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Technical Note

Variability of stream flow discharge in response to self-similar random fields of temporal fluctuations in lateral inflow rate

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SUMMARY

This article presents the use of stochastic methodology for quantitative analysis of variability in stream flow discharge in response to fluctuations in lateral inflow rate, where the lateral inflow rate is considered to be the difference between rainfall and infiltration rates. In this work, we focus on the case where the temporal correlation structure of the fluctuations in the lateral inflow rate can be characterized by the statistics of random fractals. A closed-form expression quantifying the stream flow variability is therefore developed to investigate the influence of the fractal dimension of lateral inflow process and the size of time domain. It is found that the stream flow discharge variability increases with the time domain size, while the fractal dimension of lateral inflow process plays a role in the smoothness of fluctuations in stream flow discharge around the mean.

1. Introduction

Understanding and quantifying the conversion of rainfall–runoff process into stream flow discharge is one of the major tasks in water resources engineering, especially for a long-term management of available water resources. Temporal fluctuations in rainfall are generally recognized as being affected by a wide range of natural physical processes, the details of which cannot be anticipated precisely. Hence, there is a great deal of uncertainty associated with the quantification of surface lateral inflow to the stream along its reach as produced by the rainfall–runoff process. This prompted us to investigate how the temporal fluctuations in the lateral inflow rate influence the variability in the stream flow discharge.

Note that lateral inflow refers to any water added to the stream due to effluent seepage from ground water, overland flow, interflow or via small springs and seeps (e.g., Singh, 1995). This research is primarily concerned with the case that the source of lateral inflow is dominated by the rainfall. Therefore, the lateral inflow rate in this work is defined as the difference between rainfall and infiltration rates.

Rainfall events show significant variability on temporal scales. However, some observations indicate that the temporal distributions of fluctuations in rainfall fields do exhibit the properties of long-range correlation and scale invariance. These properties greatly simplify the statistical characterization of rainfall fields at time scales by using the concept of fractal objects (e.g., De Michele and Bernardara, 2005; Hubert et al., 1993; Menabde et al., 1997; Olsson et al., 1993; Schmitt et al., 1998; Venugopal and Foufoula-Georgiou, 1996; Veneziano et al., 1996). In other words, the temporal distribution of fluctuations in rainfall fields can be modeled according to self-similar random processes and their temporal correlation satisfies a power law (e.g., Hewett, 1986; Voss, 1985).

The surface lateral inflow to the stream is a direct consequence of the rainfall–runoff process. There is a need to address the uncertainty (variability) associated with the prediction of available stream water resources, which is the task undertaken herein. In the following analysis, the temporal fluctuations in the lateral inflow rate is considered to be self-similar random fields such that the temporal variability in the lateral inflow rate can be dealt with using a fractal description, where the lateral inflow rate represents the surface runoff mainly from rainfall.

In the following we present a stochastic analysis of one-dimensional transient stream flow subject to uniformly distributed lateral inflow along the side of the stream. The application of the perturbation-based nonstationary spectral techniques will lead to a closed-form solution for quantifying the variability in stream flow discharge. This solution provides a basis for assessing the impact of input parameters on the stream flow discharge variability.

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2. Problem formulation

Unsteady flow in a stream has traditionally been formulated based on the Saint–Venant system equations (e.g., Chow et al., 1988). In practical applications, the local and convective accelerations in the system equations are often neglected to simplify the analysis. The exclusion of accelerations thus leads the system equations to a single equation, known as the diffusion wave equation (e.g., Fan and Li, 2006; Gottardi and Venutelli, 2008; Moussa, 1996; Sivapalan et al., 1997; Sulis et al., 2010)

$$\frac{\partial Q}{\partial t} = D_h\left(\frac{\partial^2 Q}{\partial X^2} - \frac{\partial q_R}{\partial X}\right) - U\left(\frac{\partial Q}{\partial X} - q_R\right) \quad (1)$$

where Q is the stream flow discharge, D_h and U are the hydraulic diffusivity and wave celerity, respectively, and q_R is the lateral inflow rate (per unit stream length), which is considered to be uniformly distributed along the stream.

Eq. (1) is highly nonlinear due to the dependence of the diffusivity and celerity coefficients on the stream flow discharge Q, the dependent variable of (1). However, it may be linearized in a perturbation form based on the steady uniform reference values of the flow discharge and flow cross-sectional area written as (e.g., Lal, 2001; Moramarco et al., 1999; Yen and Tsai, 2001)

$$\frac{\partial Q'}{\partial t} = D_{h0}\frac{\partial^2 Q'}{\partial X^2} - U_0\left(\frac{\partial Q'}{\partial X} - q\right) \quad (2)$$

where $Q' = Q - Q_0$, $q = q_R - q_0$, and Q_0 and q_0 represent the steady uniform initial values. For a wide rectangular channel, for example, the diffusivity coefficient may be expressed in the form (e.g., Yen and Tsai, 2001)

$$D_{h0} = \frac{1}{2}\left[1 - \left(1 - 2\frac{U_0}{V_0} + \frac{U_0^2}{V_0^2}\right)F_0^2\right]\frac{V_0 Y_0}{S_0} \quad (3)$$

where V_0 and Y_0 are the uniform flow velocity and depth, respectively, S_0 is the channel bed slope, $F_0 = V_0/(gY_h)^{0.5}$, and Y_h is the hydraulic depth. The celerity coefficient is given by (e.g., Yen and Tsai, 2001)

$$U_0 = \frac{3}{2}V_0 \quad (4a)$$

using Chezy's formula and

$$U_0 = \frac{5}{3}V_0 \quad (4b)$$

using Manning's formula. Note that the term $\partial q/\partial X$ has been omitted from (2) due to the assumption of uniformly distributed recharge.

In the analysis presented below, the lateral inflow representing the source of stream flow is assumed to be a temporally correlated random field (a stochastic process based on the time series). It results in temporally correlated random fluctuations in stream flow discharge. That is, the stream flow discharge, the output (dependent variable) of the stream flow equation, is also treated as a random field. As such, the perturbation Eq. (2) provides a framework for quantifying the stream flow variability in terms of the temporal variability of the lateral inflow.

3. Stream flow variability analysis

We consider a weakly stationary random lateral inflow field in time so that the fluctuations in lateral inflow may be presented in form of Fourier-Stieltjes integral as:

$$q(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_q(\omega) \quad (5)$$

where ω is the frequency and dZ_q is the complex random amplitude of the fluctuations. The perturbed quantity of stream flow discharge in (2) may be expressed by the Fourier-Stieltjes representation of a nonstationary process (e.g., Li and McLaughlin, 1991) as:

$$Q'(X,t) = \int_{-\infty}^{\infty} \Phi_{Qq}(X,t,\omega) dZ_q(\omega) \quad (6)$$

where $\Phi_{Qq}(X, t, \omega)$ is the transfer function.

Using (5) and (6), it follows from (2) that

$$\frac{\partial \Phi_{Qq}}{\partial t} = D_{h0}\frac{\partial^2 \Phi_{Qq}}{\partial X^2} - U_0\frac{\partial \Phi_{Qq}}{\partial X} + U_0 e^{i\omega t} \quad (7)$$

To gain a clear insight into the influence of the source variability on the stream flow, we focus only on the case where the boundary and initial conditions are deterministic (i.e., the case of lateral-inflow-dominated stream). Thus, the stochastic perturbation boundary and initial conditions associated with (7) take the forms:

$$\Phi_{Qq}(0,t) = 0 \quad (8a)$$

$$\Phi_{Qq}(L,t) = 0 \quad (8b)$$

$$\Phi_{Qq}(X,0) = 0 \quad (8c)$$

where L is the length of the stream. The system of Eqs. (7) and (8) admits the following solution:

$$\Phi_{Qq} = 2\pi U_0 \exp\left(\frac{\xi\mu}{2}\right)\sum_{n=1}^{\infty}\frac{n - n\exp\left(-\frac{\mu}{2}\right)\cos(n\pi)}{n^2\pi^2 + \frac{\mu^2}{4}}\sin(n\pi\xi) \times \frac{\exp(i\omega t) - \exp(-(\rho_1 n^2 + \rho_2)t)}{\rho_1 n^2 + \rho_2 + i\omega} \quad (9)$$

where $\mu = U_0 L/D_0$, $\xi = X/L$, $\rho_1 = D_0\pi^2/L^2$, $\rho_2 = U_0^2/(4D_0)$. A useful approximation may be made for the case of $\rho_1 t \gg 1$. For this case, the infinite series in (9) converges rapidly (e.g., Haberman, 1998) and (9) becomes

$$\Phi_{Qq} = 2\pi U_0\frac{1 + e^{-\mu/2}}{\pi^2 + \mu^2/4}e^{\mu\xi/2}\sin(\pi\xi)\frac{e^{i\varpi t} - e^{-\alpha t}}{\alpha + i\omega} \quad (10)$$

where $\alpha = \rho_1 + \rho_2 = (D_{h0}\pi^2 + \mu^2/4)/L^2$. In conjunction with (9), (6) is written as:

$$Q'(X,t) = 2\pi U_0\frac{1 + e^{-\mu/2}}{\pi^2 + \mu^2/4}e^{\mu\xi/2}\sin(\pi\xi)\int_{-\infty}^{\infty}\frac{e^{i\varpi t} - e^{-\alpha t}}{\alpha + i\omega}dZ_q(\omega) \quad (11)$$

Requiring from the representation theorem for Q', one obtains

$$\sigma_Q^2 = E[Q'Q'^*] = 8\pi^2 U_0^2\left(\frac{1 + e^{-\mu/2}}{\pi^2 + \mu^2/4}\right)^2 e^{\mu\xi}\sin^2(\pi\xi) \int_0^{\infty}\frac{1 + e^{-2\alpha t} - 2e^{-\alpha t}\cos(\omega t)}{\omega^2 + \alpha^2}S_{qq}(\omega)d\omega \quad (12)$$

where σ_Q^2 is the variance of the stream flow discharge, $E(-)$ stands for the ensemble average, the asterisk denotes the operation of complex conjugation, and $Sqq(\omega)$ is the spectrum of the lateral inflow perturbation.

As mentioned earlier, the temporal correlation structure of the rate of lateral inflow is assumed described by the statistics of random fractals. It has been demonstrated by Voss (1985) and Hewett (1986) that the spectral density of the fractal objects follows the power-law behavior. Hence, the spectrum of the lateral inflow $Sqq(\omega)$ in (12) has the form of

$$S_{qq}(\omega) = S_0/\omega^{\beta} \quad (13)$$

where S_0 is the spectral density at $\omega = 1$, β is the spectral exponent which can be related to the fractal dimension D. For one-dimensional fractal objects, $\beta = 5 - 2D$ and $1 < D < 2$. The reader is referred to Mandelbrot (1983) for a detailed discussion on the concept of fractals.

Note that the frequency is proportional to the inverse of the characteristic scale of time domain. In the limit when $\omega \to 0$, $Sqq(\omega)$ approaches ∞, implying that the time scale under consideration is unbounded. In reality, the extent of time series or the profile is finite, and therefore there exists a cut-off frequency ω_0 such that below ω_0 the spectral content becomes negligible. With such a cut-off limit, the modified power spectrum of the lateral inflow accounting for the finite-scale effect reads as:

$$S_{qq}(\omega) = \begin{cases} S_0/\omega^{\beta} & \text{for} \quad \omega > \omega_0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $\omega_0 = 2\pi/T_0$ and T_0 is the size of time domain. The idea of using a power spectrum with a low cut-off representing the spatial structure of the log hydraulic conductivity field at the scale of interest has been applied to solve various groundwater problems (e.g., Chang and Kemblowski, 1994; Di Federico and Neuman, 1997; Zhan and Wheatcraft, 1996). It has been mentioned in Molz et al. (1997) that a stationary function with a correlation scale larger than the domain size gives a good approximation of a nonstationary function in a finite domain. This implies that with the aid of cut-off limit, the existing stochastic models developed for the stationary process can be applied to analyze nonstationary fractal processes.

In addition, the observed variance of lateral inflow associated with (13) within the range of the time domain can be found as:

$$\sigma_q^2 = 2\int_{\omega_0}^{\infty} \frac{S_0}{\omega^{5-2D}}\, d\omega = \frac{S_0}{(2-D)\omega_0^{4-2D}} = \frac{S_0}{(2-D)}\left(\frac{T_0}{2\pi}\right)^{4-2D} \tag{15}$$

where σ_q^2 is the observed variance of the lateral inflow. Eq. (15) indicates the observed variance is scale-dependent (a function of time scale). This equation also shows that the variance of the observed lateral inflow increases with the time domain size. This agrees with our geologic intuition.

Inserting (14) into (12) and integrating over the frequency domain yields

$$\begin{aligned}\sigma_Q^2 \cong {} & 8\pi^2\sigma_q^2 U_0^2 S_0 \left(\frac{1+e^{-\mu/2}}{\pi^2+\mu^2/4}\right)^2 e^{2\mu\xi}\sin^2(\pi\xi)\tau^{6-2D} \\ & \left\{(1+e^{-2\tau})\frac{{}_2F_1(1,3-D;\,4-D;\,-\rho^{-2})}{(6-2D)(\rho\tau)^{6-2D}} - 2\frac{e^{-\tau}}{\tau^2}\left[\Gamma(2D-4)\sin\left[\frac{\pi}{2}(5-2D)\right]\right.\right. \\ & -\Gamma(2D-4)\sin\left[\frac{\pi}{2}(2D-3)\right]{}_1F_2(1;\,2.5-D,3-D;\,0.25\tau^2) \\ & \left.\left.+\frac{{}_1F_2(D-2;\,0.5,D-1;\,-0.25\rho^2\tau^2)}{(4-2D)(\rho\tau)^{4-2D}} - \frac{\pi}{2}\sec\left[\frac{\pi}{2}(2D-3)\right]\frac{\cosh(\tau)}{\tau^{4-2D}}\right]\right\}\end{aligned} \tag{16}$$

where $\tau = \alpha t$, ${}_2F_1$ is Gauss's hypergeometric function (e.g., Gradshteyn and Ryzhik, 1980), ${}_1F_2$ is the generalized hypergeometric function (e.g., Gradshteyn and Ryzhik, 1980), $\Gamma(-)$ is the Gamma function, and $\rho = 2\pi/(\alpha\, T_0)$.

Again, note that σ_q^2 increases with the size of the time domain. Thus, to investigate the influence of the fractal dimension of the lateral inflow on the variance of the stream flow discharge, we assume that the time-domain variance of the lateral inflow is constant. This assumption allows us to replace S_0 in (16) with

$$S_0 = (2-D)\left(\frac{2\pi}{T_0}\right)^{4-2D}\sigma_q^2 \tag{17}$$

Fig. 1 demonstrates how the variance in the stream flow discharge is influenced by the fractal dimension of the lateral inflow. When $D > 1.5$, the recharge process generates anti-persistence, having negatively correlated increments. Anti-persistent stochastic processes tend to show an increase in value following previous decreases, or show a decrease following previous increases. On the other hand, when $D < 1.5$, the lateral inflow processes display persistence, having positively correlated increments. That is, positive increments tend to be followed by other positive increments, while negative increments tend to be followed by other negative increments. As the value of fractal dimension is increased, the temporal correlation is decreased and the profile of stream flow discharge is smoothed (i.e., less fluctuations in the stream flow discharge around the mean). When the fractal dimension is close to 2, the increments of the stream flow discharge exhibit an anti-correlated structure and the variance of the stream flow discharge approaches zero.

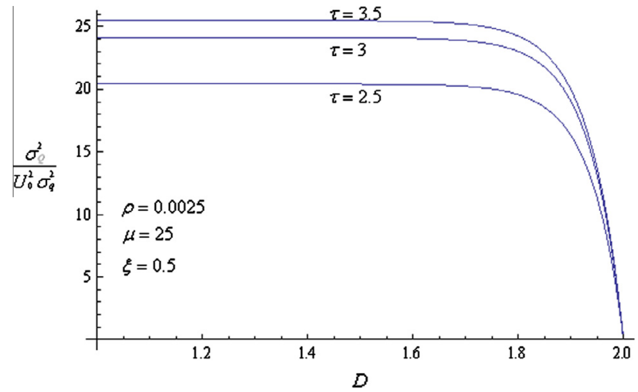

Fig. 1. Dimensionless variance of stream flow discharge as a function of fractal dimension of temporally fractal lateral inflow process.

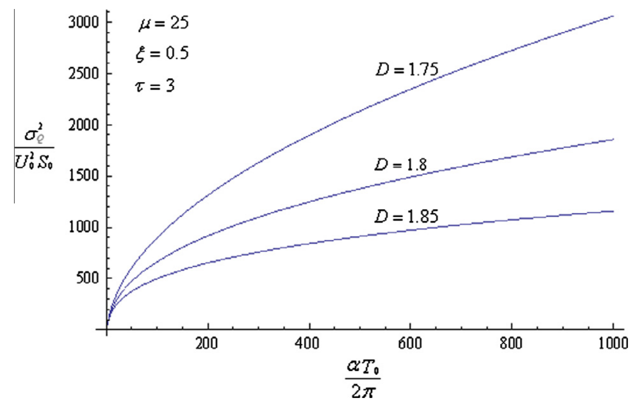

Fig. 2. Dimensionless variance of stream flow discharge as a function of time domain size of fractal lateral inflow process.

The plot of dependence of the variance of the stream flow discharge upon the scale of the time domain is illustrated in Fig. 2 based on (16). The increase in the variance of the stream flow discharge with the time domain scale is due to the fact that an increased size of the time domain introduces a wide range of variability in the lateral inflow processes which in turn increases the variability of the stream flow discharge. The rate of the increase in the variance of the stream flow discharge is more significant for a smaller D (more persistence of the process).

4. Conclusions

Stochastic analysis of the stream flow problem demonstrates that the fractal dimension of temporal lateral inflow process and the size of the time domain have an important effect on the variability in the stream flow discharge, as illustrated by the following results:

1. The variance of the stream flow discharge is influenced strongly by the fractal dimension of temporal lateral inflow process. A higher fractal dimension of temporal lateral inflow processes results in a less persistence of stream flow processes, which produces smaller fluctuations in the stream flow discharge about its mean.
2. The variability in the stream flow discharge is scale-dependent and increases with the size of time domain.

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