



Nonemptiness problems of Wang tiles with three colors



Hung-Hsun Chen, Wen-Guei Hu^{*,1}, De-Jan Lai, Song-Sun Lin²

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan

ARTICLE INFO

Article history:

Received 28 January 2014

Accepted 6 June 2014

Available online 23 June 2014

Communicated by D.-Z. Du

Keywords:

Nonemptiness

Wang tiles

Decidability

Edge coloring

Periodic patterns

Transition matrix

ABSTRACT

This investigation studies nonemptiness problems of plane edge coloring with three colors. In the edge coloring (or Wang tiles) of a plane, unit squares with colored edges that have one of p colors are arranged side by side such that the touching edges of the adjacent tiles have the same colors. Given a basic set B of Wang tiles, the nonemptiness problem is to determine whether or not $\Sigma(B) \neq \emptyset$, where $\Sigma(B)$ is the set of all global patterns on \mathbb{Z}^2 that can be constructed from the Wang tiles in B . Wang's conjecture is that for any B of Wang tiles, $\Sigma(B) \neq \emptyset$ if and only if $\mathcal{P}(B) \neq \emptyset$, where $\mathcal{P}(B)$ is the set of all periodic patterns on \mathbb{Z}^2 that can be generated by the tiles in B .

When $p \geq 5$, Wang's conjecture is known to be wrong. When $p = 2$, the conjecture is true. This study proves that when $p = 3$, the conjecture is also true. If $\mathcal{P}(B) \neq \emptyset$, then B has a subset B' of minimal cycle generators such that $\mathcal{P}(B') \neq \emptyset$ and $\mathcal{P}(B'') = \emptyset$ for $B'' \subsetneq B'$. This study demonstrates that the set $\mathcal{C}(3)$ of all minimal cycle generators contains 787,605 members that can be classified into 2,906 equivalence classes. $\mathcal{N}(3)$ is the set of all maximal non-cycle generators: if $B \in \mathcal{N}(3)$, then $\mathcal{P}(B) = \emptyset$ and $\mathcal{P}(\tilde{B}) \neq \emptyset$ for $\tilde{B} \supsetneq B$. Wang's conjecture is shown to be true by proving that $B \in \mathcal{N}(3)$ implies $\Sigma(B) = \emptyset$.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

The coloring of \mathbb{Z}^2 using unit squares has been studied for many years [1–14]. In 1961, Wang [14] started to study the square tiling of a plane to prove theorems by pattern recognition. Unit squares with colored edges are arranged side by side so that the touching edges of the adjacent tiles have the same color; the tiles cannot be rotated or reflected. Today, such tiles are called Wang tiles or Wang dominos [4,7].

The 2×2 unit square is denoted by $\mathbb{Z}_{2 \times 2}$. The set of p colors is $\{0, 1, \dots, p-1\}$. Therefore, the total set of Wang tiles is denoted by $\Sigma_{2 \times 2}(p) \equiv \{0, 1, \dots, p-1\}^{\mathbb{Z}_{2 \times 2}}$. A set B of Wang tiles is called a basic set (of Wang tiles). Let $\Sigma(B)$ and $\mathcal{P}(B)$ be the sets of all global patterns and periodic patterns on \mathbb{Z}^2 , respectively, that can be constructed from the Wang tiles in B .

* Corresponding author.

E-mail addresses: hhchen.am00g@nctu.edu.tw (H.-H. Chen), wghu@mail.nctu.edu.tw (W.-G. Hu), werre216asfe87dirk@gmail.com (D.-J. Lai), sslin@math.nctu.edu.tw (S.-S. Lin).

¹ The second author would like to thank the National Science Council, R.O.C. and the S.T. Yau Center for partially supporting this research.

² The fourth author would like to thank the National Science Council, R.O.C. (Contract No. NSC 98-2115-M-009) and the S.T. Yau Center for partially supporting this research.

Given a finite set B of Wang tiles, the nonemptiness problem concerning tiling with of Wang tiles is to determine whether or not

$$\Sigma(B) \neq \emptyset. \tag{1.1}$$

Clearly, $\mathcal{P}(B) \subseteq \Sigma(B)$, meaning that if $\mathcal{P}(B) \neq \emptyset$, then $\Sigma(B) \neq \emptyset$. In [14], Wang conjectured that any set of tiles that can tile a plane can tile the plane periodically:

$$\text{if } \Sigma(B) \neq \emptyset, \text{ then } \mathcal{P}(B) \neq \emptyset. \tag{1.2}$$

If (1.2) holds, then the nonemptiness problem that is specified by (1.1) is reduced to the much easier problem of determining whether or not

$$\mathcal{P}(B) \neq \emptyset. \tag{1.3}$$

However, in 1966, Berger [4] proved that Wang’s conjecture was wrong and the nonemptiness problem concerning Wang’s tiling is undecidable. He presented a set B of 20426 Wang tiles that could only tile the plane aperiodically:

$$\Sigma(B) \neq \emptyset \quad \text{and} \quad \mathcal{P}(B) = \emptyset. \tag{1.4}$$

Later, he reduced the number of tiles to 104. Thereafter, smaller basic sets were found by Knuth, Läuchli, Robinson, Penrose, Ammann, Culik and Kari. Currently, the smallest number of tiles that can tile a plane aperiodically is 13, with five colors: (1.4) holds and then (1.2) fails for $p = 5$ [5].

Recently, Hu and Lin [8] showed that Wang’s conjecture (1.2) holds if $p = 2$: any set of Wang tiles with two colors that can tile a plane can tile the plane periodically.

In that study, they showed that statement (1.2) can be approached by studying how periodic patterns can be generated from a given basic set. First, B is called a cycle generator if $\mathcal{P}(B) \neq \emptyset$; otherwise, B is called a non-cycle generator. Moreover, $B \subset \Sigma_{2 \times 2}(p)$ is called a minimal cycle generator (MCG) if B is a cycle generator and $\mathcal{P}(B') = \emptyset$ whenever $B' \subsetneq B$; $B \subset \Sigma_{2 \times 2}(p)$ is called a maximal non-cycle generator (MNCG) if B is a non-cycle generator and $\mathcal{P}(B'') \neq \emptyset$ for any $B'' \supsetneq B$.

Given $p \geq 2$, denote the set of all minimal cycle generators by $\mathcal{C}(p)$ and the set of maximal non-cycle generators by $\mathcal{N}(p)$. Clearly,

$$\mathcal{C}(p) \cap \mathcal{N}(p) = \emptyset. \tag{1.5}$$

Statement (1.2) follows for $p \geq 2$ if

$$\Sigma(B) = \emptyset \quad \text{for any } B \in \mathcal{N}(p) \tag{1.6}$$

can be shown. Indeed, in [8], it is shown that $\mathcal{C}(2)$ has 38 members; $\mathcal{N}(2)$ has nine members, and (1.6) holds for $p = 2$. This paper studies the case of $p = 3$. Now, $\mathcal{C}(3)$ and $\mathcal{N}(3)$ have close to a million members and cannot be handled manually. After the symmetry group D_4 of $\mathbb{Z}_{2 \times 2}$ and the permutation group S_p of colors of horizontal and vertical edges, respectively, are applied, $\mathcal{C}(3)$ still contains thousands of equivalent classes. Hence, computer programs are utilized to determine $\mathcal{C}(3)$ and $\mathcal{N}(3)$ and finally (1.6) is shown to hold for $p = 3$. Therefore, the conjecture (1.2) is true for $p = 3$. Moreover, if $\mathcal{P}(B) \neq \emptyset$, $\mathcal{P}(B)$ contains a periodic pattern with horizontal period $m \leq 35$.

For $p = 4$, $\mathcal{C}(4)$ is enormous. Therefore, the arguments and the computer program need to be much efficient to handle this situation.

Corner coloring with $p = 3$ can be treated similarly. The result will be presented elsewhere.

The rest of paper is arranged as follows. Section 2 introduces the ordering matrix of all 81 local patterns and classifies them into three groups. The recurrence formula for patterns on $\Sigma_{m \times n}$ are derived. It is important in proving (1.6) – that the maximum non-cycle generators cannot generate global patterns. Section 3 will introduce the procedure for determining the sets $\mathcal{C}(3)$ and $\mathcal{N}(3)$. The main result is proven using a computer.

2. Preliminary

This section introduces all necessary elements for proving (1.6). First, let $\Sigma_{m \times n}(B)$ be the set of all local patterns on $\mathbb{Z}_{m \times n}$ that can be generated by B . Clearly,

$$\text{if } \Sigma_{m \times n}(B) = \emptyset \quad \text{for some } m, n \geq 2, \text{ then } \Sigma(B) = \emptyset. \tag{2.1}$$

2.1. Symmetries

The symmetry of the unit square $\mathbb{Z}_{2 \times 2}$ is introduced. The symmetry group of the rectangle $\mathbb{Z}_{2 \times 2}$ is D_4 , which is the dihedral group of order eight. The group D_4 is generated by rotation ρ through $\frac{\pi}{2}$ and reflection m about the y -axis. Denote the elements of D_4 by $D_4 = \{I, \rho, \rho^2, \rho^3, m, m\rho, m\rho^2, m\rho^3\}$.

Therefore, given a basic set $B \subset \Sigma_{2 \times 2}(p)$ and any element $\tau \in D_4$, another basic set $(B)_\tau$ can be obtained by transforming the local patterns in B by τ .

Additionally, consider the permutation group S_p on $\{0, 1, \dots, p-1\}$. If $\eta \in S_p$ and $\eta(0) = i_0, \eta(1) = i_1, \dots, \eta(p-1) = i_{p-1}$, we write

$$\eta = \begin{pmatrix} 0 & 1 & \dots & p-1 \\ i_0 & i_1 & \dots & i_{p-1} \end{pmatrix}.$$

For $\eta \in S_p$ and $B \in \Sigma_{2 \times 2}(p)$, another basic set $(B)_\eta$ can be obtained.

In edge coloring, the permutations of colors in the horizontal and vertical directions are mutually independent. Denote the permutations of colors in the horizontal and vertical edges by $\eta_h \in S_p$ and $\eta_v \in S_p$, respectively. Now, for any $B \subset \Sigma_{2 \times 2}(p)$, define the equivalence class $[B]$ of B by

$$[B] = \{B' \subset \Sigma_{2 \times 2}(p) : B' = ((B)_\tau)_{\eta_h \eta_v}, \tau \in D_4 \text{ and } \eta_h, \eta_v \in S_p\}. \tag{2.2}$$

In [8], whether or not $\Sigma(B) = \emptyset$ and $\mathcal{P}(B) = \emptyset$ is shown to be independent of the choice of elements in $[B]$. Indeed, for any $B' \in [B]$,

$$\Sigma(B') \neq \emptyset \quad (\text{or } \mathcal{P}(B') \neq \emptyset) \quad \text{if and only if } \Sigma(B) \neq \emptyset \quad (\text{or } \mathcal{P}(B) \neq \emptyset).$$

Moreover, for $B' \in [B]$, B' is an MCG (MNCG) if and only if B is an MCG (MNCG). Therefore, groups D_4 and S_3 can be used efficiently to reduce the number of cases $B \subset \Sigma_{2 \times 2}(3)$ that must be considered, greatly reducing the computation time.

2.2. Ordering matrix

Now, the case $p = 3$ is considered. The vertical ordering matrix $\mathbf{Y}_{2 \times 2} = [y_{i,j}]_{9 \times 9}$ of all local patterns in $\Sigma_{2 \times 2}(p)$ is given by

$$\mathbf{Y}_{2 \times 2} = \begin{matrix} & \begin{matrix} \begin{matrix} \uparrow 0 \\ \downarrow 0 \end{matrix} & \begin{matrix} \uparrow 1 \\ \downarrow 0 \end{matrix} & \begin{matrix} \uparrow 2 \\ \downarrow 0 \end{matrix} & \begin{matrix} \uparrow 0 \\ \downarrow 1 \end{matrix} & \begin{matrix} \uparrow 1 \\ \downarrow 1 \end{matrix} & \begin{matrix} \uparrow 2 \\ \downarrow 1 \end{matrix} & \begin{matrix} \uparrow 0 \\ \downarrow 2 \end{matrix} & \begin{matrix} \uparrow 1 \\ \downarrow 2 \end{matrix} & \begin{matrix} \uparrow 2 \\ \downarrow 2 \end{matrix} \end{matrix} \\ \begin{matrix} \uparrow 0 \\ \downarrow 0 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 0 \\ \downarrow 1 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 0 \\ \downarrow 2 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 1 \\ \downarrow 0 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 1 \\ \downarrow 1 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 1 \\ \downarrow 2 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 2 \\ \downarrow 0 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 2 \\ \downarrow 1 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \\ \begin{matrix} \uparrow 2 \\ \downarrow 2 \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} & \begin{matrix} \square \\ \square \end{matrix} \end{matrix} \tag{2.3}$$

$$= \begin{bmatrix} \mathbf{Y}_{2;1} & \mathbf{Y}_{2;2} & \mathbf{Y}_{2;3} \\ \mathbf{Y}_{2;4} & \mathbf{Y}_{2;5} & \mathbf{Y}_{2;6} \\ \mathbf{Y}_{2;7} & \mathbf{Y}_{2;8} & \mathbf{Y}_{2;9} \end{bmatrix}_{3 \times 3} \tag{2.4}$$

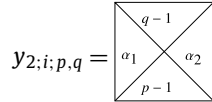
The recurrence relation of \mathbf{Y}_{m+1} is easily obtained as follows. Denote by

$$\mathbf{Y}_2 = [y_{2;p,q}]_{3 \times 3} = \sum_{i=1}^9 \mathbf{Y}_{2;i} \tag{2.5}$$

and

$$\mathbf{Y}_{2;i} = [y_{2;i;p,q}]_{3 \times 3}, \tag{2.6}$$

where



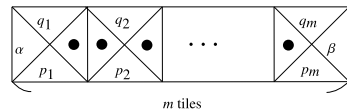
and $i = 1 + \alpha_1 \cdot 3^1 + \alpha_2 \cdot 3^0$, $\alpha_i \in \{0, 1, 2\}$. Here, the sum of (2.5) is the formal sum that means $y_{2;p,q} = \bigcup_{i=1}^9 y_{2;i;p,q}$. For $m \geq 2$, denote by

$$\mathbf{Y}_{m+1} = \sum_{i=1}^9 \mathbf{Y}_{m+1;i} \tag{2.7}$$

and

$$\mathbf{Y}_{m+1;i} = [y_{m+1;i;p,q}]_{3^m \times 3^m}, \tag{2.8}$$

where $y_{m+1;i;p,q}$ is the set of all patterns of the form



where α, β, p_k and $q_k \in \{0, 1, 2\}$, $1 \leq k \leq m$, such that

$$\begin{cases} i = 1 + \alpha \cdot 3^1 + \beta \cdot 3^0 \\ p = 1 + \sum_{k=1}^m p_k 3^{m-k} \\ q = 1 + \sum_{k=1}^m q_k 3^{m-k} \end{cases} \tag{2.9}$$

and $\bullet \in \{0, 1, 2\}$. Therefore, when $i = 1, 4, 7$,

$$\mathbf{Y}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^3 y_{2;j+i-1;1,1} \mathbf{Y}_{m;3j-2} & \sum_{j=1}^3 y_{2;j+i-1;1,2} \mathbf{Y}_{m;3j-2} & \sum_{j=1}^3 y_{2;j+i-1;1,3} \mathbf{Y}_{m;3j-2} \\ \sum_{j=1}^3 y_{2;j+i-1;2,1} \mathbf{Y}_{m;3j-2} & \sum_{j=1}^3 y_{2;j+i-1;2,2} \mathbf{Y}_{m;3j-2} & \sum_{j=1}^3 y_{2;j+i-1;2,3} \mathbf{Y}_{m;3j-2} \\ \sum_{j=1}^3 y_{2;j+i-1;3,1} \mathbf{Y}_{m;3j-2} & \sum_{j=1}^3 y_{2;j+i-1;3,2} \mathbf{Y}_{m;3j-2} & \sum_{j=1}^3 y_{2;j+i-1;3,3} \mathbf{Y}_{m;3j-2} \end{bmatrix}_{3^m \times 3^m};$$

when $i = 2, 5, 8$,

$$\mathbf{Y}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^3 y_{2;j+i-2;1,1} \mathbf{Y}_{m;3j-1} & \sum_{j=1}^3 y_{2;j+i-2;1,2} \mathbf{Y}_{m;3j-1} & \sum_{j=1}^3 y_{2;j+i-2;1,3} \mathbf{Y}_{m;3j-1} \\ \sum_{j=1}^3 y_{2;j+i-2;2,1} \mathbf{Y}_{m;3j-1} & \sum_{j=1}^3 y_{2;j+i-2;2,2} \mathbf{Y}_{m;3j-1} & \sum_{j=1}^3 y_{2;j+i-2;2,3} \mathbf{Y}_{m;3j-1} \\ \sum_{j=1}^3 y_{2;j+i-2;3,1} \mathbf{Y}_{m;3j-1} & \sum_{j=1}^3 y_{2;j+i-2;3,2} \mathbf{Y}_{m;3j-1} & \sum_{j=1}^3 y_{2;j+i-2;3,3} \mathbf{Y}_{m;3j-1} \end{bmatrix}_{3^m \times 3^m};$$

when $i = 3, 6, 9$,

$$\mathbf{Y}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^3 y_{2;j+i-3;1,1} \mathbf{Y}_{m;3j} & \sum_{j=1}^3 y_{2;j+i-3;1,2} \mathbf{Y}_{m;3j} & \sum_{j=1}^3 y_{2;j+i-3;1,3} \mathbf{Y}_{m;3j} \\ \sum_{j=1}^3 y_{2;j+i-3;2,1} \mathbf{Y}_{m;3j} & \sum_{j=1}^3 y_{2;j+i-3;2,2} \mathbf{Y}_{m;3j} & \sum_{j=1}^3 y_{2;j+i-3;2,3} \mathbf{Y}_{m;3j} \\ \sum_{j=1}^3 y_{2;j+i-3;3,1} \mathbf{Y}_{m;3j} & \sum_{j=1}^3 y_{2;j+i-3;3,2} \mathbf{Y}_{m;3j} & \sum_{j=1}^3 y_{2;j+i-3;3,3} \mathbf{Y}_{m;3j} \end{bmatrix}_{3^m \times 3^m}.$$

Given $B \subset \Sigma_{2 \times 2}(3)$, the associated vertical transition matrix $\mathbf{V}_{2 \times 2}(B)$ is defined by $\mathbf{V}_{2 \times 2}(B) = [v_{i,j}]$, where $v_{i,j} = 1$ if and only if $y_{i,j} \in B$.

The recurrence formula for a higher-order vertical transition matrix can be obtained as follows. Denote by

$$\mathbf{V}_2 = \sum_{i=1}^9 \mathbf{V}_{2;i},$$

with

$$\mathbf{V}_{2;i} = [v_{2;i;p,q}]_{3 \times 3}.$$

For $m \geq 2$, denote by

$$\mathbf{V}_{m+1} = \sum_{i=1}^9 \mathbf{V}_{m+1;i}.$$

Now, for $i = 1, 4, 7$,

$$\mathbf{V}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^3 v_{2;j+i-1;1,1} \mathbf{V}_{m;3j-2} & \sum_{j=1}^3 v_{2;j+i-1;1,2} \mathbf{V}_{m;3j-2} & \sum_{j=1}^3 v_{2;j+i-1;1,3} \mathbf{V}_{m;3j-2} \\ \sum_{j=1}^3 v_{2;j+i-1;2,1} \mathbf{V}_{m;3j-2} & \sum_{j=1}^3 v_{2;j+i-1;2,2} \mathbf{V}_{m;3j-2} & \sum_{j=1}^3 v_{2;j+i-1;2,3} \mathbf{V}_{m;3j-2} \\ \sum_{j=1}^3 v_{2;j+i-1;3,1} \mathbf{V}_{m;3j-2} & \sum_{j=1}^3 v_{2;j+i-1;3,2} \mathbf{V}_{m;3j-2} & \sum_{j=1}^3 v_{2;j+i-1;3,3} \mathbf{V}_{m;3j-2} \end{bmatrix}_{3^m \times 3^m};$$

for $i = 2, 5, 8$,

$$\mathbf{V}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^3 v_{2;j+i-2;1,1} \mathbf{V}_{m;3j-1} & \sum_{j=1}^3 v_{2;j+i-2;1,2} \mathbf{V}_{m;3j-1} & \sum_{j=1}^3 v_{2;j+i-2;1,3} \mathbf{V}_{m;3j-1} \\ \sum_{j=1}^3 v_{2;j+i-2;2,1} \mathbf{V}_{m;3j-1} & \sum_{j=1}^3 v_{2;j+i-2;2,2} \mathbf{V}_{m;3j-1} & \sum_{j=1}^3 v_{2;j+i-2;2,3} \mathbf{V}_{m;3j-1} \\ \sum_{j=1}^3 v_{2;j+i-2;3,1} \mathbf{V}_{m;3j-1} & \sum_{j=1}^3 v_{2;j+i-2;3,2} \mathbf{V}_{m;3j-1} & \sum_{j=1}^3 v_{2;j+i-2;3,3} \mathbf{V}_{m;3j-1} \end{bmatrix}_{3^m \times 3^m};$$

for $i = 3, 6, 9$,

$$\mathbf{V}_{m+1;i} = \begin{bmatrix} \sum_{j=1}^3 v_{2;j+i-3;1,1} \mathbf{V}_{m;3j} & \sum_{j=1}^3 v_{2;j+i-3;1,2} \mathbf{V}_{m;3j} & \sum_{j=1}^3 v_{2;j+i-3;1,3} \mathbf{V}_{m;3j} \\ \sum_{j=1}^3 v_{2;j+i-3;2,1} \mathbf{V}_{m;3j} & \sum_{j=1}^3 v_{2;j+i-3;2,2} \mathbf{V}_{m;3j} & \sum_{j=1}^3 v_{2;j+i-3;2,3} \mathbf{V}_{m;3j} \\ \sum_{j=1}^3 v_{2;j+i-3;3,1} \mathbf{V}_{m;3j} & \sum_{j=1}^3 v_{2;j+i-3;3,2} \mathbf{V}_{m;3j} & \sum_{j=1}^3 v_{2;j+i-3;3,3} \mathbf{V}_{m;3j} \end{bmatrix}_{3^m \times 3^m}.$$

Therefore, as in [1,8], it can be proven that

$$|\Sigma_{(m+1) \times n}(B)| = |\mathbf{V}_{m+1}^{n-1}|. \tag{2.10}$$

2.3. Periodic patterns

This subsection studies periodic patterns in detail.

For $m, n \geq 1$, a global pattern $u = (\alpha_{i,j})_{i,j \in \mathbb{Z}}$ on \mathbb{Z}^2 is called (m, n) -periodic if every $i, j \in \mathbb{Z}$,

$$\alpha_{i+mp, j+nq} = \alpha_{i,j} \tag{2.11}$$

for all $p, q \in \mathbb{Z}$.

Let $\mathcal{P}_B(m, n)$ be the set of all (m, n) -periodic patterns and B -admissible patterns. Let $\Gamma_B(m, n) = |\mathcal{P}_B(m, n)|$ be the number of all (m, n) -periodic and B -admissible patterns.

As in [3], $\mathcal{P}_B(m, n)$ can be expressed by trace operators as follows.

From (2.9), the periodic patterns in \mathbf{Y}_{m+1} are given by $\mathbf{Y}_{m+1;i}$, $i = 1, 5, 9$. Define

$$\mathbf{T}_m \equiv \sum_{i=1,5,9} \mathbf{V}_{m+1;i}. \tag{2.12}$$

\mathbf{T}_m is called the trace operator of order m , as in [3]. Therefore, the following result is obtained.

Proposition 2.1. Given $B \subseteq \Sigma_{2 \times 2}(3)$, for $m, n \geq 1$,

$$\Gamma_B(m, n) = \text{tr}(\mathbf{T}_m^n). \tag{2.13}$$

Proof. The proof is similar to that for corner coloring in [2]. The details of the proof are omitted. \square

Notably, from Proposition 2.1, $\mathcal{P}(B) \neq \emptyset$ if and only if $\Gamma_B(m, n) > 0$ for some $m, n \geq 1$.

Recall some notation and terms from matrix theory. A matrix \mathbf{A} is called nilpotent if $\mathbf{A}^k = 0$ for some $k \geq 1$. The property “nilpotent” can be used to specify whether B is a cycle generator or non-cycle generator.

Proposition 2.2. Given a basic set $B \subset \Sigma_{2 \times 2}(3)$,

- (i) B is a cycle generator if and only if \mathbf{T}_m is not nilpotent for some $m \geq 1$.
- (ii) $\Sigma(B) = \emptyset$ if and only if \mathbf{V}_m is nilpotent for some $m \geq 1$.

Proof. From (2.13) of Proposition 2.1, B is easily seen to be a cycle generator if and only if $\text{tr}(\mathbf{T}_m^n) > 0$ for some $m, n \geq 1$. Therefore, (i) follows immediately.

Similarly, from (2.10), (ii) follows.

The following proposition provides an efficient method to check the nilpotent for non-negative matrix and can be easily proven. The proof is omitted. \square

Proposition 2.3. Suppose A is a non-negative matrix. Then, \mathbf{A} is nilpotent if and only if \mathbf{A} can be reduced to a zero matrix by repeating the following process: if the i -th row (column) of \mathbf{A} is a zero row, then the i -th column (row) of \mathbf{A} is replaced with a zero column.

3. Main result

3.1. Periodic pairs

This section firstly classifies all local patterns in $\{0, 1, 2\}^{\mathbb{Z}_{2 \times 2}}$ into three groups.

First, the local pattern $\alpha = \begin{matrix} & \alpha_2 & \\ \alpha_1 & \alpha_3 & \\ & \alpha_0 & \end{matrix} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is assigned a number by

$$\varphi((\alpha_0, \alpha_1, \alpha_2, \alpha_3)) = 1 + \sum_{j=0}^3 \alpha_j 3^j. \tag{3.1}$$

Then, all 81 local patterns are listed in the following three groups G_0 , G_1 and G_2 .

$$G_0 = \{1, 11, 21, 31, 41, 51, 61, 71, 81\}$$

$$G_1 = \{2, 3, 4, 7, 10, 12, 14, 17, 19, 20, 24, 27, 28, 32, 33, 34, 38, 40, \\ 42, 44, 48, 49, 50, 54, 55, 58, 62, 63, 65, 68, 70, 72, 75, 78, 79, 80\}$$

$$G_2 = \{5, 6, 8, 9, 13, 15, 16, 18, 22, 23, 25, 26, 29, 30, 35, 36, 37, 39, \\ 43, 45, 46, 47, 52, 53, 56, 57, 59, 60, 64, 66, 67, 69, 73, 74, 76, 77\}$$

Clearly, every tile in G_0 can generate a (1, 1)-periodic pattern. Furthermore, elements in G_i , $i = 1, 2$, form periodic pairs as in Fig. 3.1: two tiles that are connected by a line can generate a (2, 2)-periodic pattern. More precisely, the diagrams in Fig. 3.1 can be interpreted as follows.

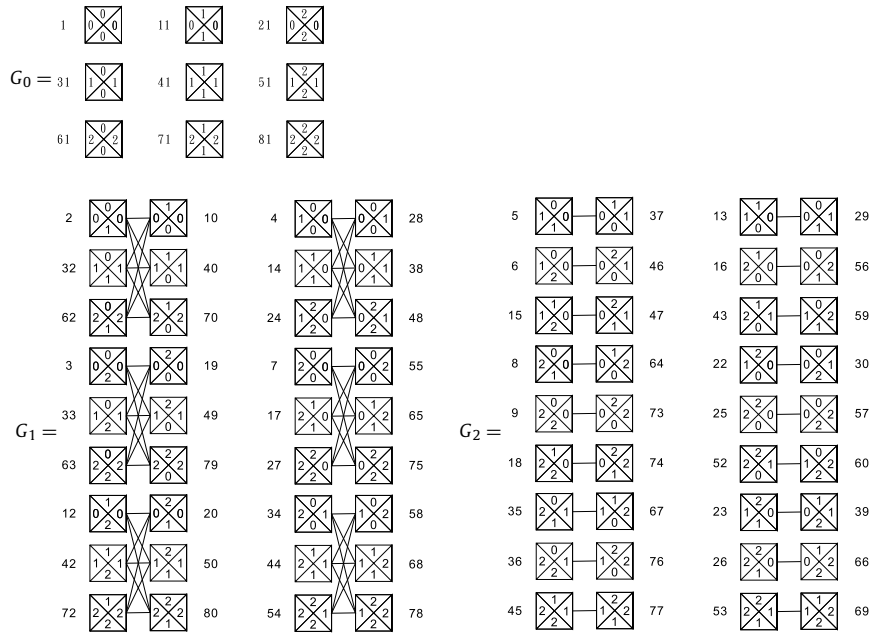


Fig. 3.1. Connections with one or two tiles in each group.

Proposition 3.1.

- (i) Each tile e in G_0 can generate a periodic pattern by repetition of itself: $\{e\}$ is then a minimum cycle generator and is $(1, 1)$ -periodic.
- (ii) For each tile e in G_1 , there exist exactly three tiles $e_1, e_2, e_3 \in G_1$ such that $\{e, e_i\}$ can form a periodic cycle, which is a $(1, 2)$ -periodic or $(2, 1)$ -periodic pattern, and $\{e, e_i\}$ is a minimum cycle generator for $1 \leq i \leq 3$.
- (iii) For each tile e in G_2 , there exists exactly one tile $e' \in G_2$ such that $\{e, e'\}$ is a minimum cycle generator and is $(2, 2)$ -periodic.

These minimum cycle generators are the simplest.

Remark 3.2. From Fig. 3.1, the set $G_1 \cup G_2$ with 72 tiles can be decomposed into 36 disjoint sets that each consists of two tiles such that each set is a minimal cycle generator. Therefore, the number of the tiles of maximal non-cycle generators in $\mathcal{N}(3)$ is equal to or less than 36. Indeed, elements in each pairs can be carefully picked up, and the maximum non-cycle generator with 36 elements thus obtained; see Table A.3. Moreover, from Proposition 2.2, for each element B in the eight equivalence classes of Table A.3, $\Sigma(B) = \emptyset$ can be verified.

3.2. Algorithms

Before the developed algorithms are presented, some notation must be introduced.

Definition 3.3.

- (i) For a set A , let $\mathbb{P}(A)$ be the power set of A .
- (ii) For $\mathbb{B} \subseteq \mathbb{P}(\Sigma_{2 \times 2}(3))$, let

$$[\mathbb{B}] = \{[B] \mid B \in \mathbb{B}\}.$$

- (iii) For $[B] \in [\mathbb{P}(\Sigma_{2 \times 2}(3))]$, let $\langle B \rangle$ be a fixed chosen element of $[B]$.
- (iv) Let $\mathcal{N}^*(3)$ be the set of all maximal non-cycle generators that cannot generate a global pattern. Indeed,

$$\mathcal{N}^*(3) = \{N \in \mathcal{N}(3) \mid \Sigma(N) = \emptyset\}.$$

- (v) For $B \subseteq \Sigma_{2 \times 2}(3)$, let $\mathcal{C}(B)$ be the set of all minimal cycle generators that are contained in B .
- (vi) For $B \subseteq \Sigma_{2 \times 2}(3)$, let $\mathcal{N}^*(B)$ be the set of all maximal non-cycle generators that cannot generate a global pattern and are contained in B .

Now, the main idea of the algorithms is introduced, as follows.

Let

$$\begin{aligned}
 N &= 2^{81}, \\
 \mathbb{P}(\Sigma_{2 \times 2}(3)) &= \{B_j \mid 0 \leq j \leq N - 1\}, \text{ where } B_0 = \emptyset, \\
 \text{Initial state for } \mathcal{C}(3) &: \mathcal{C}_I(0) = \emptyset, \\
 \text{Initial state for } \mathcal{N}^*(3) &: \mathcal{N}_I^*(0) = \emptyset, \\
 \text{Initial state for the set of aperiodic sets} &: \mathcal{U}_I(0) = \emptyset.
 \end{aligned}$$

Algorithm 1 Main Algorithm

```

Main Algorithm
j = 0
repeat
  j = j + 1
  if P(B_j) ≠ ∅,
    C_I(j) = C_I(j - 1) ∪ {B_j}
  else
    if Σ(B_j) = ∅,
      N_I^*(j) = N_I^*(j - 1) ∪ {B_j}
    else
      U_I(j) = U_I(j - 1) ∪ {B_j}
  end
end
until j = N - 1
    
```

After [Algorithm 1](#) has been executed, if $\mathcal{U}_I(N - 1) = \emptyset$, then Wang’s conjecture holds for $p = 3$. The methods to achieve the goal are introduced below:

- (I) reduce the number of cases that must be considered in the computation,
- (II) construct efficient initial states for $\mathcal{C}(3)$ and $\mathcal{N}^*(3)$,
- (III) construct an efficient process for determining whether or not $\mathcal{P}(B_j) = \emptyset$ and $\Sigma(B_j) = \emptyset$.

With respect to (I), the decomposition $\Sigma_{2 \times 2}(3) = G_0 \cup G_1 \cup G_2$ is used to reduce the number of cases that must be considered in the computation. Clearly, if $B \subseteq \Sigma_{2 \times 2}(3)$ contains a tile $e \in G_0$, then B is a cycle generator. Now, in studying Wang’s conjecture, only cases $B \subseteq G_1 \cup G_2$ have to be considered.

Given $B = A_1 \cup A_2$ with $A_1 \in \mathbb{P}(G_1)$ and $A_2 \in \mathbb{P}(G_2)$, if A_1 or A_2 is a cycle generator, then B immediately satisfies [\(1.2\)](#). By [\(2.2\)](#), the cases $B \subseteq G_1 \cup G_2$ that have to be considered can be further reduced to the cases in \mathcal{I} or \mathcal{I}' :

$$\begin{aligned}
 \mathcal{I} &\equiv \{A_1 \cup \langle A_2 \rangle \mid A_1 \in \mathcal{D}_1 \text{ and } [A_2] \in [\mathcal{D}_2]\} \\
 &= \{B_j \mid 1 \leq j \leq |\mathcal{I}|\}
 \end{aligned} \tag{3.2}$$

and

$$\mathcal{I}' \equiv \{\langle A_1 \rangle \cup A_2 \mid [A_1] \in [\mathcal{D}_1] \text{ and } A_2 \in \mathcal{D}_2\}, \tag{3.3}$$

where

$$\mathcal{D}_j = \{A \in \mathbb{P}(G_j) \mid A \not\supseteq C \text{ for any } C \in \mathcal{C}(G_j)\} \tag{3.4}$$

for $j = 1, 2$. For brevity, the proof is omitted. From [Table A.1](#), $N' \equiv |\mathcal{I}| \approx 1.35075 \times 10^{12}$ and $|\mathcal{I}'| \approx 1.38458 \times 10^{12}$. Therefore, \mathcal{I} is the better choice for reducing $B \subseteq G_1 \cup G_2$. Notably, $N' \ll |\mathbb{P}(G_1 \cup G_2)| = 2^{72} \approx 4.72237 \times 10^{21}$; the reduction is considerable. [Table A.1](#) presents the details.

With respect to (II), let $\mathcal{U}_I(0) = \emptyset$. The initial data for $\mathcal{C}(3)$ are given by the set $\mathcal{C}_I(0)$ of all minimal cycle generators that are the subsets of G_0 , G_1 , or G_2 . Indeed,

$$\mathcal{C}_I(0) = \bigcup_{j=0}^2 \mathcal{C}(G_j). \tag{3.5}$$

On the other hand, the initial data for $\mathcal{N}^*(3)$ are given by

$$\mathcal{N}_I^*(0) = \{N \in \mathcal{N}^*(G_1 \cup G_2) : |N| = 36\}. \tag{3.6}$$

From Remark 3.2, $\mathcal{N}_I^*(0)$ equals the set of all maximal non-cycle generators in $G_1 \cup G_2$ with 36 tiles. $\mathcal{C}_I(0)$ and $\mathcal{N}_I^*(0)$ can be easily found using a computer program. See [Tables A.2a and A.3](#).

With respect to (III), the flowchart, which is based on (I) and (II), is as follows.

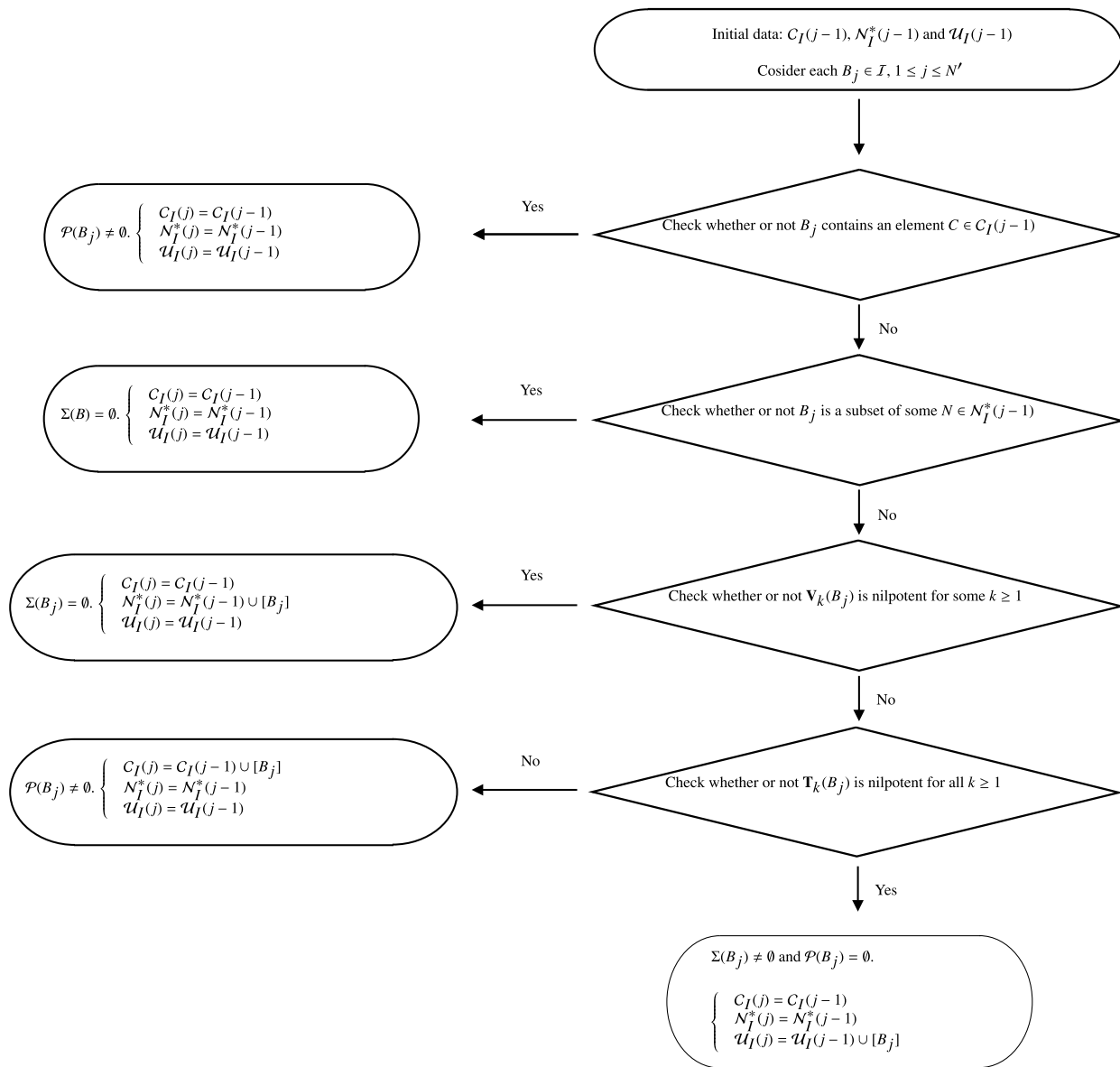


Fig. 3.2. Flowchart of the computer program.

Remark 3.4. Suppose that the computation based on the flowchart has been completed. Let $\mathcal{C} = \mathcal{C}_I(N' - 1)$, $\mathcal{N}^* = \mathcal{N}_I^*(N' - 1)$ and $\mathcal{U} = \mathcal{U}_I(N' - 1)$.

- (i) If the set $\mathcal{U} = \emptyset$, then Wang’s conjecture holds for $p = 3$; otherwise, every element in \mathcal{U} is an aperiodic set.
- (ii) It is easy to see that an element in \mathcal{C} may be not a minimal cycle generator. However, $\mathcal{C}(3)$ can be obtained from $\tilde{\mathcal{C}} \equiv \bigcup_{C \in \mathcal{C}} [C]$ by the following process. If $C_1, C_2 \in \tilde{\mathcal{C}}$ with $C_1 \subsetneq C_2$, then C_2 must be removed from $\tilde{\mathcal{C}}$. Indeed,

$$\mathcal{C}(3) = \{C \in \tilde{\mathcal{C}} \mid C \text{ does not contain any } C' \in \tilde{\mathcal{C}} \text{ except itself}\}.$$

- (iii) In a manner similar to that for (ii), let $\tilde{\mathcal{N}} \equiv \bigcup_{N \in \mathcal{N}^*} [N]$. Now,

$$\mathcal{N}^*(3) = \{N \in \tilde{\mathcal{N}} \mid N \text{ is not a proper subset of } N' \text{ for all } N' \in \tilde{\mathcal{N}} \text{ except itself}\}.$$

Moreover, if $\mathcal{U} = \emptyset$, $\mathcal{N}(3) = \mathcal{N}^*(3)$.

3.3. Main result

The computer program of Fig. 3.2 is written, and the computation is completed in finite time. Indeed, the cases that consume the most time are those in which the numbers of tiles in $B \subseteq G_1 \cup G_2$ are 18 and 19. These cases can be computed completely within a week. The main result is as follows.

Theorem 3.5. *The set $\mathcal{U} = \emptyset$, and Wang's conjecture holds for $p = 3$.*

Remark 3.6.

- (i) $\mathcal{C}(3)$ and $\mathcal{N}(3)$ can be obtained and their numbers are listed in Table A.4a.
- (ii) The computational results reveal that the maximum orders m of $\mathbf{T}_m(B)$ and $\mathbf{V}_m(B)$ in applying Proposition 2.2 are $m = 35$ and $m = 13$, respectively. More precisely, for $B' = \{2, 5, 13, 36, 53, 60, 62, 64, 77\}$, $\mathbf{T}_{35}(B')$ is not nilpotent but $\mathbf{T}_k(B')$ is nilpotent for $1 \leq k \leq 34$. On the other hand, for $B'' = \{2, 4, 5, 6, 9, 13, 14, 16, 18, 27, 32, 39, 60, 67, 78, 79\}$, $\mathbf{V}_{13}(B'')$ is nilpotent and $\mathbf{V}_k(B'')$ are not nilpotent for any $1 \leq k \leq 12$. The analytic proof that these numbers are maximal is not available. A prior estimate of the upper bound of m does not exist.

For completeness, Tables A.4a and A.4b give the numbers of minimal cycle generators and maximal non-cycle generators.

Acknowledgement

The authors want to thank Prof. Wen-Wei Lin for suggesting the use of the concept of nilpotence to identify cycle and non-cycle generators.

Appendix A

Table A.1 presents the numbers of \mathcal{D}_j and $[\mathcal{D}_j]$, $j = 1, 2$. Denote by $g_j(k)$ the number of \mathcal{D}_j with k tiles and $\bar{g}_j(k)$ the number of $[\mathcal{D}_j]$ with k tiles for $1 \leq j \leq 2$ and $1 \leq k \leq 18$.

Tables A.2a and A.2b present the equivalence classes of the minimal cycle generators in G_1 and G_2 .

Table A.3 shows the equivalence classes of maximal non-cycle generators with 36 tiles.

Tables A.4a and A.4b show the numbers of $\mathcal{C}(3)$ and $\mathcal{N}(3)$. Firstly, denote by

Table A.1
Numbers of \mathcal{D}_j and $[\mathcal{D}_j]$ with k tiles for $1 \leq j \leq 2$ and $1 \leq k \leq 18$.

k	$g_1(k)$	$\bar{g}_1(k)$	$g_2(k)$	$\bar{g}_2(k)$
1	36	1	36	1
2	576	8	612	8
3	5304	31	6504	34
4	31 032	146	47 988	219
5	122 184	475	256 320	971
6	342 204	1290	998 136	3692
7	711 288	2581	2 812 752	10 043
8	1 129 896	4092	5 771 988	20 554
9	1 397 892	5005	8 886 612	31 338
10	1 361 448	4903	10 558 368	37 319
11	1 047 816	3763	9 807 336	34 539
12	635 580	2321	7 125 612	25 253
13	300 888	1106	4 007 484	14 203
14	109 080	423	1 708 632	6162
15	29 304	118	533 664	1945
16	5 508	28	115 164	453
17	648	4	15 336	65
18	36	1	948	8

Table A.2a
Equivalence classes of the minimal cycle generators in G_1 .

k	$[C] \in [C(G_1)]$ with k tiles
2	$\{\{2, 10\}\}$
	$\{\{2, 40\}\}$
3	$\{\{2, 12, 19\}\}$
	$\{\{2, 12, 49\}\}$
	$\{\{2, 42, 79\}\}$

Table A.2b
Equivalence classes of the minimal cycle generators in G_2 .

k	$[C] \in [C(G_2)]$ with k tiles
2	{[5, 37]}
3	{[5, 45, 73]}
4	{[5, 13, 30, 46]}
	{[5, 16, 30, 73]}
	{[5, 16, 39, 74]}
5	{[5, 9, 46, 64]}
	{[5, 13, 35, 64]}
	{[5, 13, 30, 43, 74]}
	{[5, 15, 30, 52, 73]}
	{[5, 9, 39, 46, 76]}
6	{[5, 6, 35, 52, 64]}
	{[5, 15, 35, 46, 66]}
	{[5, 13, 35, 43, 57, 73]}
	{[5, 6, 43, 53, 66, 73]}
	{[5, 13, 36, 53, 66, 73]}
	{[5, 9, 39, 52, 67, 74]}
	{[5, 9, 39, 43, 74, 76]}
	{[5, 13, 30, 43, 47, 64]}
	{[5, 6, 13, 43, 47, 66]}
	{[5, 9, 13, 25, 39, 74]}
	{[5, 9, 13, 30, 47, 64]}
	{[5, 6, 16, 47, 57, 64]}
	{[5, 9, 16, 53, 66, 74]}
	{[5, 9, 13, 39, 53, 74]}
	{[5, 6, 13, 30, 52, 73]}
{[5, 9, 15, 43, 60, 74]}	
7	{[5, 13, 35, 45, 66, 74]}
	{[5, 9, 13, 30, 52, 64, 74]}
	{[5, 9, 13, 47, 52, 57, 64]}
	{[5, 6, 16, 36, 53, 66, 73]}
	{[5, 9, 16, 39, 47, 69, 76]}
	{[5, 6, 13, 35, 43, 66, 73]}
	{[5, 6, 16, 35, 39, 47, 76]}
	{[5, 9, 13, 30, 52, 56, 64]}
	{[5, 6, 16, 35, 36, 57, 73]}
	{[5, 9, 15, 22, 46, 56, 66]}
{[5, 15, 25, 35, 45, 64, 74]}	
8	{[5, 9, 13, 26, 35, 43, 57, 74]}
	{[5, 9, 13, 35, 39, 52, 74, 76]}
	{[5, 9, 13, 45, 47, 52, 56, 64]}

Table A.3
Equivalence classes of maximal non-cycle generators with 36 tiles.

1.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 42, 43, 44, 45, 52, 53, 54, 62, 63, 72]}
2.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 42, 43, 44, 45, 53, 54, 60, 62, 63, 72]}
3.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 42, 43, 44, 45, 54, 60, 62, 63, 69, 72]}
4.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 32, 33, 34, 35, 36, 42, 44, 45, 54, 59, 60, 62, 63, 69, 72]}
5.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 26, 27, 32, 33, 35, 36, 42, 45, 57, 58, 59, 60, 62, 63, 68, 69, 72, 78]}
6.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 26, 27, 32, 33, 35, 36, 42, 57, 58, 59, 60, 62, 63, 68, 69, 72, 77, 78]}
7.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 26, 27, 32, 33, 36, 42, 57, 58, 59, 60, 62, 63, 67, 68, 69, 72, 77, 78]}
8.	{[2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 27, 32, 33, 36, 42, 45, 57, 58, 59, 60, 62, 63, 66, 67, 68, 69, 72, 78]}

Table A.4a
Numbers of $C_3(k)$ and $C_{3,e}(k)$ with k tiles.

k	$ C_3(k) $	$ C_{3,e}(k) $
10	2880	10
9	84600	301
8	305388	1094
7	264384	952
6	105012	406
5	21060	102
4	3672	29
3	528	8
2	72	3
1	9	1

Table A.4b
Numbers of $\mathcal{N}_3(k)$ and $\mathcal{N}_{3,e}(k)$ with k tiles.

k	$ \mathcal{N}_3(k) $	$ \mathcal{N}_{3,e}(k) $
36	1296	8
34	720	3
32	1152	4
31	3168	11
30	576	2
29	288	1
28	3168	12
27	3456	12
26	6048	21
25	5760	20
24	5184	18
23	6624	23
22	8640	30
21	12672	44
20	20160	70
19	35280	123
18	50256	175
17	90000	313
16	93024	324
15	108720	379
14	120384	422
13	148536	522
12	163512	576
11	157536	556
10	186480	657
9	133200	483
8	42624	156
7	2160	9

$$\begin{cases} \mathcal{C}_3(k) = \{B \in \mathcal{C}(3) : |B| = k\}, \\ \mathcal{N}_3(k) = \{N \in \mathcal{N}(3) : |N| = k\}, \\ \mathcal{C}_{3,e}(k) = \{[B] \in [\mathcal{C}(3)] : |B'| = k \text{ for all } B' \in [B]\}, \\ \mathcal{N}_{3,e}(k) = \{[N] \in [\mathcal{N}(3)] : |N'| = k \text{ for all } N' \in [N]\}. \end{cases}$$

Clearly, from Proposition 3.1, $\mathcal{C}(3) = \bigcup_{k=1}^{36} \mathcal{C}_3(k)$ and $\mathcal{N}(3) = \bigcup_{k=1}^{36} \mathcal{N}_3(k)$. Only the cases for $\mathcal{C}_3(k) \neq \emptyset$ and $\mathcal{N}_3(k) \neq \emptyset$ are listed.

References

- [1] J.C. Ban, S.S. Lin, Patterns generation and transition matrices in multi-dimensional lattice models, *Discrete Contin. Dyn. Syst.* 13 (3) (2005) 637–658.
- [2] J.C. Ban, W.G. Hu, S.S. Lin, Y.H. Lin, Zeta functions for two-dimensional shifts of finite type, *Mem. Amer. Math. Soc.* 221 (1037) (2013).
- [3] J.C. Ban, S.S. Lin, Y.H. Lin, Patterns generation and spatial entropy in two dimensional lattice models, *Asian J. Math.* 11 (2007) 497–534.
- [4] R. Berger, The Undecidability of the Domino Problem, *Mem. Amer. Math. Soc.*, vol. 66, 1966.
- [5] K. Culik II, An aperiodic set of 13 Wang tiles, *Discrete Math.* 160 (1996) 245–251.
- [6] C. Goodman-Strauss, Can't decide? Undecide! *Notices Amer. Math. Soc.* 57 (2010) 343–356.
- [7] B. Grünbaum, G.C. Shephard, *Tilings and Patterns*, W.H. Freeman, New York, 1986.
- [8] W.G. Hu, S.S. Lin, Nonemptiness problems of plane square tiling with two colors, *Proc. Amer. Math. Soc.* 139 (2010) 1045–1059.
- [9] J. Kari, A small aperiodic set of Wang tiles, *Discrete Math.* 160 (1996) 259–264.
- [10] A. Lagae, P. Dutré, An alternative for Wang tiles: colored edges versus colored corners, *ACM Trans. Graph.* 25 (4) (2006) 1442–1459.
- [11] A. Lagae, J. Kari, P. Dutré, Aperiodic sets of square tiles with colored corners, Report CW 460, Department of Computer Science, K.U. Leuven, Leuven, Belgium, Aug. 2006.
- [12] R. Penrose, The role of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. Appl.* 10 (1974) 266–271.
- [13] R.M. Robinson, Undecidability and nonperiodicity for tilings of the plane, *Invent. Math.* 12 (1971) 177–209.
- [14] H. Wang, Proving theorems by pattern recognition, II, *Bell Syst. Tech. J.* 40 (1961) 1–41.