Published for SISSA by Springer

Received: April 11, 2014
Revised: May 30, 2014
Accepted: July 21, 2014
Published: August 18, 2014

Note on symmetric BCJ numerator

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Abstract: We present an algorithm that leads to BCJ numerators satisfying manifestly the three properties proposed by Broedel and Carrasco in [42]. We explicitly calculate the numerators at 4, 5 and 6-points and show that the relabeling property is generically satisfied.

Keywords: Scattering Amplitudes, Gauge Symmetry

ArXiv ePrint: 1403.6262

[1] The unusual ordering of authors in this paper is just to fit outdated requirement of Chinese University for recognition of contributions.

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Article funded by SCOAP^3.

doi:10.1007/JHEP08(2014)098

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1 Introduction

In an inspiring paper [1], Bern, Carrasco and Johansson (BCJ) have made a remarkable observation that the Yang-Mills tree-level scattering amplitude can be rearranged into a summation over Feynman-like diagrams constructed by cubic vertices only,

$$\mathcal{A}_{\text{tot}} = \sum_i \frac{c_i n_i}{D_i} \tag{1.1}$$

where its kinematic dependent numerators n_i satisfy the same algebraic identities as the color factors c_i, i.e.,

$$\begin{aligned} &\text{antisymmetry}: \quad c_i = -c_j \quad \Rightarrow n_i = -n_j \\ &\text{Jacobi}-\text{like identity}: c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0. \end{aligned} \tag{1.2}$$

This duality between the color and kinematic factors was later found to be present in a variety of Yang-Mills theories [2–10] and, perhaps most surprisingly, was shown to be valid at least for the first few loop levels [2, 3, 5–9, 11–19]. The apparently symmetrical structure also suggests mirror versions of the existing color decomposition formulations. In particular that studies of the BCJ duals of the Del Duca-Dixon-Maltoni(DDM) form [20] and the original trace form of color decomposition formulations can be found in [11, 21–26]. For example at tree-level, the full Yang-Mills amplitude were shown to be expressible in

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terms of Kleiss-Kuijf basis [27] color-ordered scalar amplitudes $\widetilde{A}(1,\sigma,n)$[1] multiplied by BCJ numerators $n_{1,\sigma,n}$ [11, 23]

$$\mathcal{A}_{\text{tot}} = \sum_{\sigma\in S_{n-2}} n_{1,\sigma,n}\widetilde{A}(1,\sigma,n), \tag{1.3}$$

This can be regarded as a result of exchanging the roles of color factors and BCJ numerators in the DDM form [20]

$$\mathcal{A}_{\text{tot}} = \sum_{\sigma\in S_{n-2}} c_{1,\sigma,n}A(1,\sigma,n), \tag{1.4}$$

where $c_{1,\sigma,n} = f^{1\sigma_1 x_1}f^{x_2\sigma_2 x_3}\dots f^{x_{n-3}\sigma_{n-2}n}$ and $A(1,\sigma,n)$ are color-ordered Yang-Mills amplitudes. The BCJ dual of the original trace form of color decomposition was studied in [22, 24–26].

In addition to the apparently equal-footing treatment of the color and the kinematic dependent factors, another interesting aspect of the BCJ duality is that once we replace the color factors c_i in the BCJ form (1.1) by yet another copy of the kinematic dependent numerator (BCJ numerator) $\widetilde{n}_i$ of the Yang-Mills theory, we get the gravity amplitude, i.e,

$$\mathcal{A}^{\text{GR}}(1,2,\dots,n-1,n) = \sum_i \frac{\widetilde{n}_i n_i}{D_i}\ . \tag{1.5}$$

The above expression of the gravity amplitude as two copies of numerators resembles an earlier discovered amplitude relation by Kawai, Lewellen and Tye (KLT) [28] in string theory, where it was suggested that the closed string tree amplitude can be expanded in terms of products of two copies of open string tree amplitudes. When taking field theory limit, KLT relation states that the tree-level gravity amplitudes can be given by the "square" of the tree-level color-ordered Yang-Mills amplitudes. More explicitly, various forms of KLT relation for gravity amplitudes have been proposed and proved in [31–33]. Among these forms, a formulation with manifest $(n-3)!$ symmetry was given in [33] as

$$\begin{aligned}
&\mathcal{A}^{\text{GR}}(1,2,\dots,n-1,n)\\
&=(-)^{n+1}\sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A_n^{\text{YM}}(1,\sigma(2,\dots,n-2),n-1,n)\mathcal{S}[\widetilde{\sigma}(2,\dots,n-2))|\sigma(2,\dots,n-2))]_{p_1}\\
&\qquad\qquad\times\widetilde{A}_n^{\text{YM}}(n-1,n,\widetilde{\sigma}(2,\dots,n-2),1),
\end{aligned}\tag{1.6}$$

where A^{YM} and $\widetilde{A}^{\text{YM}}$ are two copies of color-ordered Yang-Mills amplitudes. However, if one goes through the original derivation in [28], one finds that we can replace the closed string theory by heterotic string theory, where one part of the heterotic string is an open string while the other part carries color information [29]. Thus it is imaginable that there is a KLT-like relation for full Yang-Mills amplitudes. In fact the KLT-like construction for full Yang-Mills amplitudes was proposed long time before in [30] and has later been systematically proved in [13] using BCFW-recursion relation. Explicitly the relation is given by

$$\mathcal{A}^{\text{YM}}(1,2,\dots,n-1,n)$$

[1]The color-ordered scalar theory is ϕ^3-scalar theory with f^{abc} as coupling constant for color-ordered cubic vertices.

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$$= (-)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A_n^{\mathrm{YM}}(1,\sigma(2,\ldots,n-2),n-1,n)\mathcal{S}[\widetilde{\sigma}(2,\ldots,n-2))|\sigma(2,\ldots,n-2))]_{p_1}$$
$$\times A_n^{\mathrm{CS}}(n-1,n,\widetilde{\sigma}(2,\ldots,n-2),1), \tag{1.7}$$

where A^{CS} is, in fact, exactly the same $\widetilde{A}$ in equation (1.3).

Now let us have a look at how to understand KLT relation (1.7) from the angle of BCJ duality (1.2). In (1.6), A^{YM} and $\widetilde{A}^{\mathrm{YM}}$ are two copies of color-ordered Yang-Mills amplitudes, which are respectively expressed by two sets of BCJ numerators n_i and $\widetilde{n}_i$ in the forms $\sum_i \frac{n_i}{D_i}$ and $\sum_i \frac{\widetilde{n}_i}{D_i}$(i.e, the form of (1.1) without color factor c_i and the sum over compatible topology with given color order). From the perspective of BCJ duality, if we replace one copy of numerators $\widetilde{n}_i$ by corresponding color factors c_i, $\widetilde{A}^{\mathrm{YM}}$ will become A_n^{CS}, thus formula (1.5) goes back to (1.1), while formula (1.6) becomes the formula (1.7).

Ever since the discovery of BCJ duality, a considerable amount of endeavor has been devoted to the systematic construction of the kinematic numerators. An explicit construction was given by Mafra, Schlotterer and Stieberger using the pure spinor language [34]. Alternatively, it was shown that the kinematic factors can be interpreted in terms of diffeomorphism algebra [23, 35–37]. In a series of recent papers [38–40] another interesting construction was provided by Cachazo, He and Yuan (CHY) using the solutions to the scattering equations. At the moment of writing it is not yet clear how to construct kinematic numerators for the most generic configuration and to arbitrary loop order.

Instead of attempting to decipher the analytic structure responsible for the possible algebraic behavior, another line of thoughts is to solve the kinematic numerators reversely in terms of scattering amplitudes, and indeed, it was discussed in [36, 41] that such expression for the numerators can always be derived in suitable basis. A technical issue lies with this approach is that because of the complexity involved, along with the ambiguity introduced by generalized gauge invariance, it is practically difficult to write down an analytic expression for generic numerator. Nevertheless at tree level, explicit numerators were worked out by Broedel and Carrasco [42] at 4 and 5-points, and 6-points in the case of four dimensions, which satisfy the following three properties:

1. The numerators satisfy Jacobi identity, and this property is referred as *BCJ representation*.

2. All external state information such as particle species and helicity is coded inside the color-ordered partial amplitudes. In other words, we have $n_i = \sum_\alpha c_{i\alpha} A_\alpha$, where $c_{i\alpha}$ is helicity blind. This property is called *amplitude-encoded representation*.

3. Expressions for numerators sharing the same topology are relabeling related. In other words, for each topology, if we know the BCJ numerator for a particular ordering of external legs, we know all others simply through relabelings. This property is called *symmetric representation*.

In this paper we present a systematic construction of the kinematic numerators at generic n-points by comparing average of different Kawai-Lewellen-Tye (KLT) expressions with the dual DDM-form (1.3). The numerators under this construction satisfy all the above three

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properties of Broedel and Carrasco. As examples we derive the explicit formulas for 4, 5 and 6 points. In particular that the 6-point expression here does not assume spinor identities.

The paper is organized as follows. In section 2 we discuss the basic idea used to determine the BCJ numerators in the dual DDM-form (1.3). In sections 3 and 4 we present explicit expressions for numerators at 4 and 5-points. Due to complexity we present the 6-point result in appendix A. We show that the numerators produced by the algorithm discussed in this paper satisfy all three properties of Broedel and Carrasco in section 5. Finally, a conclusion is given in section 6.

2 General framework

Our starting point is the KLT expression (1.7) of the total Yang-Mills amplitude, which we rewrite down here (see eq. (6.3) and eq. (6.4) in [13]),

$$\begin{aligned}
&\mathcal{A}_n^{KLT}(1,2,\ldots,n-1,n)\\
&=\mathcal{A}_n^{KLT}(1,\{2,\ldots,n-2\},n-1,n)\\
&=(-)^{n+1}\sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A_n(1,\sigma(2,\ldots,n-2),n-1,n)\mathcal{S}[\widetilde{\sigma}(2,\ldots,n-2))|\sigma(2,\ldots,n-2))]_{p_1}\\
&\qquad\qquad\qquad\times \widetilde{A}_n(n-1,n,\widetilde{\sigma}(2,\ldots,n-2),1)\\
&=(-)^{n+1}\sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} \widetilde{A}_n(1,n-1,n,\widetilde{\sigma}(2,\ldots,n-2))\mathcal{S}[\widetilde{\sigma}(2,\ldots,n-2))|\sigma(2,\ldots,n-2))]_{p_1}\\
&\qquad\qquad\qquad\times A_n(1,\sigma(2,\ldots,n-2),n-1,n)
\end{aligned}\tag{2.1}$$

where A_n is the color ordered partial amplitude of Yang-Mills theory and $\widetilde{A}_n$ is the color ordered partial amplitude of color-ordered scalar theory appearing in the dual DDM-form (1.3). Notice that although the total amplitude is totally symmetric under permutations of all n legs, the expressions in the second and the third lines are not manifestly so. In particular that legs $1,n-1,n$ are kept fixed, while the ordering of the rest $(n-3)$ legs appear in $\mathcal{A}_n^{KLT}(1,2,\ldots,n-1,n)$ has no effect on the expression. To emphasize this feature, we write the parameter of amplitude $\mathcal{A}_n^{KLT}$ as $(1,\{2,\ldots,n-2\},n-1,n)$. The momentum kernel $\mathcal{S}$ appears in (2.1) is defined as [31, 43]

$$\mathcal{S}[i_1,i_2,\ldots,i_k|j_1,j_2,\ldots,j_k]_{p_1}=\prod_{t=1}^{k}(s_{i_t1}+\sum_{q>t}^{k}\theta(i_t,i_q)s_{i_ti_q})\tag{2.2}$$

where $\theta(i_t,i_q)$ is zero when pair (i_t,i_q) has same ordering at both sets $\mathcal{I}=\{i_1,i_2,\ldots,i_k\},\mathcal{J}=\{j_1,j_2,\ldots,j_k\}$, otherwise it is one. A few examples are the following:

$$\mathcal{S}[2,3,4|2,4,3]=s_{21}(s_{31}+s_{34})s_{41},\quad \mathcal{S}[2,3,4|4,3,2]=(s_{21}+s_{23}+s_{24})(s_{31}+s_{34})s_{41}.$$

In this definition momentum p_1 plays a distinct role, in the sense that for each leg i there is always one term s_{i1}. In the case when different choices of p_1 are encountered, we should write $\mathcal{S}[\mathcal{I}|\mathcal{J}]_{p_1}$ to avoid confusions.

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Since our goal is to obtain a *symmetric representation* for BCJ numerators, as a second step we symmetrize expression (2.1)[2],

$$\mathcal{A}_n^S = \frac{1}{n!} \sum_{\sigma \in S_n} \mathcal{A}_n^{KLT}(\sigma_1, \{\sigma_2, \ldots, \sigma_{n-2}\}, \sigma_{n-1}, \sigma_n) \tag{2.3}$$

However note that since KLT expression is already manifestly $(n-3)!$-symmetric, equation (2.3) reduces to averaging over n-choices for σ_1, then $(n-1)$-choices for σ_{n-1} and $(n-2)$-choices for σ_n,

$$\mathcal{A}_n^S = \frac{1}{n(n-1)(n-2)} \sum_{\substack{i,\,j,\,k\,=\,1\\ i\,\neq\, j\,\neq\, k}}^{n} \mathcal{A}_n^{KLT}(i, \{1, 2, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, \widehat{k}, \ldots, n\}, j, k), \tag{2.4}$$

where $\{1, 2, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, \widehat{k}, \ldots, n\}$ denote the permutation $\{1, 2, \ldots, n\}$ with the three legs i, j, k deleted.

As a third step, we expand $\widetilde{A}_n$ using Kleiss-Kuijf(KK)-basis by KK relation [27] (KK relation for color ordered scalar amplitude was proved in [13]), with, for example, 1, n fixed at the first and the last position

$$\widetilde{A}_n(1, \{\alpha\}, n, \{\beta\}) = (-1)^{\#(\beta)} \sum_{\sigma \in OP(\{\alpha\} \bigcup \{\beta\}^T)} \widetilde{A}_n(1, \sigma, n), \tag{2.5}$$

where we have summed over all the possible permutations with keeping ordering in $\{\alpha\}$ set and reversing the ordering in $\{\beta\}$ set, $\#(\beta)$ denotes the number of elements in $\{\beta\}$ set. Thus $\mathcal{A}_n^S$ becomes

$$\mathcal{A}_n^S = \sum_{\sigma \in S_{n-2}} \widetilde{n}_{1\sigma(2,\ldots,n-1)n} \widetilde{A}_n(1, \sigma(2, \ldots, n-1), n), \tag{2.6}$$

where the $\widetilde{n}_{1\sigma(2,\ldots,n-1)n}$ here is a collective factor of color-ordered amplitude A_n and kinematic factors. Comparing with the dual DDM-form (1.3) of the total Yang-Mills amplitude, we propose that the desired BCJ numerator n_α^S satisfying the three properties of [42] to be

$$n^s_{1\sigma(2,\ldots,n-1)n} \equiv \widetilde{n}_{1\sigma(2,\ldots,n-1)n}. \tag{2.7}$$

To summarize, the basic idea is as follows:

- We construct only the BCJ numerator n_i in dual DDM-form (1.3). The rest numerators are given by Jacobi identity. By this construction, the expression is automatically BCJ representation.
- We consider constructing the numerators using KLT relation, where the helicity information is automatically coded inside the partial amplitude.
- The $\mathcal{A}_n^S$ (2.3) is averaged over all permutations of n external legs, thus the relabeling property is manifestly constructed.

[2] Considering the average of KLT relations was firstly suggested in the work [44]. However, the numerator suggested in [44] by taking $(n-2)!$ cannot satisfy the relabeling property.

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3 Four-point construction

Having discussed the general framework, let us present a few examples explicitly. We start with averaging KLT relation over all 4! permutations,

$$\begin{aligned}\mathcal{A}_4^S(1,2,3,4) &= \frac{1}{4!}\sum_{\sigma\in S_4}\mathcal{A}_4^{KLT}(\sigma)\\ &= \frac{-1}{24}\Big(A(1,3,4,2)s_{12}\tilde{A}(1,2,3,4)+A(1,4,3,2)s_{12}\tilde{A}(1,2,4,3)\\ &\qquad +\cdots+A(1,3,2,4)s_{14}\tilde{A}(1,4,3,2)\Big)\,. \end{aligned} \tag{3.1}$$

Translating all $\tilde{A}$ into KK-basis by the KK relation (2.5), and comparing equation (3.1) with its dual DDM-form expression

$$\mathcal{A}_4(1,2,3,4)=n_{1234}\widetilde{A}(1,2,3,4)+n_{1324}\widetilde{A}(1,3,2,4), \tag{3.2}$$

we see that numerator n_{1234} can be read off from the coefficient of $\widetilde{A}(1,2,3,4)$ in the symmetrized KLT expression (3.1), giving

$$\begin{aligned}n_{1234} = \frac{1}{24}\big(&s_{12}A(2,3,4,1)-s_{12}A(2,4,3,1)+s_{13}A(3,2,4,1)-s_{14}A(4,2,3,1)-s_{21}A(1,3,4,2)\\ &+s_{21}A(1,4,3,2)-s_{23}A(3,1,4,2)+s_{24}A(4,1,3,2)+s_{31}A(1,4,2,3)-s_{32}A(2,4,1,3)\\ &+s_{34}A(4,1,2,3)-s_{34}A(4,2,1,3)-s_{41}A(1,3,2,4)+s_{42}A(2,3,1,4)-s_{43}A(3,1,2,4)\\ &+s_{43}A(3,2,1,4)\big)\end{aligned} \tag{3.3}$$

The expression above can be further translated into BCJ basis [1] by considering KK [27] and BCJ [1] relations for color-ordered Yang-Mills amplitudes,

$$n_{1234}=\frac{1}{3}[s_{12}A(1,2,3,4)-s_{23}A(1,3,2,4)]=\frac{s_{12}(s_{13}-s_{23})}{3s_{13}}A(1,2,3,4). \tag{3.4}$$

which is simply the four-point result obtained by Broedel and Carrasco in [42]. The other numerator n_{1324}, obtained by collecting the coefficient of $\widetilde{A}(1,3,2,4)$, gives the same expression as (3.3) with legs $(2,3)$ swapped.

4 The 5-point case

The calculation at 5-points follows in a straightforward manner. Here we list the result for n_{12345} in KK-basis.

$$\begin{aligned}n_{12345} = &\frac{1}{10}A(1,2,3,4,5)(s_{12}(s_{13}+s_{23})+s_{45}(s_{34}+s_{35}))+\frac{1}{10}A(1,2,4,3,5)(-s_{12}s_{34}+s_{35}s_{45})\\ &+\frac{1}{10}A(1,3,2,4,5)(s_{12}s_{13}-s_{45}s_{23})\\ &+\frac{1}{60}A(1,3,4,2,5)(s_{23}(s_{24}-s_{25})+s_{12}(3s_{13}-s_{24}+s_{25})+s_{13}(s_{14}-s_{15}+2s_{34}-2s_{35})\\ &\qquad +s_{45}(-2s_{14}-s_{45}+3s_{25}+2s_{34}+s_{35})\\ &\qquad -(s_{14}+s_{24}+s_{34})(s_{14}+s_{23}+2(s_{24}+s_{34})))\\ &+\frac{1}{60}A(1,4,2,3,5)(s_{12}(s_{13}+3s_{14}-s_{15}+2s_{23}-2s_{25})+s_{24}s_{34}-2s_{13}s_{35}-s_{15}s_{35}\end{aligned}$$

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$$
\begin{aligned}
&\qquad\qquad +2s_{23}s_{35}+s_{25}s_{35}-s_{24}s_{45}+3s_{35}s_{45}+s_{14}(-s_{34}+s_{45}))\\
&+\frac{1}{60}A(1,4,3,2,5)(s_{23}s_{24}+s_{12}(s_{13}+2s_{14}-s_{23}-2s_{25})-s_{23}s_{25}-s_{14}s_{34}\\
&\qquad\qquad +2s_{23}s_{34}+s_{24}s_{34}+s_{13}(s_{14}-s_{15}-2s_{35})-s_{15}s_{35}+s_{25}s_{35}\\
&\qquad\qquad -2s_{14}s_{45}+2s_{25}s_{45}-s_{34}s_{45}+s_{35}s_{45}),
\end{aligned}
\tag{4.1}
$$

which in BCJ basis simplifies to

$$
\begin{aligned}
n_{12345}=&\Big(A(1,2,3,4,5)(s_{14}+s_{24}+s_{34})(s_{13}^3(2s_{14}+2s_{23}-3(s_{24}+s_{34}))+\\
&\qquad s_{13}^2(3s_{14}^2+3s_{23}^2-s_{24}^2+3s_{23}s_{34}-6s_{24}s_{34}-5s_{34}^2-s_{14}(3s_{24}+2s_{34}))\\
&\qquad -s_{14}s_{23}(s_{14}^2-s_{24}^2+s_{24}s_{34}+2s_{34}^2+3s_{14}(s_{23}+s_{34})+s_{23}(-2s_{24}+3s_{34}))\\
&\qquad +s_{13}(s_{14}^3-3s_{14}^2s_{23}+s_{23}^3+2s_{23}s_{34}(s_{24}+s_{34})+s_{23}^2(2s_{24}+5s_{34})\\
&\qquad -s_{34}(s_{24}^2+3s_{24}s_{34}+2s_{34}^2)-s_{14}(3s_{23}^2+s_{24}^2+3s_{24}s_{34}+3s_{34}^2+s_{23}(3s_{24}+s_{34}))))\\
&+A(1,2,4,3,5)(s_{13}^4s_{14}-s_{14}(s_{23}+s_{34})(s_{14}+s_{24}+s_{34})(s_{14}^2-s_{24}^2+s_{24}s_{34}+2s_{34}^2\\
&\qquad +3s_{14}(s_{23}+s_{34})+s_{23}(-2s_{24}+3s_{34}))+s_{13}^2(2s_{14}^3+3s_{14}^2(s_{23}-s_{24}+3s_{34})\\
&\qquad -s_{24}(-3s_{23}^2+s_{24}^2-3s_{23}s_{34}+6s_{24}s_{34}+5s_{34}^2)+\\
&\qquad s_{14}(2s_{23}^2-6s_{24}^2-3s_{23}(s_{24}-2s_{34})-9s_{24}s_{34}+7s_{34}^2))\\
&\qquad +s_{13}^3(3s_{14}^2+3s_{14}(s_{23}-s_{24}+2s_{34})+s_{24}(2s_{23}-3(s_{24}+s_{34})))\\
&\qquad -s_{13}(s_{14}^3(s_{23}-s_{34})+s_{14}^2(s_{23}^2+(6s_{24}-s_{34})s_{34}+3s_{23}(s_{24}+s_{34}))\\
&\qquad +s_{14}(s_{23}^2(2s_{24}+s_{34})+s_{24}s_{34}(8s_{24}+9s_{34})+2s_{23}(s_{24}^2+4s_{24}s_{34}+s_{34}^2))\\
&\qquad +s_{24}(-s_{23}^3-2s_{23}s_{34}(s_{24}+s_{34})-s_{23}^2(2s_{24}+5s_{34})+s_{34}(s_{24}^2+3s_{24}s_{34}+2s_{34}^2))))\Big)\\
&\times\frac{1}{30s_{13}s_{14}(s_{13}+s_{14}+s_{34}))}
\end{aligned}
\tag{4.2}
$$

and we do find that relabeling symmetry is satisfied at 5-points. This result is also same with the one obtained by Broedel and Carrasco in [42] The explicit expression for 6-point numerator is considerably more complicated and we leave the result to appendix A.

5 Verifying symmetry properties of the numerators

In this section let us check whether the BCJ numerators constructed following the algorithm outlined at the beginning of this paper indeed satisfy the three properties proposed in [42]. As remarked at the end of section 2, the n^s_α's defined by this algorithm satisfy the BCJ-representation automatically since $n^s_{1\sigma(2,\ldots,n-1)n}$ works as a basis and other numerators are constructed through antisymmetry and Jacobi identity. The n^s_α's are also amplitude-encoded representation since n^S_α is of the form $\sum A_n\mathcal{K}$, where $\mathcal{K}$ are kinematic factors constructed by s_{ij} and all helicity information is included in A_n.

The last property, i.e., the symmetric representation, is however not trivial. Note that since all numerators corresponding to other topologies can be constructed by those in dual DDM-form (with half ladder structure), if we can show the relabeling symmetry is true for numerators in dual DDM-form, it must be true for other topologies. For the topology of half ladder structure, there are $n!$ different labelings. Among them $(n-2)!$ of the numerators are directly given by the algorithm (2.7) and others can be constructed using the Jacobi-identity and anti-symmetry. In our third step, we have fixed only two legs $1, n$,

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the relabeling property is manifestly true among $(2,3,\ldots,n-1)$ by construction. Since all permutations can be generated by successive permutations between the $(n-1)$ consequtive pairs, we can reduce our checking to the following two permutations: (12) and $(n-1)n$. Now let us consider the numerators of dual-DDM form $n_{213\ldots(n-1)n}$ and $n_{123\ldots n(n-1)}$. We have two ways to get the same n_α from basis numerators: one is by relabeling $n_{123\ldots(n-1)n}$ and another one, by using Jacobi relation and antisymmetry. If the expressions obtained by these two ways are the same, the relabeling property is fully checked.

5.1 Permutation (12)

Now we consider the relabeling property under the permutation (12) for $n_{123\ldots(n-1)n}$. The BCJ numerator $n_{213\ldots(n-1)n}$ can be obtained by two ways. The first way is by relabeling from $n_{123\ldots(n-1)n}$, which we denote as

$$(n^S_{213\ldots(n-1)n})_R \equiv n^s_{123\ldots(n-1)n}|_{1\leftrightarrow 2} \tag{5.1}$$

The second way is by antisymmetry specially for this topology and we get

$$(n^S_{213\ldots(n-1)n})_A = -n^s_{123\ldots(n-1)n} \tag{5.2}$$

To check they are the same, we notice that permutation (12) in equation (2.6) gives

$$\begin{aligned}\mathcal{A}^S_n &= \sum_{\sigma\in S_{n-2}} \widetilde{n}_{2\sigma(1,\ldots,n-1)n}\widetilde{A}_n(2,\sigma(1,\ldots,n-1),n)\\ &= (n^S_{213\ldots(n-1)n})_R\widetilde{A}_n(2,1,3,\ldots,n-1,n)+\ldots\end{aligned} \tag{5.3}$$

where we have written the expansion at the second line and identify the coefficient of $\widetilde{A}_n(2,1,3,\ldots,n-1,n)$ to be the $(n^S_{213\ldots(n-1)n})_R$ given in (5.1). The reason is simple: because at our second step, symmetry among n external legs are manifest, thus when we go to different KK-basis, their coefficients are related to each other by simple relabeling.

A relation between $(n^S_{213\ldots(n-1)n})_R$ given in (5.1) and the original $(1,n)$-basis numerators $n^S_{123\ldots(n-1)n}$ can be obtained by translating the amplitude $\widetilde{A}(1,\sigma,n)$ in equation (2.6) into $(2,n)$-basis $\widetilde{A}(2,\rho,n)$. Notice however, according to KK-relation (2.5),

$$\widetilde{A}(1,\alpha,2,\beta,n) = \widetilde{A}(2,\beta,n,1,\alpha) = (-)^{1+\#(\alpha)}\sum_{\rho\in COP(\beta\bigcup(\alpha^T,1))}\widetilde{A}(2,\rho,n) \tag{5.4}$$

On the right hand side we see that because of the presence of an extra α^T sitting on the left of leg 1, we never get $\widetilde{A}_n(2,1,3,\ldots,n-1,n)$ unless set α is empty. When that happens, there is only one contribution with a $(-)$ sign. Putting (5.4) back to (2.6) and we find the coefficient of $\widetilde{A}_n(2,1,3,\ldots,n-1,n)$ is given by $-n^s_{123\ldots(n-1)n}$. Thus we have proven that (5.1) coincides with (5.2).

5.2 Permutation $((n-1)n)$

The proof of $((n-1)n)$ invariance is similar. Using relabeling we find

$$\begin{aligned}\mathcal{A}^S_n &= \sum_{\sigma\in S_{n-2}} \widetilde{n}_{1\sigma(2,\ldots,n-2,n)(n-1)}\widetilde{A}_n(1,\sigma(2,\ldots,n-2,n),n-1)\\ &= (n^S_{123\ldots(n-2)n(n-1)})_R\widetilde{A}_n(1,2,,\ldots,n-2,n,n-1)+\ldots\end{aligned} \tag{5.5}$$

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On the other hand expanding $\widetilde{A}(1,\sigma,n)$ into $(1,n-1)$-basis gives

$$\widetilde{A}(1,\alpha,n-1,\beta,n) = (-)^{1+\#(\beta)} \sum_{\rho\in COP(\alpha\bigcup(n,\beta^T,1))} \widetilde{A}(1,\rho,n-1) \tag{5.6}$$

we see on the right hand side of the equation that because of the extra ordering β^T between legs n and 1, we never get $\widetilde{A}_n(1,2,3,\ldots,n-2,n,n-1)$ unless β is empty. When this happens, there is only one contribution with a $(-)$ sign, thus we get

$$(n^S_{123\ldots(n-2)n(n-1)})_R = -n^s_{123\ldots(n-1)n} \tag{5.7}$$

as required by anti-symmetry of the BCJ numerator.

6 Conclusion

In this paper we discussed a systematic construction of the BCJ numerator based on matching KLT and dual DDM-form of the full Yang-Mills amplitude. Using this method we explicitly calculated the numerators at 4, 5 and 6 points and verified the three symmetry properties proposed by Broedel and Carrasco [42] hold generically. Note the similarity between the expressions discussed and the prescription proposed by Cachazo, He and Yuan [40] despite the method used in this paper does not rely on the existence of the solutions to scattering equations.

There are a few things one can proceed. First, although we have the general algorithm, its computation takes a long time when the number of external leg increases. Thus it will be nice if we can have a general patten of n_α expanded into the KK-basis (or BCJ-basis). Secondly, it will be natural to generalize above results to loop level. At loop-level the current method does not seem to straightforwardly generalize because of the lack of support from KLT relation, yet naively it might be possible to solve numerators by comparing integrands in suitable basis. We leave this part of the discussion to future works.

Acknowledgments

We are pleased to thank Stephen G. Naculich for pointing out a typo in the five-point result in the previous version. Y. J. Du would like to acknowledge the EU programme Erasmus Mundus Action 2 and the International Postdoctoral Exchange Fellowship Program of China(with Fudan University as the home university) for supporting his postdoctoral research in Lund University. Y. J. Du's research is supported in parts by the NSF of China Grant No.11105118, China Postdoctoral Science Foundation No.2013M530175 and the Fundamental Research Funds for the Central Universities of Fudan University No.20520133169. C.F. would like to acknowledge the support from National Science Council, 50 billions project of Ministry of Education and National Center for Theoretical Science, Taiwan, Republic of China as well as the support from S.T. Yau center of National Chiao Tung University. B.F is supported, in part, by fund from Qiu-Shi and Chinese NSF funding under contract No.11031005, No.11135006, No. 11125523.

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A 6-point numerator

In this appendix we present the explicit formula of the 6-point BCJ numerator in KK basis. Note however, despite its complixity, this expression holds in all dimensions greater or equal to three and does not depend on helicity except through color-ordered amplitudes.

$$
\begin{aligned}
n_{123456} = \frac{1}{120}\{ & A(123456)2\,[8(s_{12}(s_{13}+s_{23})(s_{14}+s_{24}+s_{34})+(s_{34}+s_{35}+s_{36})(s_{45}+s_{46})s_{56})] \\
&+A(123546)4\,[s_{12}(s_{13}+s_{23})(s_{14}+s_{15}-s_{16}+s_{24}+s_{25}-s_{26}+s_{34}+s_{35}-s_{36}) \\
&\qquad +2(s_{34}+s_{35}+s_{36})s_{46}s_{56}] \\
&+A(124356)8\,[s_{35}+s_{36})(s_{45}+s_{46})s_{56}+4s_{12}(2s_{14}s_{23}+2s_{23}s_{24}+2s_{13}(s_{14}+s_{24}) \\
&\qquad -(s_{15}+s_{16}+s_{25}+s_{26})s_{56}] \\
&+A(124536)\,[8s_{36}(s_{45}+s_{46})s_{56}+s_{12}(2s_{15}s_{24}-2s_{16}s_{24}+4s_{23}s_{24}+2s_{24}s_{25} \\
&\qquad -2s_{24}s_{26}-s_{23}s_{35}+s_{23}s_{36}+s_{13}(4s_{14}+4s_{24}-s_{35}+s_{36})+3s_{24}s_{45} \\
&\qquad +s_{14}(2s_{15}-2s_{16}+4s_{23}+2s_{25}-2s_{26}+3s_{45}-3s_{46})-3s_{24}s_{46}-3s_{15}s_{56} \\
&\qquad -s_{16}s_{56}-3s_{25}s_{56}-s_{26}s_{56})] \\
&+A(125346)\,[8(s_{34}+s_{36})s_{46}s_{56}+s_{12}(4s_{15}s_{23}-2s_{16}s_{23}+2s_{23}s_{24}+4s_{23}s_{25} \\
&\qquad -2s_{23}s_{26}+3s_{23}s_{34}+s_{13}(2s_{14}+4s_{15}-2s_{16}+2s_{24}+4s_{25}-2s_{26}+3s_{34}-3s_{36}) \\
&\qquad -3s_{23}s_{36}-s_{15}s_{45}-s_{25}s_{45}+s_{14}(2s_{23}-3s_{46})-s_{16}s_{46}-3s_{24}s_{46} \\
&\qquad -s_{26}s_{46}+s_{15}s_{56}+s_{25}s_{56})] \\
&+A(125436)\,[8s_{36}s_{46}s_{56}+s_{12}(2s_{15}s_{23}+2s_{15}s_{24}-2s_{16}s_{24}+2s_{23}s_{24}+2s_{23}s_{25}+2s_{24}s_{25} \\
&\qquad -2s_{24}s_{26}-s_{23}s_{34}+s_{13}(2s_{14}+2s_{15}+2s_{24}+2s_{25}-s_{34}-3s_{36})-3s_{23}s_{36} \\
&\qquad -s_{15}s_{45}-s_{25}s_{45}+s_{14}(2s_{15}-2s_{16}+2s_{23}+2s_{25}-2s_{26}-3s_{46})-s_{16}s_{46} \\
&\qquad -3s_{24}s_{46}-s_{26}s_{46}-3s_{15}s_{56}-3s_{25}s_{56})] \\
&+A(132456)\,[4(2s_{12}s_{13}(s_{14}+s_{24}+s_{34})-(s_{14}+s_{15}+s_{16}-s_{24}-s_{25}-s_{26}-s_{34} \\
&\qquad -s_{35}-s_{36})(s_{45}+s_{46})s_{56})] \\
&+A(132546)\,[4(s_{12}s_{13}(s_{14}+s_{15}-s_{16}+s_{24}+s_{25}-s_{26}+s_{34}+s_{35}-s_{36}) \\
&\qquad +(-s_{14}-s_{15}-s_{16}+s_{24}+s_{25}+s_{26}+s_{34}+s_{35}+s_{36})s_{46}s_{56})] \\
&+A(134256)\,[-(s_{23}(s_{25}+s_{26})+s_{13}(s_{15}+s_{16}+3(s_{35}+s_{36}))+(3s_{14}+2s_{15}+2s_{16} \\
&\qquad -4s_{25}-4s_{26}-3s_{34}-2s_{35}-2s_{36})(s_{45}+s_{46}))s_{56}+s_{12}(8s_{13}(s_{14}+s_{34}) \\
&\qquad +(s_{25}+s_{26})s_{56})] \\
&+A(134526)\,[-s_{23}s_{24}s_{25}+s_{23}s_{24}s_{26}-s_{23}s_{26}s_{56}-2s_{14}s_{45}s_{56}-s_{15}s_{45}s_{56}-s_{16}s_{45}s_{56} \\
&\qquad +4s_{26}s_{45}s_{56}+2s_{34}s_{45}s_{56}+s_{35}s_{45}s_{56}+s_{36}s_{45}s_{56}-s_{14}s_{46}s_{56}-s_{15}s_{46}s_{56} \\
&\qquad -s_{16}s_{46}s_{56}+4s_{26}s_{46}s_{56}+s_{34}s_{46}s_{56}+s_{35}s_{46}s_{56}+s_{36}s_{46}s_{56}+s_{12}(s_{24}(s_{25}-s_{26}) \\
&\qquad +s_{13}(4s_{14}-s_{25}+s_{26}+4s_{34})+s_{26}s_{56})+s_{13}(-s_{16}s_{34}+s_{34}s_{35}-s_{34}s_{36} \\
&\qquad +2s_{34}s_{45}+s_{14}(s_{15}-s_{16}+s_{35}-s_{36}+s_{45}-s_{46})-2s_{34}s_{46}+s_{15}(s_{34}-s_{56}) \\
&\qquad -2s_{35}s_{56}-s_{36}s_{56})] \\
&+A(135246)\,[s_{12}(s_{13}(2s_{14}+4s_{15}-2s_{16}+3s_{24}-3s_{26}+2s_{34}+4s_{35}-2s_{36})+s_{24}s_{46}) \\
&\qquad -s_{13}((s_{14}+2s_{34}+s_{36})s_{46}+s_{35}(s_{45}-s_{56}))+s_{46}(-s_{23}s_{24}+(-2s_{14} \\
&\qquad -3s_{15}-2s_{16}+4s_{24}+4s_{26}+2s_{34}+3s_{35}+2s_{36})s_{56})] \\
&+A(135426)\,[-s_{23}s_{24}s_{25}+s_{23}s_{24}s_{26}-s_{23}s_{24}s_{45}-s_{23}s_{26}s_{56}+s_{15}s_{45}s_{56}-s_{35}s_{45}s_{56} \\
&\qquad -s_{14}s_{46}s_{56}-s_{15}s_{46}s_{56}-s_{16}s_{46}s_{56}+4s_{26}s_{46}s_{56}+s_{34}s_{46}s_{56}+s_{35}s_{46}s_{56} \\
&\qquad +s_{36}s_{46}s_{56}+s_{12}(s_{13}(2s_{14}+2s_{15}-s_{24}-3s_{26}+2s_{34}+2s_{35}) \\
&\qquad +s_{24}(s_{25}-s_{26}+s_{45})+s_{26}s_{56})+s_{13}(-s_{16}s_{34}+s_{34}s_{35}-s_{34}s_{36}-s_{35}s_{45}
\end{aligned}
$$

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$$
\begin{aligned}
&\quad +s_{14}(s_{15}-s_{16}+s_{35}-s_{36}-s_{46})-2s_{34}s_{46}-s_{36}s_{46}+s_{15}(s_{34}-s_{56})-2s_{35}s_{56})] \\
&+A(142356)\,[(-2s_{16}s_{35}+3s_{23}s_{35}+2s_{25}s_{35}+2s_{26}s_{35}-2s_{16}s_{36}+3s_{23}s_{36}+2s_{25}s_{36}+2s_{26}s_{36} \\
&\quad -3s_{13}(s_{35}+s_{36})-2s_{15}(s_{35}+s_{36})+s_{14}s_{45}-s_{24}s_{45}+4s_{35}s_{45}+4s_{36}s_{45}+s_{14}s_{46} \\
&\quad -s_{24}s_{46}+4s_{35}s_{46}+4s_{36}s_{46})s_{56}+s_{12}(8s_{13}s_{14}+8s_{14}s_{23} \\
&\quad -(s_{15}+s_{16}+3(s_{25}+s_{26}))s_{56})] \\
&+A(142536)\,[(-3s_{13}s_{36}-2s_{15}s_{36}-2s_{16}s_{36}+3s_{23}s_{36}+2s_{25}s_{36}+2s_{26}s_{36}+s_{14}s_{45}-s_{24}s_{45} \\
&\quad +4s_{36}s_{45}+4s_{36}s_{46})s_{56}+s_{12}(4s_{13}s_{14}-s_{23}s_{35}+s_{23}s_{36}+s_{14}(2s_{15}-2s_{16} \\
&\quad +4s_{23}+2s_{25}-2s_{26}+3s_{45}-3s_{46})-s_{15}s_{56}-2s_{25}s_{56}-s_{26}s_{56})] \\
&+A(143256)\,[-(s_{23}(s_{25}+s_{26})+2s_{15}s_{35}+2s_{16}s_{35}-2s_{25}s_{35}-2s_{26}s_{35}+2s_{15}s_{36}+2s_{16}s_{36} \\
&\quad -2s_{25}s_{36}-2s_{26}s_{36}+s_{13}(s_{15}+s_{16}+3(s_{35}+s_{36}))+3s_{14}s_{45}-2s_{25}s_{45} \\
&\quad -2s_{26}s_{45}+s_{34}s_{45}-2s_{35}s_{45}-2s_{36}s_{45}+3s_{14}s_{46}-2s_{25}s_{46}-2s_{26}s_{46}+s_{34}s_{46} \\
&\quad -2s_{35}s_{46}-2s_{36}s_{46})s_{56}+s_{12}(8s_{13}s_{14}-3(s_{25}+s_{26})s_{56})] \\
&+A(143526)\,[-s_{23}s_{24}s_{25}+s_{23}s_{24}s_{26}-s_{23}s_{25}s_{34}+s_{23}s_{26}s_{34}-s_{14}s_{34}s_{45}+s_{14}s_{34}s_{46} \\
&\quad -s_{23}s_{26}s_{56}-s_{15}s_{35}s_{56}-s_{16}s_{35}s_{56}+2s_{26}s_{35}s_{56}-s_{15}s_{36}s_{56}-s_{16}s_{36}s_{56} \\
&\quad +2s_{26}s_{36}s_{56}-2s_{14}s_{45}s_{56}+2s_{26}s_{45}s_{56}-s_{34}s_{45}s_{56}+s_{35}s_{45}s_{56}+s_{36}s_{45}s_{56} \\
&\quad -s_{14}s_{46}s_{56}+2s_{26}s_{46}s_{56}+s_{35}s_{46}s_{56}+s_{36}s_{46}s_{56}+s_{12}(4s_{13}s_{14}+s_{23}s_{25} \\
&\quad -s_{23}s_{26}+s_{14}(-s_{25}+s_{26})-3s_{26}s_{56})+s_{13}(s_{14}(s_{15}-s_{16}+s_{35}-s_{36}+s_{45}-s_{46}) \\
&\quad -(s_{15}+2s_{35}+s_{36})s_{56})] \\
&+A(145236)\,[-s_{23}s_{24}s_{36}-s_{23}s_{34}s_{36}-s_{14}s_{34}s_{45}-s_{14}s_{35}s_{45}-s_{14}s_{36}s_{46}-s_{15}s_{36}s_{56} \\
&\quad -s_{16}s_{36}s_{56}+3s_{23}s_{36}s_{56}+2s_{26}s_{36}s_{56}+2s_{14}s_{45}s_{56}-s_{24}s_{45}s_{56}-s_{25}s_{45}s_{56} \\
&\quad +3s_{36}s_{45}s_{56}+2s_{36}s_{46}s_{56}-s_{13}s_{36}(s_{14}+s_{56})+s_{12}(2s_{13}s_{14}-s_{23}s_{25}-s_{23}s_{35} \\
&\quad +2s_{23}s_{36}+s_{14}(2s_{15}-s_{16}+3s_{23}-s_{26}+3s_{45}-s_{46})-s_{15}s_{56}-s_{26}s_{56})] \\
&+A(145326)\,[-s_{23}s_{24}s_{25}+s_{23}s_{24}s_{26}-s_{23}s_{25}s_{34}+s_{23}s_{26}s_{34}-s_{23}s_{24}s_{35}-s_{23}s_{34}s_{35} \\
&\quad -s_{14}s_{34}s_{45}-s_{14}s_{35}s_{45}-s_{14}s_{36}s_{46}-s_{23}s_{26}s_{56}-s_{15}s_{36}s_{56}-s_{16}s_{36}s_{56} \\
&\quad +2s_{26}s_{36}s_{56}-2s_{14}s_{45}s_{56}+2s_{26}s_{45}s_{56}-s_{34}s_{45}s_{56}-s_{35}s_{45}s_{56}+s_{36}s_{45}s_{56} \\
&\quad +s_{26}s_{46}s_{56}+s_{36}s_{46}s_{56}+s_{12}(2s_{13}s_{14}+s_{23}s_{25}-2s_{23}s_{26}+s_{23}s_{35} \\
&\quad +s_{14}(s_{15}-s_{23}-s_{26}+2s_{45})-s_{26}s_{56})+s_{13}(s_{14}(s_{15}-s_{16}-s_{36}+s_{45}-s_{46}) \\
&\quad -(s_{15}+s_{36})s_{56})] \\
&+A(152346)\,[-s_{25}s_{35}s_{45}-2s_{13}s_{34}s_{46}-s_{14}s_{34}s_{46}-s_{16}s_{34}s_{46}+2s_{23}s_{34}s_{46}+s_{24}s_{34}s_{46} \\
&\quad +s_{26}s_{34}s_{46}-s_{13}s_{36}s_{46}-s_{14}s_{36}s_{46}-s_{16}s_{36}s_{46}+s_{23}s_{36}s_{46}+s_{24}s_{36}s_{46} \\
&\quad +s_{26}s_{36}s_{46}+s_{25}s_{35}s_{56}-s_{25}s_{46}s_{56}+4s_{34}s_{46}s_{56}+4s_{36}s_{46}s_{56} \\
&\quad +s_{12}(4s_{15}s_{23}-s_{16}s_{23}+s_{23}s_{24}-s_{23}s_{26}+2s_{23}s_{34}+s_{13}(s_{14}+4s_{15}-s_{16} \\
&\quad +s_{24}-s_{26}+s_{34}-s_{36})-2s_{23}s_{36}-s_{15}s_{45}+s_{14}(s_{23}-s_{46})-2s_{24}s_{46} \\
&\quad -s_{26}s_{46}+s_{15}s_{56})+s_{15}(s_{35}(s_{45}-s_{56})+s_{46}s_{56})] \\
&+A(152436)\,[-s_{23}s_{34}s_{36}+s_{15}s_{34}s_{45}-s_{25}s_{34}s_{45}+s_{15}s_{35}s_{45}-s_{25}s_{35}s_{45}+s_{13}s_{36}(s_{34}-s_{46}) \\
&\quad -s_{14}s_{36}s_{46}-s_{16}s_{36}s_{46}+s_{23}s_{36}s_{46}+s_{24}s_{36}s_{46}+s_{26}s_{36}s_{46}+s_{15}s_{36}s_{56} \\
&\quad -s_{25}s_{36}s_{56}-s_{15}s_{45}s_{56}+s_{25}s_{45}s_{56}+4s_{36}s_{46}s_{56}+s_{12}(2s_{15}s_{23}+2s_{15}s_{24} \\
&\quad -s_{16}s_{24}+s_{23}s_{24}-s_{24}s_{26}-s_{23}s_{34}+s_{13}(s_{14}+2s_{15}+s_{24}-s_{36})-2s_{23}s_{36} \\
&\quad -s_{15}s_{45}+s_{14}(2s_{15}-s_{16}+s_{23}-s_{26}-s_{46})-2s_{24}s_{46}-s_{26}s_{46}-3s_{15}s_{56})] \\
&+A(153246)\,[-s_{23}s_{35}s_{45}-s_{25}s_{35}s_{45}-s_{13}s_{14}s_{46}-s_{23}s_{24}s_{46}-2s_{13}s_{34}s_{46}-s_{14}s_{34}s_{46} \\
&\quad -s_{16}s_{34}s_{46}+s_{24}s_{34}s_{46}+s_{26}s_{34}s_{46}-s_{13}s_{36}s_{46}-s_{14}s_{36}s_{46}-s_{16}s_{36}s_{46} \\
&\quad +s_{24}s_{36}s_{46}+s_{26}s_{36}s_{46}+s_{23}s_{35}s_{56}+s_{25}s_{35}s_{56}+2s_{24}s_{46}s_{56}+2s_{26}s_{46}s_{56}
\end{aligned}
$$

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$$
\begin{aligned}
&+2s_{34}s_{46}s_{56}-s_{35}s_{46}s_{56}+2s_{36}s_{46}s_{56}+s_{12}(s_{23}(-s_{24}+s_{26})\\
&+s_{13}(s_{14}+4s_{15}-s_{16}+s_{24}-s_{26}+s_{34}-s_{36})-s_{15}s_{45}-2s_{24}s_{46}-s_{26}s_{46}+s_{15}s_{56})\\
&+s_{15}(s_{35}(s_{45}-s_{56})-3s_{46}s_{56})]\\
+A(153426)\,&[-s_{23}s_{24}s_{25}+s_{23}s_{24}s_{26}-s_{23}s_{24}s_{35}-s_{23}s_{24}s_{45}+s_{15}s_{35}s_{45}-s_{23}s_{35}s_{45}\\
&-s_{24}s_{35}s_{45}-s_{25}s_{35}s_{45}-s_{14}s_{34}s_{46}-s_{16}s_{34}s_{46}+s_{26}s_{34}s_{46}-s_{14}s_{36}s_{46}\\
&-s_{16}s_{36}s_{46}+s_{26}s_{36}s_{46}+s_{12}(s_{23}s_{24}+s_{23}s_{26}+s_{24}s_{26}+s_{13}(s_{14}+2s_{15}\\
&-s_{26}+s_{34})-s_{15}(s_{24}+s_{45})-s_{26}s_{46})+s_{13}(-s_{16}s_{34}-s_{34}s_{36}\\
&+s_{14}(s_{15}-s_{16}-s_{36}-s_{46})-2s_{34}s_{46}-s_{36}s_{46}+s_{15}(s_{34}-s_{56}))\\
&-s_{23}s_{26}s_{56}+s_{15}s_{35}s_{56}-s_{26}s_{35}s_{56}+s_{15}s_{45}s_{56}+s_{35}s_{45}s_{56}-s_{15}s_{46}s_{56}\\
&+2s_{26}s_{46}s_{56}+s_{34}s_{46}s_{56}+s_{36}s_{46}s_{56}]\\
+A(154236)\,&[-s_{23}s_{24}s_{36}-s_{23}s_{34}s_{36}+s_{15}s_{34}s_{45}-s_{24}s_{34}s_{45}-s_{25}s_{34}s_{45}+s_{15}s_{35}s_{45}\\
&-s_{24}s_{35}s_{45}-s_{25}s_{35}s_{45}-s_{14}s_{36}s_{46}-s_{16}s_{36}s_{46}+s_{23}s_{36}s_{46}+s_{26}s_{36}s_{46}\\
&-s_{13}s_{36}(s_{14}+s_{46})-s_{15}s_{36}s_{56}+2s_{23}s_{36}s_{56}+s_{26}s_{36}s_{56}-2s_{15}s_{45}s_{56}\\
&+s_{24}s_{45}s_{56}+s_{25}s_{45}s_{56}-s_{36}s_{45}s_{56}+2s_{36}s_{46}s_{56}+s_{12}(s_{13}(s_{14}+s_{15})\\
&+2s_{15}s_{23}-s_{23}s_{24}-s_{23}s_{34}-2s_{23}s_{36}-s_{15}s_{45}+s_{14}(2s_{15}-s_{16}+s_{23}-s_{26}-s_{46})\\
&-s_{26}s_{46}-s_{15}s_{56})]\\
+A(154326)\,&[-s_{23}s_{24}s_{25}+s_{23}s_{24}s_{26}-s_{23}s_{25}s_{34}+s_{23}s_{26}s_{34}-s_{23}s_{24}s_{35}-s_{23}s_{34}s_{35}\\
&-s_{23}s_{24}s_{45}+s_{15}s_{34}s_{45}-2s_{23}s_{34}s_{45}-s_{24}s_{34}s_{45}-s_{25}s_{34}s_{45}+s_{15}s_{35}s_{45}\\
&-s_{23}s_{35}s_{45}-s_{24}s_{35}s_{45}-s_{25}s_{35}s_{45}-s_{14}s_{36}s_{46}-s_{16}s_{36}s_{46}+s_{26}s_{36}s_{46}\\
&+s_{12}(s_{13}(s_{14}+s_{15})-s_{15}s_{23}+s_{23}s_{24}+s_{14}(s_{15}-s_{26})+2s_{23}s_{26}+s_{23}s_{34}\\
&-s_{15}s_{45}-s_{26}s_{46})-s_{23}s_{26}s_{56}-s_{15}s_{36}s_{56}+s_{26}s_{36}s_{56}+2s_{15}s_{45}s_{56}\\
&-s_{26}s_{45}s_{56}+s_{34}s_{45}s_{56}+s_{35}s_{45}s_{56}+s_{26}s_{46}s_{56}+s_{36}s_{46}s_{56}\\
&+s_{13}(s_{14}(s_{15}-s_{16}-s_{36}-s_{46})-s_{36}s_{46}-s_{15}s_{56})]\} \qquad \text{(A.1)}
\end{aligned}
$$

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