(t)}, \tag{18}$$

where $\hat{W}(t) \equiv \alpha t + W(t)$, and $\alpha \equiv \mu/\sigma$. The stock price process $\hat{S}(t)$ for the second time interval $[t_1, t_2)$ can be reexpressed as follows:

$$\hat{S}(t) = S^{'}(0)e^{k_1\sigma\hat{W}(t_1)+\sigma\hat{W}_1(t-t_1)}, \tag{19}$$

where the drifted Brownian motion $\hat{W}_1(t-t_1)$ is defined as $\alpha(t-t_1) + (W(t) - W(t_1))$, and $S^{'}(0) \equiv S(0)e^{(1-k_1)(1+\mu t_1)}$. Note that $\hat{W}(t_1)$ is F_{t_1} measurable given that the collection of σ-algebras F_τ, $0 \le \tau \le T$, is a filtration generated by the Brownian motion $W(\tau)$. Thus Theorem 2.1 can be used to derive the conditional joint density of random variables $\max_{t_1 \le t \le t_2}\left(\hat{W}_1(t-t_1)\right)$ and $\hat{W}_1(t_2-t_1)$ based on the information of F_{t_1}. The stock price process $\hat{S}(t)$ for the third time interval $t \in [t_2, t_3)$ can be reexpressed as

$$\hat{S}(t) = S^{''}(0)e^{k_1k_2\sigma\hat{W}(t_1)+k_2\sigma\hat{W}_1(t_2-t_1)+\sigma\hat{W}_2(t-t_2)}, \tag{20}$$

where $S^{''}(0) \equiv S(0)e^{(\mu t-k_1-k_2+2)-k_1k_2\mu t_1-k_2\mu(t_2-t_1)-\mu(t-t_2)}$, and $\hat{W}_2(t-t_2) \equiv \alpha(t-t_2) + (W(t) - W(t_2))$. Note that both $\hat{W}(t_1)$ and $\hat{W}_1(t_2-t_1)$ are F_{t_2} measurable. Theorem 2.1 can again be used to derive the conditional joint density of random variables $\max_{t_2 \le t \le t_3}\left(\hat{W}_2(t-t_2)\right)$ and $\hat{W}_2(t_3-t_2)$ based on the information of F_{t_2}. By combining equations (18), (19) and (20), the approximated stock price process $\hat{S}(t)$ can be rewritten as

$$\hat{S}(t) = \begin{cases} S(0)e^{\sigma\hat{W}(t)} & 0 \le t < t_1, \\ S^{'}(0)e^{k_1\sigma\hat{W}(t_1)+\sigma\hat{W}_1(t-t_1)} & t_1 \le t < t_2, \\ S^{''}(0)e^{k_1k_2\sigma\hat{W}(t_1)+k_2\sigma\hat{W}_1(t_2-t_1)+\sigma\hat{W}_2(t-t_2)} & t_2 \le t < t_3. \end{cases} \tag{21}$$

3.2. Evaluate the integration of exponential functions

The pricing formulae in this paper can be expressed in terms of a multiple integration of an exponential function, the exponent term of which is a quadratic function of integrators. Theorem 3.1 shows that such an integration can be expressed as a CDF of a multi-variate normal distribution by taking advantage of some matrix and vector calculations. For convenience, for any matrix $\daleth$, we use $|\daleth|$, $\daleth^T$ and $\daleth^{-1}$ to denote the determinant, the transpose, and the inverse of $\daleth$. $\daleth_{i,j}$ stands for the element located in the i-th row and j-th column of $\daleth$. For any vector ν, we use ν_i to denote the i-th element of ν.

Theorem 3.1 *Let* $\mathbf{x}$ *and* $\mathbf{B}$ *be an* $n \times 1$ *constant vector,* $\mathbf{C}$ *be a constant, and* $\mathbf{A}$ *be an* $n \times n$ *symmetric invertible negative-definite constant matrix. For any general quadratic formula* $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} + \mathbf{C}$, *the n-variate integral for* $e^{\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+\mathbf{C}}$

$$\int_{-\infty}^{p_n}\int_{-\infty}^{p_{n-1}}\cdots\int_{-\infty}^{p_1} e^{\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+\mathbf{C}}d\mathbf{x} \tag{22}$$

can be expressed in terms of a CDF of an n-dimensional standard normal distribution $F_{Y_1,Y_2,\ldots,Y_n}$ *with covariance matrix* Σ *as follows:*

$$e^{\mathbf{C}^{'}}\sqrt{\frac{\pi^n}{|-\mathbf{A}|}}F_{Y_1,Y_2,\ldots,Y_n} \times\left(\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}, \frac{p_2-\mathbf{m}_2}{\mathbf{S}_{2,2}}, \ldots, \frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}, \Sigma\right), \tag{23}$$

where the vector $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n) \equiv -\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}$, $\mathbf{C}^{'} \equiv \mathbf{C} - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$, $\Sigma \equiv (-2\mathbf{SAS})^{-1}$, *and S is a* $n \times n$ *diagonal matrix defined as*

$$\mathbf{S}_{i,j} \equiv \begin{cases} \sqrt{((-2\mathbf{A})^{-1})_{i,i}} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Proof See Appendix A. □

4. Analytical formulae

We will first derive the approximating analytical pricing formula for the up-and-out barrier call with a single discrete dividend in section 4.1. This approach can be extended to derive the pricing formula for the multi-dividend case as discussed in section 4.2.

4.1. The single-discrete-dividend case

Since only one dividend is paid during the life of the option, the option maturity date T is later than the first exdividend date t_1 but earlier than the second exdividend date t_2 (i.e. $t_1 < T < t_2$). The call option value $\dot{C}$ can be evaluated by applying the risk-neutral valuation method to the approximated stock price process $\hat{S}(t)$ defined in equation (21) as follows:

$$\dot{C} \equiv e^{-rT} \mathrm{E}\left[(\hat{S}(T) - K)1_{\{\dot{E}_1\cap\dot{E}_2\cap\dot{E}_3\}}\right], \tag{24}$$

where $\dot{E}_1$, $\dot{E}_2$ denote the events that the stock price process does not hit the barrier B during the time intervals $[0, t_1)$ and $[t_1, T]$, respectively, and $\dot{E}_3$ denotes the event that the option is in the money at the maturity date. Specifically, the three events $\dot{E}_1$, $\dot{E}_2 and \dot{E}_3$ are defined as follows:

$$\begin{aligned} \dot{E}_1 &\equiv \left\{\hat{S}(t) < B, \quad \forall t \in [0, t_1)\right\}, \\ \dot{E}_2 &\equiv \left\{\hat{S}(t) < B, \quad \forall t \in [t_1, T]\right\}, \\ \dot{E}_3 &\equiv \left\{\hat{S}(T) > K\right\}. \end{aligned} \tag{25}$$

To evaluate the option value, we derive the joint density of the maximum stock prices over the time interval $[0, t_1)$ and $[t_1, T]$ and the stock price at time t_1 and T by Theorem 2.1. Define $\hat{M}(t_1) \equiv \max_{0\le t<t_1} \hat{W}(t)$ as the maximum value of $\hat{W}(t)$ over the time interval $[0, t_1)$, and $\hat{M}_1(T - t_1) \equiv \max_{t_1\le t\le T} \hat{W}_1(t-t_1)$ as the maximum value of $\hat{W}_1(t-t_1)$ over the time interval $[t_1, T]$. Thus the three events $\dot{E}_1$, $\dot{E}_2 and \dot{E}_3$ can be rewritten by substituting the definition of $\hat{S}(t)$ defined in equation (21) into equation (25) to obtain

$$\begin{aligned} \dot{E}_1 &= \left\{S(0)e^{\sigma\hat{M}(t_1)} < B\right\} = \left\{\hat{M}(t_1) < b\right\}, \\ \dot{E}_2 &= \left\{S^{'}(0)e^{k_1\sigma\hat{W}(t_1)+\sigma\hat{M}_1(T-t_1)} < B\right\} \\ &= \left\{\hat{M}_1(T-t_1) < b^{'} - k_1\hat{W}(t_1)\right\}, \\ \dot{E}_3 &= \left\{S^{'}(0)e^{k_1\sigma\hat{W}(t_1)+\sigma\hat{W}_1(T-t_1)} > K\right\} \\ &= \left\{\hat{W}_1(T-t_1) > k^{'} - k_1\hat{W}(t_1)\right\}, \end{aligned}$$

where b, $b^{'}$ and $k^{'}$ represent $\frac{1}{\sigma}\log\frac{B}{S(0)}$, $\frac{1}{\sigma}\log\frac{B}{S^{'}(0)}$ and $\frac{1}{\sigma}\log\frac{K}{S^{'}(0)}$, respectively. The joint density functions $f_{\hat{M}(t_1),\hat{W}(t_1)}$ and $f_{\hat{M}_1(T-t_1),\hat{W}_1(T-t_1)}$ can be derived from Theorem 2.1 as follows:

$$\dot{f}_{\hat{M}(t_1),\hat{W}(t_1)}(m, w) = \begin{cases} \frac{2(2m-w)}{t_1\sqrt{2\pi t_1}} e^{\alpha w - \frac{1}{2}\alpha^2 t_1 - \frac{1}{2t_1}(2m-w)^2} & \text{if } m \ge w^+, \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

$$\dot{f}_{\hat{M}_1(T-t_1),\hat{W}_1(T-t_1)}(m_1, w_1) = \begin{cases} \frac{2(2m_1-w_1)}{(T-t_1)\sqrt{2\pi(T-t_1)}} \\ \quad\times e^{\alpha w_1 - \frac{1}{2}\alpha^2(T-t_1) - \frac{1}{2(T-t_1)}(2m_1-w_1)^2} & \text{if } m_1 \ge w_1^+, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

For simplicity, we will use the symbols $\dot{f}_0$ and $\dot{f}_1$ to represent $f_{\hat{M}(t_1),\hat{W}(t_1)}$ and $f_{\hat{M}_1(T-t_1),\hat{W}_1(T-t_1)}$, respectively. Since the two drifted Brownian motions $\hat{W}(t)$ for $t \in [0, t_1)$ and $\hat{W}_1(t - t_1)$ for $t \in [t_1, T]$ are independent due to the Markov property of the Brownian motion, the joint density function of maximum stock prices over $[0, t_1)$ and $[t_1, T]$ and the stock prices at time t_1 and T can be calculated by directly multiplying $\dot{f}_0$ by $\dot{f}_1$. By substituting this joint density function into equation (24), the analytical pricing formula can be derived by the law of iterated expectation as follows:

$$\begin{aligned} &\dot{C} \\ &= e^{-rT} E\left[E\left[(\hat{S}(T) - K)1_{\{\dot{E}_1\cap\dot{E}_2\cap\dot{E}_3\}}\middle|\hat{W}(t_1), \hat{M}(t_1)\right]\right] \\ &= e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{b}\int_{k^{'}-k_1 w}^{\infty}\int_{-\infty}^{b^{'}-k_1 w} \\ &\times\left(S^{'}(0)e^{k_1\sigma w+\sigma w_1} - K\right)\dot{f}_1(m_1, w_1)\dot{f}_0(m, w)dm_1dw_1dmdw \end{aligned} \tag{28}$$

$$\begin{aligned} &= e^{-rT}\int_{-\infty}^{b}\int_{w^+}^{b}\int_{k^{'}-k_1 w}^{b^{'}-k_1 w}\int_{w_1^+}^{b^{'}-k_1 w} \\ &\times\left(S^{'}(0)e^{k_1\sigma w+\sigma w_1} - K\right)\dot{f}_1(m_1, w_1)\dot{f}_0(m, w)dm_1dw_1dmdw \end{aligned} \tag{29}$$

where the domain of the integral in equation (29) is obtained by mimicking the analysis in figure 1; it is derived by taking the intersection of the supports of $\dot{f}_1(m_1, w_1)$ and $\dot{f}_0(m_0, w_0)$ with the integral domain in equation (28). In the integrand in equation (29), only $\dot{f}_0(m, w)$ contains the integrator m and $\dot{f}_1(m_1, w_1)$ contains the integrator m_1. Therefore, $\int_{w^+}^{b}\dot{f}_0(m, w)dm$ and $\int_{w_1^+}^{b^{'}-k_1 w}\dot{f}_1(m_1, w_1)dm_1$ can be integrated separately by Lemma 2.2 as follows:

$$\begin{aligned} \dot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{k^{'}-k_1 w}^{b^{'}-k_1 w}\left(S^{'}(0)e^{k_1\sigma w+\sigma w_1} - K\right) \\ &\times\left(\int_{w_1^+}^{b^{'}-k_1 w}\frac{2(2m_1-w_1)}{(T-t_1)\sqrt{2\pi(T-t_1)}}\right. \\ &\quad\left.\times e^{\alpha w_1 - \frac{1}{2}\alpha^2(T-t_1) - \frac{1}{2(T-t_1)}(2m_1-w_1)^2}dm_1\right) \\ &\times\left(\int_{w^+}^{b}\frac{2(2m-w)}{t_1\sqrt{2\pi t_1}}e^{\alpha w - \frac{1}{2}\alpha^2 t_1 - \frac{1}{2t_1}(2m-w)^2}dm\right)dw_1dw. \end{aligned} \tag{30}$$

The variables in the lower and the upper limits for the above integral can be eliminated by the change of variables; that is, we substitute $x = w_1 + k_1 w$ and $y = w$ into equation (30) to obtain†

$$\begin{aligned} \dot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{k^{'}}^{b^{'}}\left(S^{'}(0)e^{\sigma x} - K\right) \\ &\times\frac{1}{\sqrt{2\pi(T-t_1)}}e^{\alpha(x-k_1 y) - \frac{1}{2}\alpha^2(T-t_1) - \frac{(x-k_1 y)^2}{2(T-t_1)}} \\ &\times\left(1 - e^{\frac{2\left(b^{'}-k_1 y\right)\left(x-k_1 y-\left(b^{'}-k_1 y\right)\right)}{T-t_1}}\right) \end{aligned} \tag{31}$$

†Note that the Jacobian determinant $\frac{\partial(w_1,w)}{\partial(x,y)} = 1$.

$$\times \frac{1}{\sqrt{2\pi t_1}} e^{\alpha y - \frac{1}{2}\alpha^2 t_1 - \frac{y^2}{2t_1}} \left(1 - e^{\frac{2b(y-b)}{t_1}}\right) dxdy.$$

$$= e^{-rT} \int_{-\infty}^{b} \int_{k'}^{b'} I(1) + I(2) + I(3) + I(4) + I(5) + I(6) + I(7) + I(8) dxdy, \quad (32)$$

Proof It can be easily derived from Theorem 3.1 and is proved in Appendix B. □

With Corollary 4.1, the double integrals of $I(1), I(2), \cdots, I(8)$ can be converted into CDFs of bivariate normal distributions. For example, the integral of $I(1)$ can be rewritten as

$$\int_{-\infty}^{b} \int_{k'}^{b'} I(1) dxdy = \int_{-\infty}^{b} \int_{k'}^{b'} -\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}} e^{-\frac{y^2}{2t_1} + \alpha y + \alpha(x - yk_1) - \frac{1}{2}\alpha^2(T-t_1) - \frac{\alpha^2 t_1}{2} - \frac{(x-yk_1)^2}{2(T-t_1)}} dxdy$$
$$= -\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}} \left(\int_{-\infty}^{b} \int_{-\infty}^{b'} e^{-\frac{1}{2(T-t_1)}x^2 + \frac{k_1}{T-t_1}xy - \left(\frac{k_1^2}{2(T-t_1)} + \frac{1}{2t_1}\right)y^2 + \alpha x + (\alpha - \alpha k_1) y - \frac{T\alpha^2}{2}} dxdy \right.$$
$$\left. - \int_{-\infty}^{b} \int_{-\infty}^{k'} e^{-\frac{1}{2(T-t_1)}x^2 + \frac{k_1}{T-t_1}xy - \left(\frac{k_1^2}{2(T-t_1)} + \frac{1}{2t_1}\right)y^2 + \alpha x + (\alpha - \alpha k_1) y - \frac{T\alpha^2}{2}} dxdy \right). \quad (35)$$

where the integrand $I(1)$ is defined as

$$I(1) \equiv -\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}} \times e^{-\frac{y^2}{2t_1} + \alpha y + \alpha(x - yk_1) - \frac{1}{2}\alpha^2(T-t_1) - \frac{\alpha^2 t_1}{2} - \frac{(x-yk_1)^2}{2(T-t_1)}},$$

$I(2) \equiv -I(1)e^{\frac{2(x-b')(b'-yk_1)}{T-t_1}}$, $I(3) \equiv -I(1)e^{\frac{2b(y-b)}{t_1}}$ and $I(4) \equiv I(1)e^{\frac{2(x-b')(b'-yk_1)}{T-t_1} + \frac{2b(y-b)}{t_1}}$. These four terms are obtained by multiplying the strike price K (in the first line) by the terms in the following two lines of equation (31). $I(5)$, $I(6)$, $I(7)$ and $I(8)$, which are obtained by multiplying $S'(0)e^{\sigma x}$ (in the first line) by the terms in the following two lines of equation (31) , are defined as

$$I(i) = -\frac{S'(0)}{K} I(i-4) e^{\sigma x}, \quad i = 5, \ldots, 8. \quad (33)$$

Since each exponent term of the integrands $I(1), I(2), \cdots, I(8)$ is a quadratic polynomial of integrators x and y, the double integral of each integrand can be expressed in terms of a bivariate normal CDF by the following Corollary:

Corollary 4.1 *The double integral G with the following format can be expressed in terms of the CDF of a bivariate standard normal distribution F_{Y_1,Y_2} as follows:*

$$G(p, q, a_1, a_2, a_3, a_4, a_5, a_6) \equiv \int_{-\infty}^{q} \int_{-\infty}^{p} e^{a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6} dxdy$$
$$= \frac{2\pi}{\sqrt{\Delta}} \exp\left(a_6 + \frac{a_2 a_4 a_5 - a_3 a_4^2 - a_1 a_5^2}{\Delta}\right) \times F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta}p + \frac{2a_3a_4 - a_2a_5}{\sqrt{\Delta}}}{\sqrt{-2a_3}}, \frac{\sqrt{\Delta}q + \frac{2a_1a_5 - a_2a_4}{\sqrt{\Delta}}}{\sqrt{-2a_1}}, \Sigma\right), \quad (34)$$

where $\Delta \equiv 4a_1a_3 - a_2^2$, *and* $\Sigma \equiv \begin{pmatrix} 1 & \frac{a_2}{2\sqrt{a_1a_3}} \\ \frac{a_2}{2\sqrt{a_1a_3}} & 1 \end{pmatrix}$.

Define $a_1(1) \equiv -\frac{1}{2(T-t_1)}$, $a_2(1) \equiv \frac{k_1}{T-t_1}$, $a_3(1) \equiv -\left(\frac{k_1^2}{2(T-t_1)} + \frac{1}{2t_1}\right)$, $a_4(1) \equiv \alpha$, $a_5(1) \equiv (\alpha - \alpha k_1)$, $a_6(1) \equiv -\frac{T\alpha^2}{2}$ as the coefficients of x^2, xy, y^2, x, y and the constant term, respectively, of the exponential term of the integrand $I(1)$ in equation (35). By Corollary 4.1, equation (35) can be rewritten in terms of bivariate normal CDFs as follows:

$$-\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}} \frac{2\pi}{\sqrt{\Delta(1)}} \exp\left(a_6(1) + \frac{a_2(1)a_4(1)a_5(1) - a_3(1)a_4(1)^2 - a_1(1)a_5(1)^2}{\Delta(1)}\right)$$
$$\times \left[F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta(1)}b' + \frac{2a_3(1)a_4(1) - a_2(1)a_5(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_3(1)}}, \frac{\sqrt{\Delta(1)}b + \frac{2a_1(1)a_5(1) - a_2(1)a_4(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_1(1)}}, \Sigma(1)\right) \right.$$
$$\left. - F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta(1)}k' + \frac{2a_3(1)a_4(1) - a_2(1)a_5(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_3(1)}}, \frac{\sqrt{\Delta(1)}b + \frac{2a_1(1)a_5(1) - a_2(1)a_4(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_1(1)}}, \Sigma(1)\right) \right], \quad (36)$$

where $\Delta(1)$, $\Sigma(1)$ are obtained by substituting $a_1(1), a_2(1), \cdots, a_6(1)$ into Corollary 4.1 as follows:

$$\Delta(1) \equiv 4a_1(1)a_3(1) - a_2(1)^2,$$
$$\Sigma(1) \equiv \begin{pmatrix} 1 & \frac{a_2(1)}{2\sqrt{a_1(1)a_3(1)}} \\ \frac{a_2(1)}{2\sqrt{a_1(1)a_3(1)}} & 1 \end{pmatrix}.$$

For convenience, we rewrite equation (36) as follows:

$$D(1)[G(b', b, a_1(1), a_2(1), a_3(1), a_4(1), a_5(1), a_6(1)) - G(k', b, a_1(1), a_2(1), a_3(1), a_4(1), a_5(1), a_6(1))],$$

where $D(1) \equiv -\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$, and G is defined in equation (34). Similarly, the double integrals for $I(2), I(3), \ldots, I(8)$ in equation (32) can all be expressed as

$$\int_{-\infty}^{b}\int_{k'}^{b'} I(i)dxdy$$
$$= D(i)[G(b', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)) - G(k', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i))],$$

where $a_1(i)$, $a_2(i)$, $a_3(i)$, $a_4(i)$, $a_5(i)$ and $a_6(i)$ denote the coefficients of x^2, xy, y^2, x, y and the constant term of the exponential term of $I(i)$. Specifically, the parameters are given by $a_1(i) = a_1(1)$, $a_3(i) = a_3(1)$, $a_2(i) = (-1)^{i+1}a_2(1)$ for $i = 2, 3, \cdots, 8$, and $D(i)$, $a_4(i)$, $a_5(i)$ and $a_6(i)$ are given by the following table:

i	$D(i)$	$a_4(i)$	$a_5(i)$	$a_6(i)$
2	$\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b'}{T-t_1}+\alpha$	$\alpha+k_1\left(\frac{2b'}{T-t_1}-\alpha\right)$	$-\frac{4b'^2+T^2\alpha^2-T\alpha^2t_1}{2T-2t_1}$
3	$\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$	α	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
4	$-\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b'}{T-t_1}+\alpha$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b'}{T-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b'^2}{T-t_1}$
5	$\frac{S'(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\alpha+\sigma$	$\alpha-\alpha k_1$	$-\frac{T\alpha^2}{2}$
6	$-\frac{S'(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b'}{T-t_1}+\alpha+\sigma$	$\alpha+k_1\left(\frac{2b'}{T-t_1}-\alpha\right)$	$-\frac{4b'^2+T^2\alpha^2-T\alpha^2t_1}{2T-2t_1}$
7	$-\frac{S'(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\alpha+\sigma$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
8	$\frac{S'(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b'}{T-t_1}+\alpha+\sigma$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b'}{T-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b'^2}{T-t_1}$

Thus, the option pricing formula in equation (32) can be rewritten as

$$\dot{C}$$
$$= e^{-rT}\int_{-\infty}^{b}\int_{k'}^{b'} I(1)+I(2)+\cdots+I(8)dxdy = e^{-rT}$$
$$\times\sum_{i=1}^{8}\Big[D(i)[G(b', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)) - G(k', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i))]\Big].$$

Note that if the upper barrier B tends to infinity, the up-and-out call degenerates into a vanilla call option. Indeed, both $b\left(=\frac{1}{\sigma}\log\frac{B}{S(0)}\right)$ and $b'\left(=\frac{1}{\sigma}\log\frac{B}{S'(0)}\right)$ tend to infinity as $B\to\infty$, and our pricing formula degenerates into the approximating formula for pricing vanilla stock call options with one discrete dividend derived in Dai and Lyuu (2009).

4.2. *Multi-discrete-dividend case*

The above approach can be repeatedly applied to derive approximated pricing formulae for barrier stock options with multiple discrete dividends. For simplicity, we derive the pricing formula for the two-dividend case in this section. The extensions for cases involving three or more dividends are straightforward. Note that $t_1 < t_2 < T < t_3$ in the two-dividend case.

To evaluate the option, we need to derive the joint density function of the maximum stock prices over the time intervals $[0, t_1)$, $[t_1, t_2)$ and $[t_2, T]$ and the stock price at the maturity date T. Define $\hat{M}_1(t_2-t_1) \equiv \max_{t_1\le t<t_2}\hat{W}_1(t-t_1)$ as the maximum value of $\hat{W}_1(t)$ over the time interval $[t_1, t_2)$ and $\hat{M}_2(T-t_2) \equiv \max_{t_2\le t\le T}\hat{W}_2(t-t_2)$ as the maximum value of $\hat{W}_2(t)$ over the time interval $[t_2, T]$. The joint density function of $\hat{M}_1(t_2-t_1)$ and $\hat{W}_1(t_2-t_1)$, and the joint density function of $\hat{M}_2(T-t_2)$ and $\hat{W}_2(T-t_2)$ can be derived by applying Theorem 2.1 as follows:

$$f_{\hat{M}_1(t_2-t_1),\hat{W}_1(t_2-t_1)}(m_1, w_1) = \begin{cases} \frac{2(2m_1-w_1)}{(t_2-t_1)\sqrt{2\pi(t_2-t_1)}} \times e^{\alpha w_1-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{1}{2(t_2-t_1)}(2m_1-w_1)^2} & \text{if } m_1\ge w_1^+, \\ 0 & \text{otherwise,} \end{cases} \quad (37)$$

$$f_{\hat{M}_2(T-t_2),\hat{W}_2(T-t_2)}(m_2, w_2) = \begin{cases} \frac{2(2m_2-w_2)}{(T-t_2)\sqrt{2\pi(T-t_2)}} \times e^{\alpha w_2-\frac{1}{2}\alpha^2(T-t_2)-\frac{1}{2(T-t_2)}(2m_2-w_2)^2} & \text{if } m_2\ge w_2^+, \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity, we use $\ddot{f}_0$, $\ddot{f}_1$ and $\ddot{f}_2$ to represent the density functions $f_{\hat{M}(t_1),\hat{W}(t_1)}$ (see equation (26)), $f_{\hat{M}_1(t_2-t_1),\hat{W}_1(t_2-t_1)}$ and $f_{\hat{M}_2(T-t_2),\hat{W}_2(T-t_2)}$, respectively. Note that the drifted Brownian motions $\hat{W}(t)$ for $t\in[0, t_1)$, $\hat{W}_1(t-t_1)$ for $t\in[t_1, t_2)$ and $\hat{W}_2(t-t_2)$ for $t\in[t_2, t_3]$ are independent due to the Markov property of the Brownian motion; therefore, the joint density function of maximum stock prices over $[0, t_1)$, $[t_1, t_2)$ and $[t_2, T]$ and the stock prices at time t_1, t_2 and T can be calculated by directly multiplying $\ddot{f}_0$ by $\ddot{f}_1$ and $\ddot{f}_2$.

The option value can be evaluated by the risk-neutral variation method as follows:

$$\ddot{C} \equiv e^{-rT}E\left[(\hat{S}(T)-K)1_{\{\ddot{E}_1\cap\ddot{E}_2\cap\ddot{E}_3\cap\ddot{E}_4\}}\right], \quad (38)$$

where $\ddot{E}_1$, $\ddot{E}_2$ and $\ddot{E}_3$ represent the events that the stock price process does not hit the barrier B during the time intervals $[0, t_1)$, $[t_1, t_2)$ and $[t_2, T]$, respectively, and $\ddot{E}_4$ denotes the event that the stock price at maturity is greater than the strike price. Specifically, $\ddot{E}_1$, $\ddot{E}_2$, $\ddot{E}_3$ and $\ddot{E}_4$ are defined as

$$\ddot{E}_1 = \left\{ S(0)e^{\sigma \hat{M}(t_1)} < B \right\} = \left\{ \hat{M}(t_1) < b \right\},$$
$$\ddot{E}_2 = \left\{ S^{'}(0)e^{k_1\sigma \hat{W}(t_1)+\sigma \hat{M}_1(t_2-t_1)} < B \right\}$$
$$= \left\{ \hat{M}_1(t_2-t_1) < b^{'} - k_1\hat{W}(t_1) \right\},$$
$$\ddot{E}_3 = \left\{ S^{''}(0)e^{k_1k_2\sigma \hat{W}(t_1)+k_2\sigma \hat{W}_1(t_2-t_1)+\sigma \hat{M}_2(T-t_2)} < B \right\}$$
$$= \left\{ \hat{M}_2(T-t_2) < b^{''} - k_1k_2\hat{W}(t_1) - k_2\hat{W}_1(t_2-t_1) \right\},$$
$$\ddot{E}_4 = \left\{ S^{''}(0)e^{k_1k_2\sigma \hat{W}(t_1)+k_2\sigma \hat{W}_1(t_2-t_1)+\sigma \hat{W}_2(T-t_2)} < K \right\}$$
$$= \left\{ \hat{W}_2(T-t_2) < k^{''} - k_1k_2\hat{W}(t_1) - k_2\hat{W}_1(t_2-t_1) \right\},$$

where $k^{''} \equiv \frac{1}{\sigma}\log\frac{K}{S^{''}(0)}$, and $b^{''} \equiv \frac{1}{\sigma}\log\frac{B}{S^{''}(0)}$, respectively. Thus, we can compute the pricing formula in equation (38) by applying the law of iterated expectation as follows:

$$\ddot{C} \tag{39}$$
$$= e^{-rT}E\left[E\left[E\left[(\hat{S}(T)-K)1_{\{\ddot{E}_1\cap\ddot{E}_2\cap\ddot{E}_3\cap\ddot{E}_4\}} \middle| \hat{W}(t_1), \hat{M}(t_1), \hat{W}_1(t_2-t_1), \hat{M}_1(t_2-t_1)\right] \middle| \hat{W}(t_1), \hat{M}(t_1)\right]\right]$$
$$= e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{b}\int_{-\infty}^{\infty}\int_{-\infty}^{b^{'}-k_1w}\int_{k^{''}-k_1k_2w-k_2w_1}^{\infty} \times \int_{-\infty}^{b^{''}-k_1k_2w-k_2w_1}\left(S^{''}(0)e^{k_1k_2\sigma w+k_2\sigma w_1+\sigma w_2}-K\right)$$
$$\times \ddot{f}_2(m_2,w_2)\cdot\ddot{f}_1(m_1,w_1)\cdot\ddot{f}_0(m,w)dm_2dw_2dm_1dw_1dmdw \tag{40}$$
$$= e^{-rT}\int_{-\infty}^{b}\int_{w^+}^{b}\int_{-\infty}^{b^{'}-k_1w}\int_{w_1^+}^{b^{'}-k_1w}\int_{k^{''}-k_1k_2w-k_2w_1}^{b^{''}-k_1k_2w-k_2w_1} \times \int_{w_2^+}^{b^{''}-k_1k_2w-k_2w_1}\left(S^{''}(0)e^{k_1k_2\sigma w+k_2\sigma w_1+\sigma w_2}-K\right)$$
$$\times \ddot{f}_2(m_2,w_2)\cdot\ddot{f}_1(m_1,w_1)\cdot\ddot{f}_0(m,w)dm_2dw_2dm_1dw_1dmdw, \tag{41}$$

where the domain of the integral in equation (41) is obtained by taking the intersection of the supports of $\ddot{f}_2(m_2,w_2)$, $\ddot{f}_1(m_1,w_1)$ and $\ddot{f}_0(m,w)$ with the integral domain in equation (40). Since only $\ddot{f}_2(m_2,w_2)$ contains the integrator m_2, $\ddot{f}_1(m_1,w_1)$ contains m_1, and $\ddot{f}_0(m,w)$ contains m in the integrand in equation (41), $\int \ddot{f}_0(m,w)dm$, $\int \ddot{f}_1(m_1,w_1)dm_1$ and $\int \ddot{f}_2(m_2,w_2)dm_2$ can be simplified by applying Lemma 2.2 as follows:

$$\ddot{C} = e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b^{'}-k_1w}\int_{k^{''}-k_1k_2w-k_2w_1}^{b^{''}-k_1k_2w-k_2w_1}\left(S^{''}(0)e^{k_1k_2\sigma w+k_2\sigma w_1+\sigma w_2}-K\right)$$
$$\times\left(\int_{w_2^+}^{b^{''}-k_1k_2w-k_2w_1}\frac{2(2m_2-w_2)}{(T-t_2)\sqrt{2\pi(T-t_2)}} e^{\alpha w_2-\frac{1}{2}\alpha^2(T-t_2)-\frac{1}{2(T-t_2)}(2m_2-w_2)^2}dm_2\right)$$
$$\times\left(\int_{w_1^+}^{b^{'}-k_1w}\frac{2(2m_1-w_1)}{(t_2-t_1)\sqrt{2\pi(t_2-t_1)}} e^{\alpha w_1-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{1}{2(t_2-t_1)}(2m_1-w_1)^2}dm_1\right)$$
$$\times\left(\int_{w^+}^{b}\frac{2(2m-w)}{t_1\sqrt{2\pi t_1}} e^{\alpha w-\frac{1}{2}\alpha^2t_1-\frac{1}{2t_1}(2m-w)^2}dm\right)dw_2dw_1dw.$$

To eliminate the variables in the lower and the upper limits for the integrals on w_1 and w_2, the equations $x = w_2 + k_2w_1 + k_1k_2w$, $y = w_1 + k_1w$ and $z = w$ are substituted into the aforementioned formula to obtain†

$$\ddot{C} = e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b^{'}}\int_{k^{''}}^{b^{''}}\left(S^{''}(0)e^{\sigma x}-K\right)$$
$$\times\frac{1}{\sqrt{2\pi(T-t_2)}}e^{\alpha(x-k_2y)-\frac{1}{2}\alpha^2(T-t_2)-\frac{(x-k_2y)^2}{2(T-t_2)}}$$
$$\times\left(1-e^{\frac{2\left(b^{''}-k_2y\right)\left(x-k_2y-\left(b^{''}-k_2y\right)\right)}{(T-t_2)}}\right)$$
$$\times\frac{1}{\sqrt{2\pi(t_2-t_1)}}e^{\alpha(y-k_1z)-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{(y-k_1z)^2}{2(t_2-t_1)}}$$
$$\times\left(1-e^{\frac{2\left(b^{'}-k_1z\right)\left(y-k_1z-\left(b^{'}-k_1z\right)\right)}{(t_2-t_1)}}\right)$$
$$\times\frac{1}{\sqrt{2\pi t_1}}e^{\alpha z-\frac{1}{2}\alpha^2t_1-\frac{z^2}{2t_1}}\left(1-e^{\frac{2b(z-b)}{t_1}}\right)dxdydz$$
$$= e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b^{'}}\int_{k^{''}}^{b^{''}}\sum_{i=1}^{16}J(i)dxdydz, \tag{42}$$

where $J(1), J(2), \cdots, J(8)$ are defined in table 1, and $J(9), J(10), \cdots, J(16)$ are defined as

$$J(i) = -\frac{S''(0)}{K}J(i-8)e^{x\sigma}, \quad i = 9,\ldots,16.$$

Since the exponent term of each of the integrands $J(1)$, $J(2), \cdots, J(16)$ is a quadratic form of the integrators x, y and z, the triple integral of each integrand can be expressed in terms of a trivariate normal CDF by the following corollary:

COROLLARY 4.2 *The triple integral with the following format can be expressed in terms of a CDF of a trivariate standard normal distribution F_{Y_1,Y_2,Y_3} as follows:*

$$p, q, r, a_1, a_2, \cdots, a_{10}) \equiv \int_{-\infty}^{r}\int_{-\infty}^{q}\int_{-\infty}^{p} e^{a_1x^2+a_2y^2+a_3z^2+a_4xy+a_5yz+a_6xz+a_7x+a_8y+a_9z+a_{10}}dxdydz$$
$$= e^{C^{'}}\sqrt{\frac{\pi^3}{|-A|}}F_{Y_1,Y_2,Y_3}\left(\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}, \frac{p_2-\mathbf{m}_2}{\mathbf{S}_{2,2}}, \frac{p_3-\mathbf{m}_3}{\mathbf{S}_{3,3}}, \Sigma\right), \tag{43}$$

where

$$A = \begin{pmatrix} a_1 & \frac{a_4}{2} & \frac{a_6}{2} \\ \frac{a_4}{2} & a_2 & \frac{a_5}{2} \\ \frac{a_6}{2} & \frac{a_5}{2} & a_3 \end{pmatrix}, B = \begin{pmatrix} a_7 \\ a_8 \\ a_9 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} \mathbf{S}_{1,1} & 0 & 0 \\ 0 & \mathbf{S}_{2,2} & 0 \\ 0 & 0 & \mathbf{S}_{3,3} \end{pmatrix},$$

$\mathbf{S}_{j,j} = \sqrt{((-2A)^{-1})_{j,j}}$, $\mathbf{m} = -\frac{1}{2}A^{-1}B$, $C^{'} = a_{10} - \frac{1}{4}B^TA^{-1}B$, *and* $\Sigma = (-2\mathbf{S}A\mathbf{S})^{-1}$.

†Note that the Jacobian determinant $\frac{\partial(w_2,w_1,w)}{\partial(x,y,z)}$ is 1.

Table 1. The definitions of $J(1), J(2), \ldots, J(8)$.

Define ζ as $\sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1}$, and η as $-\frac{z^2}{2t_1}+\alpha z+\alpha(y-zk_1)+\alpha(x-yk_2)-\frac{\alpha^2 t_1}{2}-\frac{1}{2}\alpha^2(T-t_2)-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{(x-yk_2)^2}{2(T-t_2)}-\frac{(y-zk_1)^2}{2(t_2-t_1)}$	
$J(1)=-\frac{K}{\zeta}e^{\eta}$,	$J(2)=\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}}$,
$J(3)=\frac{K}{\zeta}e^{\eta+\frac{2\left(x-b''\right)\left(b''-yk_2\right)}{T-t_2}}$,	$J(4)=-\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}+\frac{2\left(x-b''\right)\left(b''-yk_2\right)}{T-t_2}}$,
$J(5)=\frac{K}{\zeta}e^{\eta+\frac{2\left(y-b'\right)\left(b'-zk_1\right)}{t_2-t_1}}$,	$J(6)=-\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}+\frac{2\left(y-b'\right)\left(b'-zk_1\right)}{t_2-t_1}}$,
$J(7)=-\frac{K}{\zeta}e^{\eta+\frac{2\left(x-b''\right)\left(b''-yk_2\right)}{T-t_2}+\frac{2\left(y-b'\right)\left(b'-zk_1\right)}{t_2-t_1}}$,	$J(8)=\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}+\frac{2\left(x-b''\right)\left(b''-yk_2\right)}{T-t_2}+\frac{2\left(y-b'\right)\left(b'-zk_1\right)}{t_2-t_1}}$,

Proof This corollary can be easily derived from Theorem 3.1. □

Let $a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i), a_7(i), a_8(i), a_9(i)$ and $a_{10}(i)$ be the coefficients of x^2, y^2, z^2, xy, yz, xz, x, y, z and the constant term, respectively, of the exponential term of the integrand $J(i)$ in table 1. These coefficients are listed in Appendix C. The triple integrals of $J(i)$ in equation (42) can be expressed in terms of CDFs of trivariate normal distributions by applying Corollary 4.2 as follows:

$$\int_{-\infty}^{b}\int_{-\infty}^{b'}\int_{k''}^{b''} J(i)dxdydz = E(i)[H(b'', b', b, a_1(i), a_2(i), \cdots, a_{10}(i)) - H(k'', b', b, a_1(i), a_2(i), \cdots, a_{10}(i))], \quad (44)$$

where the function H is defined in equation (43), and the function $E(i)$ is defined as follows:

$$E(2)=E(3)=E(5)=E(8)=\frac{K}{\sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1}},$$

$$E(1)=E(4)=E(6)=E(7)=-E(2),$$

$$E(9)=E(12)=E(14)=E(15)=\frac{S''(0)}{\sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1}},$$

$$E(10)=E(11)=E(13)=E(16)=-E(9),$$

By substituting equation (44) into equation (42), the option price formula for the two-dividend case is derived as follows:

$$\ddot{C} = e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b'}\int_{k''}^{b''} J(1)+J(2)+\cdots+J(16)dxdydz$$
$$= e^{-rT}\sum_{i=1}^{16}\Big[E(i)[H(b'', b', b, a_1(i), a_2(i), \cdots, a_{10}(i)) - H(k'', b', b, a_1(i), a_2(i), \cdots, a_{10}(i))]\Big].$$

5. Numerical results

Unlike most numerical pricing approaches that might generate unstable pricing results or hedging parameters (i.e. the Greek letters) as mentioned in Figlewski and Gao (1999) and Dai and Lyuu (2010), our approximate pricing formulae can generate stable pricing results and hedging parameters as illustrated in figure 2. In panel (a), an up-and-out call option value increases with the increment of the initial stock price when the stock price is low. However, the increment of the initial stock price also increases the probability for the option to knock out (i.e. the stock price path goes upward to reach the barrier). Therefore, when the initial stock price is higher than a certain level, say 52 in this numerical example, the option value decreases with the increment of the initial stock price. This phenomenon can be confirmed by checking the delta, i.e. the rate of change of the option price with the price of the underlying stock, as illustrated in panel (b). The delta smoothly decreases with the increment of the initial stock price and becomes negative when the stock price exceeds 52.

To examine the superiority of our pricing formulae, we will compare the accuracy among our approximation pricing formulae and other approximation formulae in the following tables. `Ours` denote the values generated by the approximation pricing formulae proposed in this paper. We also follow the Chiras and Manaster (1978) assumption by approximating the discrete dividends paid over the life of the option with the equivalent continuous dividend yield q being derived as follows:

$$S(0)e^{-qT} = S(0) - \sum_{i=1}^{n} c_i e^{-rt_i},$$

where n denotes the number of dividends paid during the life of the option. Then the discrete-dividend barrier option can be approximately priced by the barrier option pricing formula with a continuous dividend yield proposed by Reiner and Rubinstein (1991), and the pricing results generated by this approach are listed under the `ContDiv` columns. Besides, we can follow Model 1 (see Roll 1977) by assuming that the process of the net-of-dividend stock price $S_N(t)$ follows a lognormal diffusion process. In addition, the initial net-of-dividend stock price is defined as

$$S_N(0) \equiv S(0) - \sum_{i=1}^{n} c_i e^{-rt_i}.$$

Thus the discrete-dividend barrier option can be approximately priced by the Reiner and Rubinstein (1991) formula with the initial stock price being replaced by $S_N(0)$. The prices generated by this approach are listed under the `Model1` columns.†

†Frishling (2002) argues that Model 1 could incorrectly render a down-and-out barrier option worthless because the net-of-dividend stock price may reach the barrier for a large present value of future dividend payments.

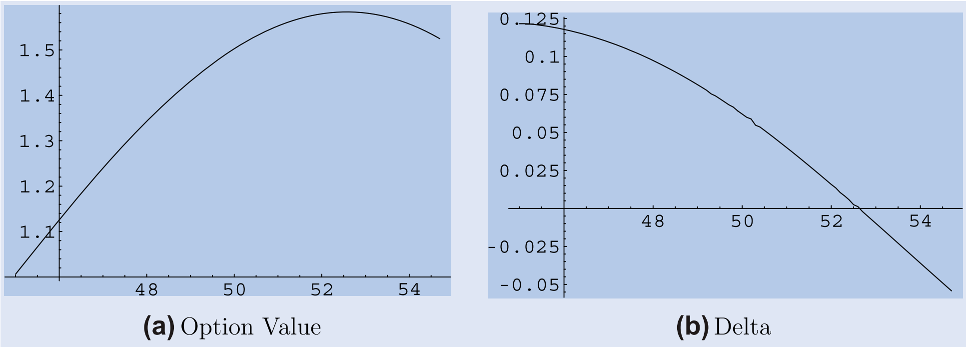

Figure 2. Option value and delta. Notes: The x-axes in both panels denote the initial stock price. The y-axes in panel (a) and (b) denote the up-and-out call price and the delta, respectively. The risk-free rate is 3%, the volatility is 20%, the strike price is 50, the barrier is 65 and the time to maturity is 1 year. A discrete dividend 1 is paid at 0.5 year.

Table 2 illustrates how the changes in the initial stock prices influence the option values and the accuracy of the aforementioned three pricing formulae. Similar to figure 2(a), the option value first increases and then decreases with the increment of the initial stock price. Similar to the phenomenon observed in Frishling (2002) for pricing vanilla options, different dividend-approximation models would generate very different prices. Here, we use a Monte Carlo simulation (denoted as `MC`) with 1 000 000 trials and a binomial lattice† (denoted as `L`) as proxy benchmarks. Recall that Frishling (2002), Bender and Vorst (2001), and Bos and Vandermark (2002) argue that only Model 3 can reflect the reality and generate consistent option prices. Thus, we use the Monte Carlo simulation as the first benchmark since it can faithfully model the downward jumps of the underlying stock price defined in **Model 3**. However, Baldi *et al.* (1999) argue that it might be difficult to obtain very precise results with the Monte Carlo simulation. Thus, we add the binomial lattice as another benchmark. It can be observed that the benchmark values produced by these two methods are close and coherent. By using the Monte Carlo simulation as the benchmark, it can be observed that our formula is more accurate than the other two formulae since the maximum absolute error (`MAE`) 0.0089 and the root-mean-squared error (`RMSE`) 0.0054 are lower than those for the other two formulae. In addition, the pricing errors of `Model1` are much more significant than the errors of the other two formulae. `Model1` produces very inaccurate results (the percentage of error = $\frac{0.1765}{0.1932} \approx 90\%$) when the initial stock price is high, say, 64. Using the binomial lattice as the benchmark also produces the same result. The `MAE` and the `RMSE` for `ContDiv` are 0.0169 and 0.01076, respectively‡. These two values for `Model1` are 0.1832 and 0.1163, respectively. They are much higher than the `MAE` 0.0013 and the `RMSE` 0.0010 for `Ours`. For simplicity, we will only use the Monte Carlo simulation with 1 000 000 trials as the benchmark (denoted by `Benchmark`) in the following experiments.

Table 3 compares the pricing results under different amounts of discrete dividend payout. It can be observed that the pricing errors of both `ContDiv` and `Model1` increase with the amount of the dividend payout, while the pricing errors of `Ours` are much smaller. `MAE` and `RMSE` of `Ours` are also smaller than those of `ContDiv` and `Model1`. Table 4 illustrates the pricing results under different stock price volatilities. Note that the value of an up-and-out call decreases with the increment in the stock price volatility since a higher volatility implies a higher 'knock out' probability. It can also be observed that `MAE` and `RMSE` of `Ours` are all smaller than 10^{-2}, while `MAE` and `RMSE` of both `ContDiv` and `Model1` are much higher. Table 5 analyses the impacts of changing the exdividend date on the option value. By observing the `Benchmark` column, the benchmark value decreases as the first exdividend date t_1 increases. Our formula successfully captures this phenomenon, while the other two approaches fail.

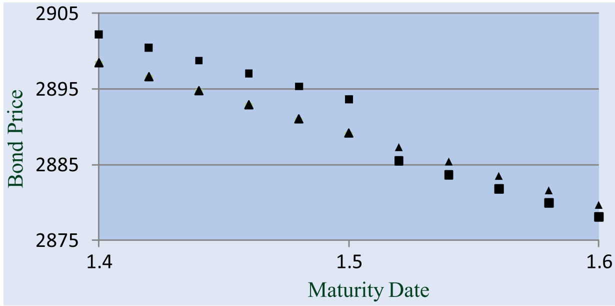

Figure 3. Evaluating vulnerable bonds. Notes: The issuing firm value is 5000, the volatility of the firm value is 25%, an exogenously given default boundary is 2400, the firm repays a debt amounting to \$150 at year 1.5, and the risk-free rate is 2%. The x-axis denotes the maturity of another unsecured debt with a face value of 3000 and the y-axis denotes the price of that debt. The pricing results for the unsecured debt generated by our formulae are marked by solid squares. Solid triangles denote the pricing results generated under the constant continuous payout ratio assumption.

Next, we extend our comparison to the two-dividend case. The underlying stock is assumed to pay two dividends at year 0.5 and year 1 and the time to maturity is set to 1.5 years. Tables 6 and 7 illustrate the impacts of changing the initial stock price and the amount of dividend payments on the option value. Again, `MAE` and `RMSE` of `Ours` are also smaller than those of `ContDiv` and `Model1`. Thus, we conclude that our pricing formulae can provide more accurate and consistent pricing results than other models.

Our option pricing model can be applied to extend the applicability of the first-passage model. A hypothetical example to analyse the impact of selling the firm's asset to finance the repayment of one junior debt on the value of another unse-

†We thank the anonymous reviewer for providing the benchmark values generated by the binomial lattice with 20 000 steps.
‡The pricing errors, `MAE`, and `RMSE` for `ContDiv` and `Model1` are not listed in table 2 for simplicity.

Table 2. Comparing the effect of changing initial stock prices on pricing barrier calls with a single discrete dividend.

$S(0)$	MC	L	Ours	error(MC)	error(L)	ContDiv	error(MC)	Model1	error(MC)
46	1.1265	1.1247	1.1260	0.0005	0.0013	1.1124	0.0141	1.1336	0.0071
48	1.3456	1.3416	1.3427	0.0029	0.0011	1.3317	0.0139	1.3641	0.0184
50	1.5054	1.5015	1.5026	0.0028	0.0011	1.4952	0.0102	1.5417	0.0363
52	1.5829	1.5785	1.5796	0.0033	0.0011	1.5767	0.0062	1.6401	0.0572
54	1.5661	1.5561	1.5571	0.0089	0.0010	1.5598	0.0063	1.6422	0.0762
56	1.4389	1.4299	1.4310	0.0079	0.0011	1.4395	0.0005	1.5423	0.1034
58	1.2112	1.2083	1.2093	0.0019	0.0010	1.2228	0.0116	1.3463	0.1352
60	0.9164	0.9097	0.9106	0.0059	0.0009	0.9266	0.0102	1.0700	0.1536
62	0.5667	0.5597	0.5602	0.0065	0.0005	0.5745	0.0078	0.7358	0.1691
64	0.1932	0.1865	0.1868	0.0065	0.0003	0.1932	0.0000	0.3697	0.1765
MAE				0.0089	0.0013		0.0141		0.1765
RMSE				0.0054	0.0010		0.0093		0.1109

Notes: All other numerical settings are the same as those in figure 2 except that the initial stock prices are listed in the first column. MC and L denote the Monte Carlo simulation and the lattice method, respectively. 'error(MC)' (or 'error(L)') denote the absolute pricing error between each pricing formula and the Monte Carlo simulation (or the lattice method). MAE denotes the maximum absolute error and RMSE denotes the root-mean-squared error.

Table 3. Comparing the effect of changing the amount of the dividend payout on pricing barrier calls with a single discrete dividend.

c_1	Benchmark	Ours	error	ContDiv	error	Model1	error
0.3	1.5759	1.5730	0.0029	1.5705	0.0054	1.5857	0.0098
0.6	1.5438	1.5435	0.0003	1.5387	0.0051	1.5680	0.0242
0.9	1.5202	1.5129	0.0073	1.5062	0.0140	1.5486	0.0283
1.2	1.4868	1.4815	0.0053	1.4729	0.0139	1.5273	0.0405
1.5	1.4478	1.4493	0.0015	1.4390	0.0088	1.5044	0.0566
1.8	1.4147	1.4163	0.0017	1.4045	0.0102	1.4798	0.0652
2.1	1.3843	1.3828	0.0015	1.3694	0.0150	1.4538	0.0695
2.4	1.3459	1.3488	0.0030	1.3338	0.0121	1.4262	0.0804
MAE			0.0073		0.0150		0.0804
RMSE			0.0036		0.0112		0.0523

Notes: All settings are the same as the settings in Table 2 except that the initial stock price is set as 50 and that the dividend c_1 is listed in the first column. Benchmark denotes the benchmark value generated by the Monte Carlo simulation. error denotes the absolute pricing error between each pricing formula and the Monte Carlo simulation.

Table 4. Comparing the effect of changing the stock price volatility on pricing barrier calls with a single discrete dividend.

Volatility	Benchmark	Ours	error	ContDiv	error	Model1	error
0.1	2.0707	2.0756	0.0049	2.0552	0.0154	2.0612	0.0094
0.2	1.5054	1.5026	0.0028	1.4952	0.0102	1.5417	0.0363
0.3	0.7215	0.7167	0.0047	0.7172	0.0043	0.7534	0.0320
0.4	0.3625	0.3611	0.0014	0.3627	0.0002	0.3846	0.0221
0.5	0.2035	0.1998	0.0037	0.2013	0.0022	0.2144	0.0109
0.6	0.1205	0.1197	0.0007	0.1209	0.0004	0.1292	0.0087
0.7	0.0767	0.0764	0.0003	0.0773	0.0006	0.0827	0.0060
0.8	0.0526	0.0511	0.0014	0.0518	0.0007	0.0556	0.0030
0.9	0.0366	0.0356	0.0010	0.0361	0.0005	0.0388	0.0021
1.0	0.0255	0.0255	0.0000	0.0260	0.0005	0.0279	0.0024
MAE			0.0079		0.0929		0.1389
RMSE			0.0042		0.0430		0.0625

Notes: All numerical settings are the same as those settings in table 2 except that the initial stock price is 50 and that the volatility of the stock price is listed in the first column.

cured senior debt that has an asset sale clause† is illustrated in figure 3. The firm is assumed to repay the former debt by the amount $150 at year 1.5, and we vary the maturity of the latter debt to analyse the effect of repaying the former debt on the value of the latter debt. Similar arguments have been studied empirically in Linn and Stock (2005). They find strong support for the following hypothesis: When the junior debt matures prior to the senior unsecured debt, the security of the senior unsecured debt is threatened and the default spread (of the senior debt) increases. One possible explanation is that the

†In section 1, we show that allowing the sale of the issuer's asset to finance the loan repayments is much more common than putting restrictions on the asset sale as discussed in Eom *et al.* (2004) and Billett *et al.* (2007).

Table 5. Comparing the effect of changing the exdividend date on pricing barrier calls with a single discrete dividend.

t_1	Benchmark	Ours	error	ContDiv	error	Model1	error
0.1	1.5425	1.5378	0.0047	1.4938	0.0486	1.5408	0.0016
0.2	1.5347	1.5335	0.0012	1.4942	0.0405	1.5410	0.0063
0.3	1.5291	1.5262	0.0029	1.4945	0.0346	1.5412	0.0121
0.4	1.5236	1.5160	0.0076	1.4948	0.0287	1.5415	0.0179
0.5	1.5054	1.5026	0.0028	1.4952	0.0102	1.5417	0.0363
0.6	1.4903	1.4861	0.0042	1.4955	0.0053	1.5419	0.0516
0.7	1.4737	1.4658	0.0079	1.4958	0.0222	1.5421	0.0684
0.8	1.4391	1.4399	0.0007	1.4962	0.0571	1.5423	0.1032
0.9	1.4036	1.4029	0.0007	1.4965	0.0929	1.5425	0.1389
MAE			0.0050		0.0154		0.0363
RMSE			0.0027		0.0061		0.0178

Notes: All numerical settings are the same as those settings in table 2, except that the initial stock price is 50 and the exdividend date is listed in the first column.

Table 6. Comparing the effect of changing initial stock prices on pricing barrier calls with two dividends.

$S(0)$	Benchmark	Ours	error	ContDiv	error	Model1	error
46	0.9156	0.9122	0.0034	0.9003	0.0153	0.9493	0.0338
48	1.0033	1.0028	0.0005	0.9964	0.0069	1.0619	0.0586
50	1.0538	1.0493	0.0045	1.0481	0.0058	1.1322	0.0783
52	1.0484	1.0438	0.0046	1.0479	0.0005	1.1519	0.1035
54	0.9880	0.9843	0.0037	0.9934	0.0055	1.1178	0.1298
56	0.8771	0.8737	0.0035	0.8872	0.0101	1.0316	0.1545
58	0.7241	0.7192	0.0049	0.7357	0.0116	0.8990	0.1749
60	0.5364	0.5315	0.0049	0.5485	0.0122	0.7287	0.1924
62	0.3249	0.3233	0.0016	0.3371	0.0122	0.5317	0.2068
64	0.1104	0.1077	0.0032	0.1131	0.0027	0.3192	0.2088
MAE			0.0049		0.0153		0.2088
RMSE			0.0038		0.0094		0.1470

Notes: All settings are the same as the settings in table 2 except that the underlying stock is assumed to pay a 1 dollar dividend at year 0.5 and year 1, and the time to maturity is 1.5 years.

Table 7. Comparing the effect of changing the amounts of dividends on pricing barrier calls with two dividends.

$c_1 = c_2$	Benchmark	Ours	error	ContDiv	error	Model1	error
0.3	1.1305	1.1238	0.0067	1.1232	0.0073	1.1514	0.0210
0.6	1.0948	1.0933	0.0015	1.0923	0.0025	1.1462	0.0514
0.9	1.0585	1.0606	0.0021	1.0594	0.0010	1.1364	0.0780
1.2	1.0279	1.0269	0.0010	1.0248	0.0031	1.1222	0.0943
1.5	0.9864	0.9897	0.0033	0.9885	0.0021	1.1038	0.1174
1.8	0.9552	0.9535	0.0018	0.9508	0.0045	1.0812	0.1260
2.1	0.9147	0.9156	0.0009	0.9118	0.0030	1.0548	0.1401
2.4	0.8769	0.8772	0.0003	0.8715	0.0055	1.0247	0.1478
MAE			0.0068		0.0073		0.1478
RMSE			0.0029		0.0041		0.1056

Notes: All settings are the same as the settings in table 6, except that the initial stock price is 50, and the underlying stock is assumed to pay dividend c_1 at year 0.5 and dividend c_2 at year 1. The amounts of the dividend payout are listed in the first column.

repayment of the former debt may weaken the financial status of the issuing firm and increase the credit risk of the latter debt, if the former debt matures prior to the latter one. The prices of the latter debt generated by our formulae (marked in solid squares in figure 3) do catch this feature by generating a significant price drop from 2893.77 (with a time to maturity of 1.5 years) to 2885.52 (with a time to maturity of 1.52 years). On the other hand, many structural credit risk models (like Kim *et al.* (1993) and Longstaff and Schwartz (1995)) use a constant continuous payout ratio instead of a discrete payment. In this experiment, the continuous payout ratio used to approximate the discrete payout at year 1.5 is estimated by the formula proposed in Geske and Shastri (1985). Under the continuous payout setting, the pricing results for the latter debt (marked in solid triangles) simply decrease smoothly with the increment in the latter debt maturity to reflect the change of time value and can not precisely reflect the risk that the latter debt holders might suffer due to the repayment of the former debt.

6. Conclusions

Most stock dividends are paid discretely rather than continuously. However, no satisfactory analytical formulae for pricing barrier stock options with discrete dividends are announced. This paper provides accurate analytical formulae for pricing barrier stock options with discrete dividend payouts. Numerical results are given to confirm the superiority of our formulae over other analytical formulae. Our formulae can also extend the applicability of the first passage model, a popular credit risk model. The falls of the stock price due to the discrete dividend payouts are analogous to selling the firm's assets to finance the debt or dividend payments. Thus, our formulae can estimate how the firm's repayments influence its financial status and the credit qualities of other outstanding debts.

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Appendix A: Reexpress the integration of exponential functions in terms of the CDF of a multi-variate normal distribution

The pricing formulae in this paper can be expressed in terms of multiple integrations of an exponential function, the exponent term of which is a quadratic function of integrators $x_1, x_2, \cdots$. These multiple integrations can be reexpressed in terms of CDFs of multivariate normal distributions that can be easily solved by mathematical softwares such as Matlab or Mathematica. Theorem 3.1 shows how to rewrite the general multivariate integration with n integrators: $x_1, x_2, \cdots, x_n$ into the CDF of a multivariate standard normal distribution. For convenience, let $\mathbf{x} \equiv (x_1, x_2, \cdots, x_n)^T$ denote a column vector of n integrators. The proof for the Theorem is given as follows.

Proof The multivariate integral in equation (22) can be reexpressed in terms of the CDF of a standard normal distribution by reexpressing the exponent term $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} + \mathbf{C}$ in terms of the exponent term of a multivariate standard normal distribution using the completing the square technique as follows:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} + \mathbf{C} = -\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y} + \mathbf{C}', \tag{45}$$

where $\mathbf{C}'$ denotes a scaler that does not depend on $\mathbf{x}$, $\mathbf{y}$ denotes a vector and Σ denotes a covariance matrix. The above equation and the values of $\mathbf{C}'$, $\mathbf{y}$ and Σ can be derived by the following lemma:

LEMMA A.1 *Under the premises that* $\mathbf{A}$ *is a symmetric invertible* $n \times n$ *matrix, and* $\mathbf{x}$, $\mathbf{B}$ *are both* $n \times 1$ *vectors, we have*

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} = \left(\mathbf{x} + \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right)^T \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right) - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}. \tag{46}$$

Proof By expanding the right-hand side of equation (46), we have

$$\left(\mathbf{x}+\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right)^T \mathbf{A}\left(\mathbf{x}+\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right)-\frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$$
$$=\mathbf{x}^T\mathbf{A}\mathbf{x}+\frac{1}{2}\mathbf{B}^T\left(\mathbf{A}^{-1}\right)^T\mathbf{A}\mathbf{x}+\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{A}^{-1}\mathbf{B}$$
$$+\frac{1}{4}\mathbf{B}^T\left(\mathbf{A}^{-1}\right)^T\mathbf{A}\mathbf{A}^{-1}\mathbf{B}-\frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \tag{47}$$
$$=\mathbf{x}^T\mathbf{A}\mathbf{x}+\frac{1}{2}\mathbf{B}^T\mathbf{x}+\frac{1}{2}\mathbf{x}^T\mathbf{B}+\frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}-\frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \tag{48}$$
$$=\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}=\text{the left-hand side of equation (46)}, \tag{49}$$

where the equation $\left(\mathbf{A}^{-1}\right)^T=\mathbf{A}^{-1}$ due to the symmetry of $\mathbf{A}$ is substituted into equation (47). Since $\mathbf{B}^T\mathbf{x}$ is a scalar, we have $\mathbf{B}^T\mathbf{x}=\left(\mathbf{B}^T\mathbf{x}\right)^T=\mathbf{x}^T\mathbf{B}$. equation (48) is obtained by substituting the aforementioned equalities into equation (47). □

By applying Lemma A.1 to equation (45), we obtain

$$\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{x}^T\mathbf{B}+\mathbf{C}=(\mathbf{x}-\mathbf{m})^T\mathbf{A}(\mathbf{x}-\mathbf{m})+\mathbf{C}', \tag{50}$$

where $\mathbf{m}\equiv-\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}$, and $\mathbf{C}'\equiv\mathbf{C}-\frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. By equating the right-hand sides of equation (45) and equation (50), we have

$$(\mathbf{x}-\mathbf{m})^T\mathbf{A}(\mathbf{x}-\mathbf{m})+\mathbf{C}'=-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}+\mathbf{C}'. \tag{51}$$

It can be observed that $\mathbf{y}$ should have the form

$$\mathbf{y}=\mathbf{S}^{-1}(\mathbf{x}-\mathbf{m}), \tag{52}$$

where $\mathbf{S}$ denotes a diagonal matrix. To solve $\mathbf{S}$, we first subtract $\mathbf{C}'$ from both sides of equation (51) to yield

$$(\mathbf{x}-\mathbf{m})^T\mathbf{A}(\mathbf{x}-\mathbf{m})=-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y} \tag{53}$$
$$=-\frac{1}{2}\left(\mathbf{S}^{-1}(\mathbf{x}-\mathbf{m})\right)^T\Sigma^{-1}\left(\mathbf{S}^{-1}(\mathbf{x}-\mathbf{m})\right) \tag{54}$$
$$=-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{S}^{-1}\Sigma^{-1}\mathbf{S}^{-1}(\mathbf{x}-\mathbf{m})$$
$$=-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T(\mathbf{S}\Sigma\mathbf{S})^{-1}(\mathbf{x}-\mathbf{m}), \tag{55}$$

where equation (52) is substituted into the right-hand side of equation (53). $\left(\mathbf{S}^{-1}\right)^T$ is equal to $\mathbf{S}^{-1}$ due to the symmetry of $\mathbf{S}$ and this equation is substituted into equation (54). By comparing the left-hand side of equation (53) and equation (55), we have $-\frac{1}{2}(\mathbf{S}\Sigma\mathbf{S})^{-1}=\mathbf{A}$, which can be rewritten as $\mathbf{S}\Sigma\mathbf{S}=(-2\mathbf{A})^{-1}$. Recall that $\mathbf{S}$ is a diagonal matrix. All diagonal elements of Σ are 1 since Σ is a covariance matrix of multivariate standard normal random variables. Thus, we have $(\mathbf{S}\Sigma\mathbf{S})_{i,i}=\mathbf{S}^2_{i,i}$, which leads us to obtain

$$\mathbf{S}_{i,j}\equiv\begin{cases}\sqrt{((-2\mathbf{A})^{-1})_{i,i}} & \text{if } i=j\\ 0 & \text{otherwise}\end{cases},$$

and $\Sigma\equiv(-2\mathbf{SAS})^{-1}$.

Now we can express equation (22) in terms of $\mathbf{C}'$, $\mathbf{m}$, $\mathbf{S}$ and Σ defined above. By applying the change of variable defined in equation (52), equation (22) can be rewritten as

$$\int_{x_n=-\infty}^{x_n=p_n}\int_{x_{n-1}=-\infty}^{x_{n-1}=p_{n-1}}\cdots\int_{x_1=-\infty}^{x_1=p_1}e^{\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+\mathbf{C}}d\mathbf{x}$$
$$=\int_{x_n=-\infty}^{x_n=p_n}\int_{x_{n-1}=-\infty}^{x_{n-1}=p_{n-1}}\cdots\int_{x_1=-\infty}^{x_1=p_1}e^{-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}+\mathbf{C}'}\left|\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right|d\mathbf{y}. \tag{56}$$

Since the elements in vector $\mathbf{y}$ can be represented as $\left(\frac{x_1-\mathbf{m}_1}{\mathbf{S}_{1,1}},\frac{x_2-\mathbf{m}_2}{\mathbf{S}_{2,2}},\cdots,\frac{x_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}\right)^T$, the Jacobian determinant can be straightforwardly computed to obtain $\left|\frac{\partial\mathbf{x}}{\partial\mathbf{y}}\right|=\prod_{i=1}^n\mathbf{S}_{i,i}=|\mathbf{S}|$. Thus, equation (56) can be further rewritten as the following closed form formula:

$$e^{\mathbf{C}'}|\mathbf{S}|\int_{-\infty}^{\frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}}\int_{-\infty}^{\frac{p_{n-1}-\mathbf{m}_{n-1}}{\mathbf{S}_{n-1,n-1}}}\cdots\int_{-\infty}^{\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}}e^{-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}}d\mathbf{y}$$
$$=e^{\mathbf{C}'}|\mathbf{S}|\sqrt{|\Sigma|}\sqrt{(2\pi)^n}\int_{-\infty}^{\frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}}\int_{-\infty}^{\frac{p_{n-1}-\mathbf{m}_{n-1}}{\mathbf{S}_{n-1,n-1}}}\cdots$$
$$\int_{-\infty}^{\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}}\frac{1}{\sqrt{|\Sigma|(2\pi)^n}}e^{-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}}d\mathbf{y} \tag{57}$$
$$=e^{\mathbf{C}'}\sqrt{\frac{\pi^n}{|-\mathbf{A}|}}F_{Y_1,Y_2,\cdots,Y_n}\left(\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}},\frac{p_2-\mathbf{m}_2}{\mathbf{S}_{2,2}},\cdots,\frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}},\Sigma\right),$$

where $|\mathbf{S}|\sqrt{|\Sigma|}=\sqrt{|\mathbf{S}\Sigma\mathbf{S}|}=\sqrt{|-2\mathbf{A}|^{-1}}$, and $|-2\mathbf{A}|=2^n|-\mathbf{A}|$ are substituted into equation (57). □

Appendix B: Proof of Corollary 3.1

Corollary 4.1 can be derived from Theorem 3.1 by setting $n=2$ as follows:

First, to make $a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6$ equal $\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{x}^T\mathbf{B}+\mathbf{C}$, we set

$$\mathbf{A}\equiv\begin{pmatrix}a_1 & \frac{a_2}{2}\\ \frac{a_2}{2} & a_3\end{pmatrix},\mathbf{B}\equiv\begin{pmatrix}a_4\\ a_5\end{pmatrix},\mathbf{C}\equiv a_6.$$

By substituting the above equations into Theorem 3.1, we have

$$\mathbf{m}=-\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}=-\frac{1}{4a_1a_3-a_2^2}\begin{pmatrix}2a_3a_4-a_2a_5\\ 2a_1a_5-a_2a_4\end{pmatrix},$$
$$\mathbf{C}'=\mathbf{C}-\frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}=a_6+\frac{a_2a_4a_5-a_3a_4^2-a_1a_5^2}{4a_1a_3-a_2^2},$$
$$\mathbf{S}_{1,1}=\sqrt{((-2\mathbf{A})^{-1})_{1,1}}=\sqrt{\frac{-2a_3}{4a_1a_3-a_2^2}},$$
$$\mathbf{S}_{2,2}=\sqrt{((-2\mathbf{A})^{-1})_{2,2}}=\sqrt{\frac{-2a_1}{4a_1a_3-a_2^2}},$$
$$\Sigma=(-2\mathbf{SAS})^{-1}=\begin{pmatrix}1 & \frac{a_2}{2\sqrt{a_1a_3}}\\ \frac{a_2}{2\sqrt{a_1a_3}} & 1\end{pmatrix},$$
$$|-\mathbf{A}|=\frac{4a_1a_3-a_2^2}{4}.$$

By substituting the above into equation (23), we obtain

$$\int_{-\infty}^{q}\int_{-\infty}^{p}e^{a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6}dxdy$$
$$=e^{\mathbf{C}'}\sqrt{\frac{\pi^2}{|-\mathbf{A}|}}F_{Y_1,Y_2}\left(\frac{p-\mathbf{m}(1)}{\mathbf{S}_{1,1}},\frac{q-\mathbf{m}(2)}{\mathbf{S}_{2,2}},\Sigma\right)$$
$$=\frac{2\pi}{\sqrt{\Delta}}\exp\left(a_6+\frac{a_2a_4a_5-a_3a_4^2-a_1a_5^2}{\Delta}\right)$$
$$F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta}p+\frac{2a_3a_4-a_2a_5}{\sqrt{\Delta}}}{\sqrt{-2a_3}},\frac{\sqrt{\Delta}q+\frac{2a_1a_5-a_2a_4}{\sqrt{\Delta}}}{\sqrt{-2a_1}},\Sigma\right),$$

where $\Delta=4a_1a_3-a_2^2$.

i	$a_7(i)$	$a_8(i)$	$a_9(i)$	$a_{10}(i)$
1	α	$\alpha-\alpha k_2$	$\alpha-\alpha k_1$	$-\frac{T\alpha^2}{2}$
2	α	$\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
3	$\frac{2b''}{T-t_2}+\alpha$	$\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\alpha-\alpha k_1$	$-\frac{4b''^2+T^2\alpha^2-T\alpha^2 t_2}{2T-2t_2}$
4	$\frac{2b''}{T-t_2}+\alpha$	$\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$\frac{4(t_2-T)b^2+t_1\left(-4b''^2-T^2\alpha^2+T\alpha^2 t_2\right)}{2t_1(T-t_2)}$
5	α	$\frac{2b'}{t_2-t_1}+\alpha-\alpha k_2$	$\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$\frac{4b'^2-T\alpha^2 t_1+T\alpha^2 t_2}{2t_1-2t_2}$
6	α	$\frac{2b'}{t_2-t_1}+\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$\frac{4t_2 b^2-T\alpha^2 t_1^2+t_1\left(-4b^2+4b'^2+T\alpha^2 t_2\right)}{2t_1(t_1-t_2)}$
7	$\frac{2b''}{T-t_2}+\alpha$	$\frac{2b'}{t_2-t_1}+\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$\frac{2b'^2}{t_1-t_2}-\frac{T\alpha^2}{2}-\frac{2b''^2}{T-t_2}$
8	$\frac{2b''}{T-t_2}+\alpha$	$\frac{2b'}{t_2-t_1}+\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b''^2}{T-t_2}+\frac{2b'^2}{t_1-t_2}$
9	$\alpha+\sigma$	$\alpha-\alpha k_2$	$\alpha-\alpha k_1$	$-\frac{T\alpha^2}{2}$
10	$\alpha+\sigma$	$\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
11	$\frac{2b''}{T-t_2}+\alpha+\sigma$	$\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\alpha-\alpha k_1$	$-\frac{4b''^2+T^2\alpha^2-T\alpha^2 t_2}{2T-2t_2}$
12	$\frac{2b''}{T-t_2}+\alpha+\sigma$	$\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$\frac{4(t_2-T)b^2+t_1\left(-4b''^2-T^2\alpha^2+T\alpha^2 t_2\right)}{2t_1(T-t_2)}$
13	$\alpha+\sigma$	$\frac{2b'}{t_2-t_1}+\alpha-\alpha k_2$	$\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$\frac{4b'^2-T\alpha^2 t_1+T\alpha^2 t_2}{2t_1-2t_2}$
14	$\alpha+\sigma$	$\frac{2b'}{t_2-t_1}+\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$\frac{4t_2 b^2-T\alpha^2 t_1^2+t_1\left(-4b^2+4b'^2+T\alpha^2 t_2\right)}{2t_1(t_1-t_2)}$
15	$\frac{2b''}{T-t_2}+\alpha+\sigma$	$\frac{2b'}{t_2-t_1}+\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$\frac{2b'^2}{t_1-t_2}-\frac{T\alpha^2}{2}-\frac{2b''^2}{T-t_2}$
16	$\frac{2b''}{T-t_2}+\alpha+\sigma$	$\frac{2b'}{t_2-t_1}+\alpha+k_2\left(\frac{2b''}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b'}{t_2-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b''S^2}{T-t_2}+\frac{2b'^2}{t_1-t_2}$

Appendix C: Coefficients of the exponential terms of $J(i)$

The coefficients $a_1(i), a_2(i), a_3(i), a_4(i), a_5(i)$ and $a_6(i)$ are defined by the following formulae.

$$a_1(i) = -\frac{1}{2(T-t_2)},$$

$$a_2(i) = -\left(\frac{k_2^2}{2(T-t_2)} + \frac{1}{2(t_2-t_1)}\right),$$

$$a_3(i) = -\left(\frac{k_1^2}{2(t_2-t_1)} + \frac{1}{2t_1}\right),$$

$$a_6(i) = 0,$$

$$a_4(i) = \begin{cases} \frac{k_2}{T-t_2} & \text{if } i = 1, 2, 5, 6, 9, 10, 13, 14 \\ -\frac{k_2}{T-t_2} & \text{otherwise} \end{cases},$$

$$a_5(i) = \begin{cases} \frac{k_2}{t_2-t_1} & \text{if } i = 1, 2, 3, 4, 9, 10, 11, 12 \\ -\frac{k_2}{t_2-t_1} & \text{otherwise} \end{cases},$$

The remaining parameters are given by the following table: