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Constrained abductive reasoning with fuzzy parameters in Bayesian networks

Han-Lin Li, Han-Ying Kao*

Institute of Information Management, National Chiao Tung University, Hsinchu, 30050, Taiwan

Abstract

This work proposes a novel approach for solving abductive reasoning problems in Bayesian networks involving fuzzy parameters and extra constraints. The proposed method formulates abduction problems using nonlinear programming. To maximize the sum of the fuzzy membership functions subjected to various constraints, such as boundary, dependency and disjunctive conditions, unknown node belief propagation is completed. The model developed here can be built on any exact propagation methods, including clustering, joint tree decomposition, etc.

Keywords: Abductive reasoning; Bayesian networks; Fuzzy parameters; Optimization; Constraints

1. Introduction

Bayesian networks are widely used knowledge representation and reasoning tools for various domains under uncertainty [1–7]. Since an expert system requires both predictive and diagnostic information, two types of reasoning commonly are employed, namely deduction and abduction. Deduction is a logical process from a hypothesis to deduce evidence where probabilistic relationships are involved, and abduction is a logical process that hypothetically explains experimental observations [7].

Several methods have been developed for solving abductive reasoning problems in Bayesian networks. Exact methods exploit the independence structure contained in the network to efficiently propagate uncertainty [1,6,7]. Meanwhile, stochastic simulation methods provide an alternative approach suitable for highly connected networks, in which exact algorithms can be inefficient [7]. Recently, search-based approximate algorithms, which search for high probability configurations through a space of possible values, have emerged as a new alternative [8]. On the other hand, two key

* Corresponding author.
E-mail addresses: hlli@cc.nctu.edu.tw (H.-L. Li), teresak_hk@yahoo.com.tw, u8734807@cc.nctu.edu.tw (H.-Y. Kao).

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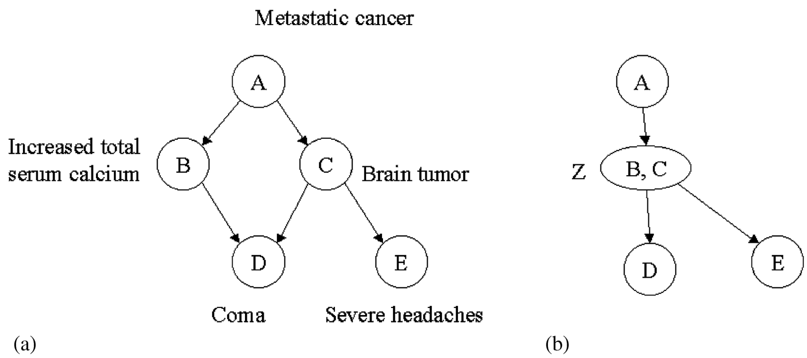

Fig. 1. (a) A Bayesian Network in Metastatic Cancer and (b) its tree structure as clustering B and C into a compound node Z [7].

approaches have been proposed for symbolic inference in Bayesian networks, namely: the symbolic probabilistic inference algorithm (SPI) [9] and symbolic calculations based on slight modifications of standard numerical propagation algorithms [1,10].

The above methods have two limitations for abductive reasoning:

(i) All relevant parameters are assumed to be crisp.
(ii) Extra constraints or knowledge regarding belief propagation in Bayesian networks are difficult to embed.

Those limitations restrict the usefulness of reasoning in Bayesian networks. First, the conditional probabilities between a node and its parents could be fuzzy parameters because of the difficulties of learning accurately the causal relationships among the nodes. Additionally, knowledge workers often acquire additional information regarding inferences in Bayesian networks, particularly when facing diverse diagnostic scenarios. This information can relate to boundary, dependency or disjunctive conditions. The above limitations are illustrated below using an example from Pearl [7].

> Metastatic cancer is a possible cause of a brain tumor and is an explanation for increased total serum calcium. Either of these could explain a patient falling into a coma. Severe headache is also possibly associated with a brain tumor.

Fig. 1(a) shows a Bayesian network representing the above cause and effect relationships. Table 1 lists the causal influences in terms of conditional probability distributions. Each variable is characterized by the probability given the state of its parents. For instance: $C \in \{1,0\}$ represents the dichotomy between having a brain tumor and not having one, $+c$ denotes the assertion $C=1$ or "Brain tumor is present", and $-c$ is the negation of $+c$, namely, $C=0$. The root node, A, which has no parent, is characterized by its prior probability distribution. The above information can be used to solve the following reasoning problems.

Problem 1. Compute the posterior probability of every A, B, and C, given the conditional probabilities in Table 1, and a situation involving a patient who is suffering from a severe headache

Table 1
The associated conditional probability distribution of Fig. 1

$P(+a) = 0.20$	
$P(+b\| + a) = 0.80$	$P(+b\| - a) = 0.20$
$P(+c\| + a) = 0.20$	$P(+c\| - a) = 0.05$
$P(+d\| + b, +c) = 0.80$	$P(+d\| - b, +c) = 0.80$
$P(+d\| + b, -c) = 0.80$	$P(+d\| - b, -c) = 0.05$
$P(+e\| + c) = 0.80$	$P(+e\| - c) = 0.60$

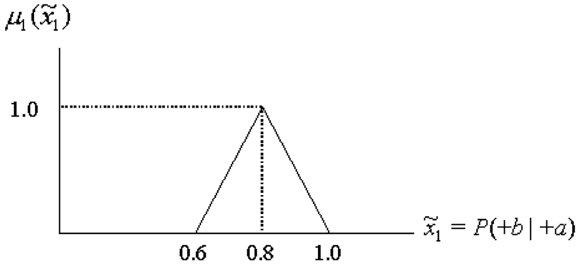

Fig. 2. The membership function $\mu_1(\tilde{x}_1)$ of $\tilde{x}_1$.

$(E = 1)$ but has not fallen into a coma $(D = 0)$; that is, compute $P(a| - d, +e)$, $P(b| - d, +e)$ and $P(c| - d, +e)$.

Current abductive reasoning methods [1,6–10] can solve Problem 1 successfully. However, if the parameters in Table 1 are fuzzy numbers, conventional methods may have difficulty in answering the queries. For instance, $P(+b| + a)$ cannot be 0.8 but rather is a fuzzy number, say $\tilde{x}_1$, where $\tilde{x}_1 = P(+b| + a)$, and is associated with a membership function $\mu_1(\tilde{x}_1)$, represented as follows (see Fig. 2).

$$\tilde{\mu}_1(\tilde{x}_1) = 5(\tilde{x}_1 - 0.6) - 5(|\tilde{x}_1 - 0.8| + \tilde{x}_1 - 0.8), \quad 0.6 \leqslant \tilde{x}_1 \leqslant 1,$$

where "$| * |$" denotes the absolute value of a term $*$.

The above expression and Fig. 2 mean that the interval of $\tilde{x}_1$ is between 0.6 and 1.0. If $\tilde{x}_1 = 0.8$ then $\mu_1(\tilde{x}_1) = 1$, implying that $\tilde{x}_1 = 0.8$ is the most possible situation. If $\tilde{x}_1 \leqslant 0.6$ or $\tilde{x}_1 \geqslant 1$ then $\mu_1(\tilde{x}_1) = 0$, the least-possible manifestation of $\tilde{x}_1$. If $\tilde{x}_1 = 0.7$, then $\mu_1(\tilde{x}_1) = 0.5$.

Fuzzy membership functions can be expressed in various ways. For example, denote $\tilde{x}_7 = P(+d| + b, -c)$ and express $\mu_7(\tilde{x}_7)$ as the following function (Fig. 3).

$$\mu_7(\tilde{x}_7) = 10(\tilde{x}_7 - 0.7) - 5(|\tilde{x}_7 - 0.8| + \tilde{x}_7 - 0.8) - 5(|\tilde{x}_7 - 0.85| + \tilde{x}_7 - 0.85), \quad 0.7 \leqslant \tilde{x}_7 \leqslant 0.95.$$

Here $\mu_7(\tilde{x}_7)$ is a trapezoid membership function and comprises four line segments, where $0.8 \leqslant \tilde{x}_7 \leqslant 0.85$ has the maximal membership. A fuzzy membership function is frequently a concave function.

This work defines the fuzzy parameters $\tilde{x}_i$, $i = 1, 2, \ldots, 8$, where $P(+b| + a) = \tilde{x}_1$, $P(+b| - a) = \tilde{x}_2$, $P(+c| + a) = \tilde{x}_3$, $P(+c| - a) = \tilde{x}_4$, $P(+d| + b, +c) = \tilde{x}_5$, $P(+d| - b, +c) = \tilde{x}_6$, $P(+d| + b, -c) = \tilde{x}_7$, and $P(+d| - b, -c) = \tilde{x}_8$. Table 2 lists the membership functions of the fuzzy parameters, among which

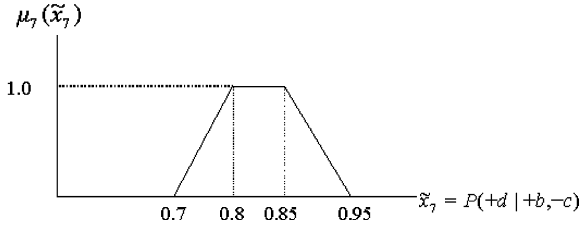

Fig. 3. The membership function $\mu_7(\tilde{x}_7)$ of $\tilde{x}_7$.

Table 2
The membership functions of fuzzy probabilities

Fuzzy parameter	$\mu_i(\tilde{x}_i)$	Domain of $\tilde{x}_i$
$\tilde{x}_1 = P(+b\|+a)$	$5(\tilde{x}_1 - 0.6) - 5(\|\tilde{x}_1 - 0.8\| + \tilde{x}_1 - 0.8)$	$[0.6, 1]$
$\tilde{x}_2 = P(+b\|-a)$	$10(\tilde{x}_2 - 0.1) - 10(\|\tilde{x}_2 - 0.2\| + \tilde{x}_2 - 0.2)$	$[0.1, 0.3]$
$\tilde{x}_3 = P(+c\|+a)$	$10(\tilde{x}_3 - 0.1) - 15(\|\tilde{x}_3 - 0.2\| + \tilde{x}_3 - 0.2)$	$[0.1, 0.25]$
$\tilde{x}_4 = P(+c\|-a)$	$25(\tilde{x}_4 - 0.01) - 17.5(\|\tilde{x}_4 - 0.05\| + \tilde{x}_4 - 0.05)$	$[0.01, 0.15]$
$\tilde{x}_5 = P(+d\|z_1)$	$5(\tilde{x}_5 - 0.6) - 5(\|\tilde{x}_5 - 0.8\| + \tilde{x}_5 - 0.8)$	$[0.6, 1]$
$\tilde{x}_6 = P(+d\|z_2)$	$10(\tilde{x}_6 - 0.7) - 10(\|\tilde{x}_6 - 0.8\| + \tilde{x}_6 - 0.8)$	$[0.7, 0.9]$
$\tilde{x}_7 = P(+d\|z_3)$	$10(\tilde{x}_7 - 0.7) - 5(\|\tilde{x}_7 - 0.8\| + \tilde{x}_7 - 0.8)$ $-5(\|\tilde{x}_7 - 0.85\| + \tilde{x}_7 - 0.85)$	$[0.7, 0.95]$
$\tilde{x}_8 = P(+d\|z_4)$	$25(\tilde{x}_8 - 0.01) - 12.5(\|\tilde{x}_8 - 0.05\| + \tilde{x}_8 - 0.05)$ $-25(\|\tilde{x}_8 - 0.07\| + \tilde{x}_8 - 0.07)$	$[0, 01.09]$

$\mu_7(\tilde{x}_7)$ and $\mu_8(\tilde{x}_8)$ are trapezoid membership functions while the remainder are triangular functions. Next, consider Problem 2.

Problem 2. Compute the belief distributions $P(a|-d,+c)$, $P(b|-d,+c)$, and $P(c|-d,+c)$, given the fuzzy membership functions in Table 2 and some constraints related to belief propagation.

Current abductive reasoning methods have difficulties in solving Problem 2 since it involves fuzzy information and extra constraints.

Consider abductive reasoning with constraints. For a given Bayesian network, knowledge workers (such as clinicians) may have professional judgments regarding the features of certain nodes and the relationships among them in particular diagnostic backgrounds. These features and relationships can take the form of various constraints.

(i) *Boundary constraints*: From additional information or observations, clinicians can infer that the posterior probability of A given $E=1$ and $D=0$ should be higher than 0.1 but lower than 0.3, which is expressed as

$$0.1 \leqslant P(+a|-d,+e) \leqslant 0.3. \tag{1.1}$$

(ii) *Functional dependency*: The beliefs of certain nodes are functionally dependent. For example, clinicians can judge that the posterior probability of B is roughly a certain multiple of that of

A given $E=1$ and $D=0$, which is expressed as

$$P(+a|-d,+e) \leqslant 2P(+b|-d,+e). \quad (1.2)$$

(iii) *Disjunctive constraints*: Sometimes disjunction may occur between nodes. For example, a doctor may estimate that either $P(+a|-d,+e)$ or $P(+b|-d,+e)$ is equal to or below 0.2, which is expressed as

$$\text{Either } P(+a|-d,+e) \leqslant 0.2 \quad \text{or} \quad P(+b|-d,+e) \leqslant 0.2. \quad (1.3)$$

By introducing these constraints into the reasoning system, the following problems are formulated.

Problem 2.1. Compute the belief distributions $P(a|-d,+c)$, $P(b|-d,+c)$, and $P(c|-d,+c)$, given the fuzzy membership functions in Table 2 and the following constraints.

$$0.1 \leqslant P(+a|-d,+e) \leqslant 0.3,$$

$$P(+b|-d,+e) \leqslant 2P(+c|-d,+e)$$

$$\text{Either } P(+a|-d,+e) \leqslant 0.2 \text{ or } P(+b|-d,+e) \leqslant 0.2.$$

Problem 2.1 is more complicated and difficult than Problem 1 when solved using current propagation methods.

This study develops a new approach for solving Problem 2.1 based on optimization techniques. Section 2 first presents the mathematical expressions of reasoning with fuzzy parameters as well as the techniques for linearizing the nonlinear absolute terms. Section 3 then formulates Problem 2.1 as a nonlinear program and introduces some constraints relating to belief propagation. Next, Section 4 illustrates several numerical examples. The final section presents some concluding remarks.

2. Posterior probabilities with fuzzy parameters

First this study reviews the conventional methods for computing the posterior probabilities with crisp parameters. Consider the Bayesian network in Fig. 1(a) with the crisp information in Table 1. Clustering [1,7] can transform Fig. 1(a) into an equivalent tree structure in Fig. 1(b), where nodes B and C are collapsed into a compound node $Z=B\&C$. Let $Z=\{z_1,z_2,z_3,z_4\}$ be a set of cardinalities of Z and $z_1=(+b,+c)$, $z_2=(-b,+c)$, $z_3=(+b,-c)$, and $z_4=(-b,-c)$. Moreover, let W_Y denote the state of all variables except Y; for example, $W_A=\{(z_1,-d+e),(z_2,-d+e),(z_3,-d+e),(z_4,-d+e)\}$. From Pearl [7], the value of $P(y|W_Y)$, which is the distribution of y conditioned on the value W_Y, can be calculated as below considering every instance of y.

$$P(+a|W_A)=\alpha_A P(+a)\sum_{i=1}^{4} P(z_i|+a)P(-d|z_i)P(+e|z_i),$$

$$P(-a|W_A)=\alpha_A P(-a)\sum_{i=1}^{4} P(z_i|-a)P(-d|z_i)P(+e|z_i),$$

$$P(+b|W_B) = \alpha_B \sum_{a=0}^{1} \left[P(a) \sum_{i=1,3} P(z_i|a)P(-d|z_i)P(+e|z_i) \right],$$

$$P(-b|W_B) = \alpha_B \sum_{a=0}^{1} \left[P(a) \sum_{i=2,4} P(z_i|a)P(-d|z_i)P(+e|z_i) \right],$$

$$P(+c|W_C) = \alpha_C \sum_{a=0}^{1} \left[P(a) \sum_{i=1,2} P(z_i|a)P(-d|z_i)P(+e|z_i) \right],$$

$$P(-c|W_C) = \alpha_C \sum_{a=0}^{1} \left[P(a) \sum_{i=3,4} P(z_i|a)P(-d|z_i)P(+e|z_i) \right], \tag{2.1}$$

where α_A, α_B, and α_C are the normalizing constant ensuring that

$$P(+a|W_A) + P(-a|W_A) = 1,$$

$$P(+b|W_B) + P(-b|W_B) = 1,$$

$$P(+c|W_C) + P(-c|W_C) = 1. \tag{2.2}$$

From (2.2), then intuitively

$$\alpha = \alpha_A = \alpha_B = \alpha_C = \frac{1}{\sum_{a=0}^{1} P(a) \sum_{i=1}^{4} P(z_i|a)P(-d|z_i)P(+e|z_i)} \tag{2.3}$$

and

$$\alpha \sum_{a} \sum_{z_i} P(a)P(z_i|a)P(-d|z_i)P(+e|z_i) = 1. \tag{2.4}$$

The value of $P(+a|W_A)$ in (2.1) is obtained below for the data in Table 1:

$$P(+a|W_A) = \alpha(0.2)[(0.8)(0.2)(1-0.8)(0.8) + (1-0.8)(0.2)(1-0.8)(0.8) + (0.8)(1-0.2)(1-0.8)(0.6) + (1-0.8)(1-0.2)(1-0.05)(0.6)].$$

Similarly,

$$P(-a|W_A) = \alpha(1-0.2)[(0.2)(0.05)(1-0.8)(0.8) + (1-0.2)(0.05)(1-0.8)(0.8) + (0.2)(1-0.05)(1-0.8)(0.6) + (1-0.2)(1-0.05)(1-0.05)(0.6)].$$

From (2.1) and (2.3), then $\alpha = 2.432$, $P(+a|W_A) = 0.097$, and $P(-a|W_A) = 0.903$.

The answers to Problem 1 are

$$P(a|-d,+e) = (0.097, 0.903),$$

$$P(b|-d,+e) = (0.097, 0.903),$$

$$P(c|-d,+e) = (0.031, 0.969).$$

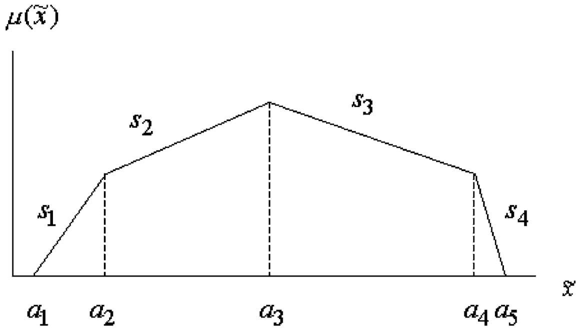

Fig. 4. A membership function of fuzzy probability.

The following illustrates another approach for calculating the posterior probabilities with fuzzy parameters.

Consider a membership function $\mu(\tilde{x})$ of $\tilde{x}$, as shown in Fig. 4. This piecewise linear function generally is expressed as

$$\mu(\tilde{x}) = \begin{cases} s_1(a_2 - a_1), & a_1 < \tilde{x} \leqslant a_2, \\ \mu(a_2) + s_2(a_3 - a_2), & a_2 < \tilde{x} \leqslant a_3, \\ \mu(a_3) + s_3(a_4 - a_3), & a_3 < \tilde{x} \leqslant a_4, \\ \mu(a_4) + s_4(a_5 - a_4), & a_4 < \tilde{x} \leqslant a_5, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.5}$$

Computing the above expression is complex. Consequently, this work employs an efficient method of expressing a piecewise linear function. Consider the following proposition.

Proposition 1. *Let $\mu(\tilde{x})$ denote the membership function of fuzzy variable $\tilde{x}$, as displayed in Fig. 4, where $a_j, j = 1, 2, \ldots, m$ represent the break points of $\mu(\tilde{x})$, and $s_j, j = 1, 2, \ldots, n$ are the slopes of line segments between a_j and a_{j+1}, and $\mu(\tilde{x})$ is the sum of absolute terms* [11,12]:

$$\mu(\tilde{x}) = f(a_1) + s_1(x - a_1) + \sum_{j=2}^{m} \frac{s_i - s_{i-1}}{2} (|x - a_j| + x - a_j). \tag{2.6}$$

Remark 1. For $\mu(\tilde{x})$ in (2.6), if and only if $s_i - s_{i-1} < 0$ for all $i = 2, 3, \ldots, n$, then $\mu(\tilde{x})$ is a concave function.

Remark 2. Generally, a fuzzy membership function $\mu(\tilde{x})$ in the reasoning system is expected to be maximized and is assumed to be a concave function.

If $\mu(\tilde{x})$ in (2.5) is a concave function and is to be maximized, then the following proposition is used for convenient linearization.

Proposition 2. *Maximizing a concave function $\mu(\tilde{x})$ in (2.6) requires solving the following linear program* [11,12]:

$$\begin{aligned}
\mathit{max}\quad & z = s_1(x-a_1) + 2\sum_{j=2}^{m} \frac{s_i - s_{i-1}}{2}\left(x - a_j + \sum_{k=1}^{j} d_k\right) \\
\mathit{s.t.}\quad & x + d_1 \geqslant a_2, \\
& x + d_1 + d_2 \geqslant a_3, \\
& \qquad \vdots \\
& x + d_1 + d_2 + \cdots + d_{m-1} \geqslant a_m, \\
& 0 \leqslant d_1 \leqslant a_2, \\
& 0 \leqslant d_{k-1} \leqslant a_k - a_{k-1}, \quad \mathit{for}\ k = 2,3,\ldots,m, \\
& x \in F(\mathit{feasible\ set}).
\end{aligned} \tag{2.7}$$

Proof. Since $d_k \leqslant a_k - a_{k-1}$, then clearly $x \geqslant a_k - (d_1 + d_2 + \cdots + d_{k-1} + d_k) \geqslant a_{k-1} - (d_1 + d_2 + \cdots + d_{k-1})$, so constraint $x + d_1 + d_2 + \cdots + d_{k-2} \geqslant a_{k-1}$ is converted by constraint $x + d_1 + d_2 + \cdots + d_{k-2} + d_{k-1} \geqslant a_k$, for $k = 2,3,\ldots,m$. □

From Proposition 2, the concave nonlinear membership functions are transformed into equivalent linear functions.

3. Abductive models with fuzzy parameters

Building upon Section 2, the abductive model for solving Problem 2.1 is formulated below.

Model 1.

$$\max \quad \sum_{i=1}^{8} \mu_i(\tilde{x}_i) \tag{3.1}$$

$$\begin{aligned}
\text{s.t.}\quad & (2.4), \\
& 0.1 \leqslant P(+a|-d,+e) \leqslant 0.3, \\
& P(+b|-d,+e) \leqslant 2P(+c|-d,+e), \\
& \text{Either } P(+a|-d,+e) \leqslant 0.2 \text{ or } P(+b|-d,+e) \leqslant 0.2,
\end{aligned} \tag{3.2}$$

where the objective function maximizes the sum of all fuzzy membership functions. Since (2.4) contains numerous nonseparate nonlinear terms, Model 1 is a highly nonlinear and nonconvex program. This work deals with the disjunctive constraint first and takes care of the nonlinear issue in Section 4.

Proposition 3. *A disjunctive constraint* $f(\tilde{x}) \leqslant 0$ *or* $g(\tilde{x}) \leqslant 0$ *can be expressed by the following inequalities*:

$$M(\theta_1 - 1) \leqslant f(\tilde{x}) \leqslant M\theta_1 + M(1 - \theta_2),$$

$$M(\theta_2 - 1) \leqslant g(\tilde{x}) \leqslant M\theta_2 + M(1 - \theta_1),$$

$$\varepsilon \leqslant \theta_2 + \theta_1 \leqslant 1, \tag{3.3}$$

where θ_1 *and* θ_2 *are* 0–1 *variables*, M *is a relatively large number*, *and* ε *is a relatively small positive number.*

The four possible combinations of θ_1 and θ_2 can be checked as follows: (1) for $\theta_1 = 1$, $\theta_2 = 1$ the constraints are $0 \leqslant f(\tilde{x}) \leqslant M$ and $0 \leqslant g(\tilde{x}) \leqslant M$, which are inactive constraints; (2) for $\theta_1 = 0$, $\theta_2 = 1$ then $-M \leqslant f(\tilde{x}) \leqslant 0$ and $0 \leqslant g(\tilde{x}) \leqslant 2M$, meaning that when $g(\tilde{x}) \geqslant 0$, $f(\tilde{x})$ must be 0 or less; (3) for $\theta_1 = 1$, $\theta_2 = 0$, the constraints are $0 \leqslant f(\tilde{x}) \leqslant 2M$ and $-M \leqslant g(\tilde{x}) \leqslant 0$, which implies that when $f(\tilde{x}) \geqslant 0$, $g(\tilde{x})$ must be 0 or less; (4) for $\theta_1 = 0$, $\theta_2 = 0$ the constraints become $-M \leqslant f(\tilde{x}) \leqslant M$ and $-M \leqslant g(\tilde{x}) \leqslant M$, which are inactive constraints. The third constraint in (3.3) excludes the combinations $\theta_1 = 1$, $\theta_2 = 1$ and $\theta_1 = 0$, $\theta_2 = 0$. To summarize, (3.3) implies that either $f(\tilde{x}) \leqslant 0$ or $g(\tilde{x}) \leqslant 0$ must be satisfied.

4. Numerical examples

Abductive reasoning problems in certain applications are solved below using the proposed constrained optimization approach.

Example 1. Problem 2.1 is solved using the following program:

$$\max \quad \sum_{i=1}^{8} \mu_i(\tilde{x}_i)$$

$$\text{s.t.} \quad \mu_1(\tilde{x}_1) = 5(\tilde{x}_1 - 0.6) - 5(|\tilde{x}_1 - 0.8| + \tilde{x}_1 - 0.8),$$

$$\mu_2(\tilde{x}_2) = 10(\tilde{x}_2 - 0.1) - 10(|\tilde{x}_2 - 0.2| + \tilde{x}_2 - 0.2),$$

$$\mu_3(\tilde{x}_3) = 10(\tilde{x}_3 - 0.1) - 15(|\tilde{x}_3 - 0.2| + \tilde{x}_3 - 0.2),$$

$$\mu_4(\tilde{x}_4) = 25(\tilde{x}_4 - 0.01) - 17.5(|\tilde{x}_4 - 0.05| + \tilde{x}_4 - 0.05),$$

$$\mu_5(\tilde{x}_5) = 5(\tilde{x}_5 - 0.6) - 5(|\tilde{x}_5 - 0.8| + \tilde{x}_5 - 0.8),$$

$$\mu_6(\tilde{x}_6) = 10(\tilde{x}_6 - 0.7) - 10(|\tilde{x}_6 - 0.8| + \tilde{x}_6 - 0.8),$$

$$\mu_7(\tilde{x}_7) = 10(\tilde{x}_7 - 0.7) - 5(|\tilde{x}_7 - 0.8| + \tilde{x}_7 - 0.8) - 5(|\tilde{x}_7 - 0.85| + \tilde{x}_7 - 0.85),$$

$$\mu_8(\tilde{x}_8) = 25(\tilde{x}_8 - 0.01) - 12.5(|\tilde{x}_8 - 0.05| + \tilde{x}_8 - 0.05) - 25(|\tilde{x}_8 - 0.07| + \tilde{x}_8 - 0.07), \tag{4.1}$$

$$\alpha[0.2\tilde{x}_1\tilde{x}_3(1-\tilde{x}_5)0.8+0.2(1-\tilde{x}_1)\tilde{x}_3(1-\tilde{x}_6)0.8$$
$$+0.2\tilde{x}_1(1-\tilde{x}_3)(1-\tilde{x}_7)0.6+0.2(1-\tilde{x}_1)(1-\tilde{x}_3)(1-\tilde{x}_8)0.6$$
$$+0.8\tilde{x}_2\tilde{x}_4(1-\tilde{x}_5)0.8+0.8(1-\tilde{x}_2)\tilde{x}_4(1-\tilde{x}_6)0.8$$
$$+0.8\tilde{x}_2(1-\tilde{x}_4)(1-\tilde{x}_7)0.6+0.8(1-\tilde{x}_2)(1-\tilde{x}_4)(1-\tilde{x}_8)0.6]=1,$$
$$0.1\leqslant P(+a|-d,+e)\leqslant 0.3,$$
$$P(+b|-d,+e)\leqslant 2P(+c|-d,+e),$$
$$\text{Either } P(+a|-d,+e)\leqslant 0.2 \text{ or } P(+b|-d,+e)\leqslant 0.2,$$
$$\tilde{x}_i\in F(\text{ feasible set}). \tag{4.2}$$

First (4.1) is linearized using Proposition 2 and then the initial program is altered into the equivalent program as follows.

$$\max \quad \sum_{i=1}^{8}\mu_i(\tilde{x}_i)$$
$$\text{s.t.} \quad \mu_1(\tilde{x}_1)=5(\tilde{x}_1-0.6)-2[5(\tilde{x}_1-0.8+d_1)],$$
$$\mu_2(\tilde{x}_2)=10(\tilde{x}_2-0.1)-2[10(\tilde{x}_2-0.2+d_2)],$$
$$\mu_3(\tilde{x}_3)=10(\tilde{x}_3-0.1)-2[15(\tilde{x}_3-0.2+d_3)],$$
$$\mu_4(\tilde{x}_4)=25(\tilde{x}_4-0.01)-2[17.5(\tilde{x}_4-0.05+d_4)],$$
$$\mu_5(\tilde{x}_5)=5(\tilde{x}_5-0.6)-2[5(\tilde{x}_5-0.8+d_5)],$$
$$\mu_6(\tilde{x}_6)=10(\tilde{x}_6-0.7)-2[10(\tilde{x}_6-0.8+d_6)],$$
$$\mu_7(\tilde{x}_7)=10(\tilde{x}_7-0.7)-2[5(\tilde{x}_7-0.8+d_{71})+5(\tilde{x}_7-0.85+d_{71}+d_{72})],$$
$$\mu_8(\tilde{x}_8)=25(\tilde{x}_8-0.01)-2[12.5(\tilde{x}_8-0.05+d_{81})+25(\tilde{x}_{82}-0.07+d_{81}+d_{82})],$$
$$\tilde{x}_1+d_1\geqslant 0.8, \quad 0\leqslant d_1\leqslant 0.8,$$
$$\tilde{x}_2+d_2\geqslant 0.2, \quad 0\leqslant d_2\leqslant 0.2,$$
$$\tilde{x}_3+d_3\geqslant 0.2, \quad 0\leqslant d_3\leqslant 0.2,$$
$$\tilde{x}_4+d_4\geqslant 0.05, \quad 0\leqslant d_4\leqslant 0.05,$$
$$\tilde{x}_5+d_5\geqslant 0.8, \quad 0\leqslant d_5\leqslant 0.8,$$
$$\tilde{x}_6+d_6\geqslant 0.8, \quad 0\leqslant d_6\leqslant 0.8,$$
$$\tilde{x}_7+d_{71}+d_{72}\geqslant 0.85, \quad 0\leqslant d_{71}\leqslant 0.8, \quad 0\leqslant d_{72}\leqslant 0.05,$$
$$\tilde{x}_8+d_{81}+d_{82}\geqslant 0.07, \quad 0\leqslant d_{81}\leqslant 0.05, \quad 0\leqslant d_{82}\leqslant 0.02,$$
$$(4.2) \tag{4.3}$$

Table 3
Solution table of Example 1

Objective value		3.0240	
$BEL(a+)$		0.1058	
$BEL(b+)$		0.20	
$BEL(c+)$		0.10	
$\tilde{x}_1$	0.9600	$\mu(\tilde{x}_1)$	0.2000
$\tilde{x}_2$	0.2800	$\mu(\tilde{x}_2)$	0.2000
$\tilde{x}_3$	0.2168	$\mu(\tilde{x}_3)$	0.6643
$\tilde{x}_4$	0.1300	$\mu(\tilde{x}_4)$	0.2000
$\tilde{x}_5$	0.6400	$\mu(\tilde{x}_5)$	0.2000
$\tilde{x}_6$	0.7200	$\mu(\tilde{x}_6)$	0.2000
$\tilde{x}_7$	0.7360	$\mu(\tilde{x}_7)$	0.3596
$\tilde{x}_8$	0.0700	$\mu(\tilde{x}_8)$	1.0000

To ensure belief propagation the lower bound of the membership functions is set at 0.2; that is, the membership of every fuzzy parameter must equal or exceed 0.2, which excludes scenarios involving poorly estimated parameters.

LINGO 8.0 solves (4.3) in less than 1 s. The solutions are $\alpha = 2.6642$ and

$$P(+a|+d,-e) = \alpha P(+a)\sum_{i=1}^{4} P(z_i|+a)P(-d|z_i)P(+e|z_i) = 0.1058,$$

$$P(+b|+d,-e) = \alpha \sum_{a=0,1}\left[P(a)\sum_{i=1,3} P(z_i|a)P(-d|z_i)P(+e|z_i)\right] = 0.20,$$

$$P(+c|+d,-e) = \sum_{a=0,1}^{+a}\left[P(a)\sum_{i=1,2} P(z_i|a)P(-d|z_i)P(+e|z_i)\right] = 0.1.$$

Table 3 lists the detailed solutions.

The results of this model differ from those for Problem 1. For example, $P(+a|+d,-e)$ changes from 0.097 to 0.1058, $P(+b|+d,-e)$ from 0.097 to 0.2, and $P(+b|+d,-e)$ from 0.031 to 0.1. This variance results from the constraints that dominate the belief propagation. Readers may have deduced that Problem 1 can be considered a special case in which every membership of the fuzzy parameters converges on 1. Under certain circumstances, knowledge workers may need to compromise among diverse, even conflicting information sources, causing fuzzy parameters to differ from their most possible values.

Example 2 (Bacterial infections). The following is another example involving urinary tract infections simplified from Leibovici et al. [13]. From Fig. 5, this example uses a Bayesian network as the

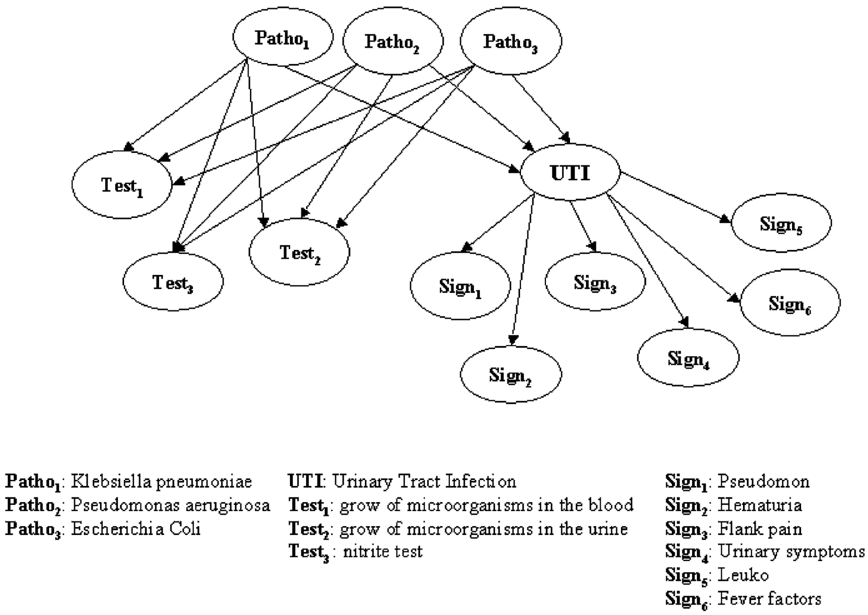

Fig. 5. A Bayesian network of Urinary Tract Infection [13].

knowledge base for the bacterial infections. To ensure simplicity without loss of generality, all random nodes are assumed to be binary and organized accordingly. Table 4 lists the conditional probability distributions of the variables, including eight fuzzy parameters listed in Table 5. The random nodes and their states are briefly introduced below.

Pathogens. $patho_i$: They are microorganisms that cause urinary tract infection. This example involves three pathogens: $patho_1$ (*Klebsiella pneumoniae*) $patho_2$ (*Pseudomonas aeruginosa*) and $patho_3$ (*Escherichia coli*). The manifestation states are severe ($patho_i = 1$) and mild ($patho_i = 0$).

Urinary tract infection. uti: The states are severe (uti $= 1$) and mild (uti $= 0$).

Signs and symptoms of urinary tract infection. $sign_i$: These signs are manifestations possibly originating from uti. Fig. 2 presents six possible signs: $sign_1$ (suprapubic pain), $sign_2$ (frequent micturition), $sign_3$ (flank pain), $sign_4$ (Urinary symptoms), $sign_5$ (serum albumin) and $sign_6$ (fever). The manifestation states are present ($sign_i = 1$) and absent ($sign_i = 0$).

Bacteriological tests. $test_i$: $test_1$ (growth of microorganisms in the blood), $test_2$ (growth of microorganisms in the urine) and $test_3$ (nitrite test). The outcomes of bacteriological tests are positive ($test_i = 1$) and negative ($test_i = 0$).

Consider a scenario in which a patient is suffering frequent micturition ($sign_2 = 1$), flank pain ($sign_3 = 1$) and urinary symptoms ($sign_4 = 1$), but has not fallen into suprapubic pain ($sign_1 = 0$), serum albumin ($sign_5 = 0$) or fever ($sign_6 = 0$). Moreover, the bacteriological tests display positive results in both $test_1$ and $test_3$, but negative results in $test_2$. The evidence set $E = \{e\} = \{sign_1 = 0, sign_2 = 1, sign_3 = 1, sign_4 = 1, sign_5 = 0, sign_6 = 0, test_1 = 1, test_2 = 1, test_3 = 0\}$. The next step is to compute the belief distribution of every $patho_1$, $patho_2$, $patho_3$ and uti.

Table 4
The conditional probability distribution of Example 2

$P(+patho_1) = 0.1$	
$P(+patho_2) = 0.09$	
$P(+patho_3) = 0.09$	
$P(+uti \mid +patho_1, +patho_2, +patho_3) = \tilde{x}_{21}$	
$P(+uti \mid +patho_1, -patho_2, +patho_3) = \tilde{x}_{22}$	
$P(+uti \mid +patho_1, +patho_2, -patho_3) = \tilde{x}_{23}$	
$P(+uti \mid +patho_1, -patho_2, -patho_3) = \tilde{x}_{24}$	
$P(+uti \mid -patho_1, +patho_2, +patho_3) = \tilde{x}_{25}$	
$P(+uti \mid -patho_1, -patho_2, +patho_3) = \tilde{x}_{26}$	
$P(+uti \mid -patho_1, +patho_2, -patho_3) = \tilde{x}_{27}$	
$P(+uti \mid -patho_1, -patho_2, -patho_3) = \tilde{x}_{28}$	
$P(+sign_1 \mid +uti) = 0.6$	$P(+sign_1 \mid -uti) = 0.01$
$P(+sign_2 \mid +uti) = 0.9$	$P(+sign_2 \mid -uti) = 0.10$
$P(+sign_3 \mid +uti) = 0.6$	$P(+sign_3 \mid -uti) = 0.05$
$P(+sign_4 \mid +uti) = 0.8$	$P(+sign_4 \mid -uti) = 0.05$
$P(+sign_5 \mid +uti) = 0.6$	$P(+sign_5 \mid -uti) = 0.10$
$P(+sign_6 \mid +uti) = 0.7$	$P(+sign_6 \mid -uti) = 0.01$
$P(+test_i \mid +patho_1) = 0.7$	$P(+test_i \mid -patho_1) = 0.1$
$P(+test_i \mid +patho_2) = 0.9$	$P(+test_i \mid -patho_2) = 0.1$
$P(+test_i \mid +patho_3) = 0.8$	$P(+test_i \mid -patho_3) = 0.2$

Table 5
The membership functions of fuzzy probabilities in Example 2

Fuzzy parameter	$\mu_{2i}(\tilde{x}_{2i})$	Domain of $\tilde{x}_{2i}$
$\tilde{x}_{21}$	$5(\tilde{x}_{21} - 0.6) - 5(\lvert\tilde{x}_{21} - 0.8\rvert + \tilde{x}_{21} - 0.8)$	$[0.6, 1]$
$\tilde{x}_{22}$	$10(\tilde{x}_{22} - 0.7) - 10(\lvert\tilde{x}_{22} - 0.8\rvert + \tilde{x}_{22} - 0.2)$	$[0.7, 0.9]$
$\tilde{x}_{23}$	$20(\tilde{x}_{23} - 0.7) - 20(\lvert\tilde{x}_{23} - 0.75\rvert + \tilde{x}_{23} - 0.75)$	$[0.7, 0.8]$
$\tilde{x}_{24}$	$10(\tilde{x}_{24} - 0.5) - 10(\lvert\tilde{x}_{24} - 0.7\rvert + \tilde{x}_{24} - 0.6)$	$[0.5, 0.7]$
$\tilde{x}_{25}$	$10(\tilde{x}_{25} - 0.7) - 10(\lvert\tilde{x}_{25} - 0.8\rvert + \tilde{x}_{25} - 0.8)$	$[0.7, 0.9]$
$\tilde{x}_{26}$	$20(\tilde{x}_{26} - 0.55) - 20(\lvert\tilde{x}_{26} - 0.6\rvert + \tilde{x}_{26} - 0.6)$	$[0.55, 0.65]$
$\tilde{x}_{27}$	$10(\tilde{x}_{27} - 0.4) - 10(\lvert\tilde{x}_{27} - 0.5\rvert + \tilde{x}_{27} - 0.5)$	$[0.4, 0.6]$
$\tilde{x}_{28}$	$100(\tilde{x}_{28}) - 100(\lvert\tilde{x}_{28} - 0.01\rvert + \tilde{x}_{28} - 0.01)$	$[0, 0.02]$

Based on observation, the clinician inferred that the belief that pathogen 3 is active ranges from 0.3 to 0.5. Moreover, since pathogens 1 and 2 are complementary, either pathogen 1 or 2 has the belief of 0.5 to be active. The following implements the above problem as the following model.

$$\max \quad \sum_{i=1}^{8} \mu_{2i}(\tilde{x}_{2i})$$

$$
\begin{aligned}
\text{s.t.}\quad & \mu_{21}(\tilde{x}_{21}) = 5(\tilde{x}_{21} - 0.6) - 5(|\tilde{x}_{21} - 0.8|\tilde{x}_{21} - 0.8),\\
& \mu_{22}(\tilde{x}_{22}) = 10(\tilde{x}_{22} - 0.7) - 10(|\tilde{x}_{22} - 0.8| + \tilde{x}_{22} - 0.8),\\
& \mu_{23}(\tilde{x}_{23}) = 20(\tilde{x}_{23} - 0.7) - 20(|\tilde{x}_{23} - 0.75| + \tilde{x}_{23} - 0.75),\\
& \mu_{24}(\tilde{x}_{24}) = 10(\tilde{x}_{24} - 0.5) - 10(|\tilde{x}_{24} - 0.6| + \tilde{x}_{24} - 0.6),\\
& \mu_{25}(\tilde{x}_{25}) = 10(\tilde{x}_{25} - 0.7) - 10(|\tilde{x}_{25} - 0.8| + \tilde{x}_{25} - 0.8),\\
& \mu_{26}(\tilde{x}_{26}) = 20(\tilde{x}_{26} - 0.55) - 20(|\tilde{x}_{26} - 0.6| + \tilde{x}_{26} - 0.6),\\
& \mu_{27}(\tilde{x}_{27}) = 10(\tilde{x}_{27} - 0.4) - 10(|\tilde{x}_{27} - 0.5| + \tilde{x}_{27} - 0.5),\\
& \mu_{28}(\tilde{x}_{28}) = 100(\tilde{x}_{28} - 0) - 100(|\tilde{x}_{28} - 0.01| + \tilde{x}_{28} - 0.01),\\
& \alpha \sum_{patho_1} \sum_{patho_2} \sum_{patho_3} \sum_{uti} \Bigg[P(patho_1)P(patho_2)P(patho_3)\\
& \quad \times P(uti \mid patho_1, patho_2, patho_3) \prod_j P(sign_j \mid patho_1, patho_2, patho_3)\\
& \quad \times \prod_k P(test_k \mid patho_1, patho_2, patho_3) \Bigg] = 1,\\
& 0.3 \leqslant P(patho_3 \mid e) \leqslant 0.5,\\
& \text{Either } P(patho_1 \mid e) \geqslant 0.5 \text{ or } P(patho_2 \mid e) \geqslant 0.5.
\end{aligned} \tag{4.4}
$$

After linearizing the nonlinear concave membership functions, (4.4) is converted into (4.5).

$$
\begin{aligned}
\max\quad & \sum_{i=1}^{8} \mu_{2i}(\tilde{x}_{2i})\\
\text{s.t.}\quad & \mu_{\tilde{x}_{21}}(\tilde{x}_{21}) = 5(\tilde{x}_{21} - 0.6) - 2[5(\tilde{x}_{21} - 0.8 + d_{21})],\\
& \mu_{\tilde{x}_{22}}(\tilde{x}_{22}) = 10(\tilde{x}_{22} - 0.7) - 2[10(\tilde{x}_{22} - 0.8 + d_{22})],\\
& \mu_{\tilde{x}_{23}}(\tilde{x}_{23}) = 20(\tilde{x}_{23} - 0.7) - 2[20(\tilde{x}_{23} - 0.75 + d_{23})],\\
& \mu_{\tilde{x}_{24}}(\tilde{x}_{24}) = 10(\tilde{x}_{24} - 0.5) - 2[10(\tilde{x}_{24} - 0.6 + d_{24})],\\
& \mu_{\tilde{x}_{25}}(\tilde{x}_{25}) = 10(\tilde{x}_{25} - 0.7) - 2[10(\tilde{x}_{25} - 0.8 + d_{25})],\\
& \mu_{\tilde{x}_{26}}(\tilde{x}_{26}) = 20(\tilde{x}_{26} - 0.55) - 2[20(\tilde{x}_{26} - 0.6 + d_{26})],\\
& \mu_{\tilde{x}_{27}}(\tilde{x}_{27}) = 10(\tilde{x}_{27} - 0.4) - 2[10(\tilde{x}_{27} - 0.5 + d_{27})],\\
& \mu_{\tilde{x}_{28}}(\tilde{x}_{28}) = 100(\tilde{x}_{28} - 0) - 2[100(\tilde{x}_{28} - 0.01 + d_{28})],\\
& \qquad \vdots
\end{aligned}
$$

Table 6
Solution table of Example 2

Objective value		8	
$BEL(patho_1+)$		0.7784	
$BEL(patho_2+)$		0.5630	
$BEL(patho_3+)$		0.3396	
$BEL(uti+)$	0.9924		
$\tilde{x}_{21}$	0.8	$\mu_{21}(\tilde{x}_{21})$	1
$\tilde{x}_{22}$	0.8	$\mu_{22}(\tilde{x}_{22})$	1
$\tilde{x}_{23}$	0.75	$\mu_{23}(\tilde{x}_{23})$	1
$\tilde{x}_{24}$	0.6	$\mu_{24}(\tilde{x}_{24})$	1
$\tilde{x}_{25}$	0.8	$\mu_{25}(\tilde{x}_{25})$	1
$\tilde{x}_{26}$	0.6	$\mu_{26}(\tilde{x}_{26})$	1
$\tilde{x}_{27}$	0.5	$\mu_{27}(\tilde{x}_{27})$	1
$\tilde{x}_{28}$	0.01	$\mu_{28}(\tilde{x}_{28})$	1

$$\alpha \sum_{patho_1} \sum_{patho_2} \sum_{patho_3} \sum_{uti} \Big[P(patho_1)P(patho_2)P(patho_3)$$

$$\times P(uti|patho_1, patho_2, patho_3) \prod_j P(sign_j|patho_1, patho_2, patho_3)$$

$$\times \prod_k P(test_k|patho_1, patho_2, patho_3) \Big] = 1,$$

$$0.3 \leqslant P(patho_3|e) \leqslant 0.5,$$

$$\text{Either } P(patho_1|e) \geqslant 0.5 \text{ or } P(patho_2|e) \geqslant 0.5. \tag{4.5}$$

With LINGO 8.0, approximately 5 s are required to obtain the solution. The calculation results are

$$\alpha = 5266017,$$

$$P(patho_1+|e) = 0.7784,$$

$$P(patho_2+|e) = 0.5630,$$

$$P(patho_3+|e) = 0.3396,$$

$$P(uti+|e) = 0.9924.$$

Table 6 summarizes the detailed results.

Example 3 (Just-in-time techniques and firm performance). The third example uses the Bayesian network to model the relationship between just-in-time purchasing (JITP) techniques and firm performance [14]. JITP is an important component of supply chain management in managing inventory flows. Several key factors link the JITP process and firm performance, and Fig. 6 models the relationships among these factors. Tables 7 and 8 summarize the probability distributions of the nodes

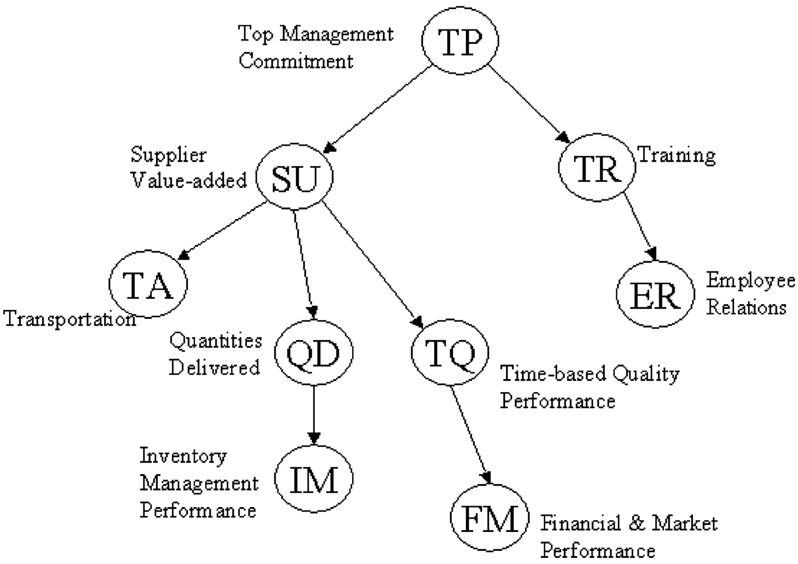

Fig. 6. A Bayesian network of the relationships between JITP techniques and performance measures [14].

Table 7
The conditional probability distribution of Example 3

$P(tp+) = \tilde{x}_{31}$	
$P(su+\|tp+) = \tilde{x}_{32}$	$P(su+\|tp-) = \tilde{x}_{33}$
$P(tr+\|tp+) = \tilde{x}_{34}$	$P(tr+\|tp-) = \tilde{x}_{35}$
$P(ta+\|su+) = 0.7$	$P(ta+\|su-) = 0.1$
$P(qd+\|su+) = 0.8$	$P(qd+\|su-) = 0.3$
$P(im+\|qd+) = 0.3$	$P(im+\|qd-) = 0.1$
$P(tq+\|su+) = 0.4$	$P(tq+\|su-) = 0.05$
$P(fm+\|tq+) = 0.7$	$P(fm+\|tq-) = 0.1$
$P(er+\|tr+) = 0.6$	$P(er+\|tr-) = 0.1$

Table 8
The membership functions of fuzzy probabilities in Example 3

Fuzzy parameter	$\mu_{3i}(\tilde{x}_{3i})$	Domain of $\tilde{x}_{3i}$
$\tilde{x}_{31}$	$5(\tilde{x}_{31} - 0.1) - 7.5(\lvert\tilde{x}_{31} - 0.3\rvert + \tilde{x}_{31} - 0.3)$	$[0.1, 0.4]$
$\tilde{x}_{32}$	$5(\tilde{x}_{32} - 0.4) - 5(\lvert\tilde{x}_{32} - 0.6\rvert + \tilde{x}_{32} - 0.6)$	$[0.4, 0.8]$
$\tilde{x}_{33}$	$20(\tilde{x}_{33} - 0.05) - 20(\lvert\tilde{x}_{33} - 0.75\rvert + \tilde{x}_{33} - 0.1)$	$[0.05, 0.15]$
$\tilde{x}_{34}$	$10(\tilde{x}_{34} - 0.5) - 5(\lvert\tilde{x}_{34} - 0.6\rvert + \tilde{x}_{34} - 0.6) - 5(\lvert\tilde{x}_{34} - 0.7\rvert + \tilde{x}_{34} - 0.7)$	$[0.5, 0.8]$
$\tilde{x}_{35}$	$10(\tilde{x}_{35} - 0.1) - 5(\lvert\tilde{x}_{35} - 0.2\rvert + \tilde{x}_{35} - 0.2) - 5(\lvert\tilde{x}_{35} - 0.3\rvert + \tilde{x}_{35} - 0.3)$	$[0.1, 0.4]$

and fuzzy parameters. This study hypothesizes a scenario in which inventory management performance is good ($im+$), employ relationship is poor ($er-$), transportation performance is good ($ta+$), and financial and market performance is poor ($fm-$). The problem involves calculating the belief distribution of all unknown nodes, top management commitment (tp), supplier value-added (su), training (tr), quantity delivered (qd), and time-based quality performance (tq). The reasoning model is formulated as (4.6).

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{5} \mu_{3i}(\tilde{x}_{3i}) \\
\text{s.t.} \quad & \mu_{31}(\tilde{x}_{31}) = 5(\tilde{x}_{31} - 0.1) - 7.5(|\tilde{x}_{31} - 0.3| + \tilde{x}_{31} - 0.3), \\
& \mu_{32}(\tilde{x}_{32}) = 5(\tilde{x}_{32} - 0.4) - 5(|\tilde{x}_{32} - 0.6| + \tilde{x}_{32} - 0.6), \\
& \mu_{33}(\tilde{x}_{33}) = 20(\tilde{x}_{33} - 0.05) - 20(|\tilde{x}_{33} - 0.75| + \tilde{x}_{33} - 0.1), \\
& \mu_{34}(\tilde{x}_{34}) = 10(\tilde{x}_{34} - 0.5) - 5(|\tilde{x}_{34} - 0.6| + \tilde{x}_{34} - 0.6) - 5(|\tilde{x}_{34} - 0.7| + \tilde{x}_{34} - 0.7), \\
& \mu_{35}(\tilde{x}_{35}) = 10(\tilde{x}_{35} - 0.1) - 5(|\tilde{x}_{35} - 0.2| + \tilde{x}_{35} - 0.2) - 5(|\tilde{x}_{35} - 0.3| + \tilde{x}_{35} - 0.3), \\
& \alpha \sum_{tp}\sum_{su}\sum_{tr}\sum_{qd}\sum_{tq} [P(tp)P(tr|tp)P(su|tp)P(ta+|su)P(qd|su)P(tq|su) \\
& \quad \times P(er-|tr)P(im+|qd)P(fm-|tq)] = 1, \\
& P(tp+|ta+, er-, im+, fm-) > 0.6, \\
& P(su+|ta+, er-, im+, fm-) > 0.8.
\end{aligned} \tag{4.6}
$$

First the nonlinear concave membership functions are linearized, yielding (4.7).

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{5} \mu_{\tilde{x}_{3i}}(\tilde{x}_{3i}) \\
\text{s.t.} \quad & \mu_{\tilde{x}_{31}}(\tilde{x}_{31}) = 5(\tilde{x}_{31} - 0.1) - 2[7.5(\tilde{x}_{31} - 0.3 + d_{31})], \\
& \mu_{\tilde{x}_{32}}(\tilde{x}_{32}) = 5(\tilde{x}_{32} - 0.4) - 2[7.5(\tilde{x}_{32} - 0.6 + d_{32})], \\
& \mu_{\tilde{x}_{33}}(\tilde{x}_{33}) = 20(\tilde{x}_{33} - 0.05) - 2[20(\tilde{x}_{33} - 0.1 + d_{33})], \\
& \mu_{\tilde{x}_{34}}(\tilde{x}_{34}) = 10(\tilde{x}_{34} - 0.5) - 2[5(\tilde{x}_{34} - 0.6 + d_{341}) + 5(\tilde{x}_{34} - 0.7 + d_{342})], \\
& \mu_{\tilde{x}_{35}}(\tilde{x}_{35}) = 10(\tilde{x}_{35} - 0.1) - 2[5(\tilde{x}_{35} - 0.2 + d_{351}) + 5(\tilde{x}_{35} - 0.3 + d_{352})], \\
& \quad \vdots \\
& \alpha \sum_{tp}\sum_{su}\sum_{tr}\sum_{qd}\sum_{tq} [P(tp)P(tr|tp)P(su|tp)P(ta+|su)P(qd|su)P(tq|su) \\
& \quad \times P(er-|tr)P(im+|qd)P(fm-|tq)] = 1,
\end{aligned}
$$

Table 9
Solution table of Example 3

Objective value		3.4101	
$BEL(tp+)$		0.6480	
$BEL(su+)$		0.8	
$BEL(tr+)$		0.3155	
$BEL(qd+)$		0.8510	
$BEL(tq+)$		0.1489	
$\tilde{x}_{31}$	0.3680	$\mu_{31}(\tilde{x}_{31})$	0.3196
$\tilde{x}_{32}$	0.7819	$\mu_{32}(\tilde{x}_{32})$	0.0905
$\tilde{x}_{33}$	0.1	$\mu_{33}(\tilde{x}_{33})$	1
$\tilde{x}_{34}$	0.6	$\mu_{34}(\tilde{x}_{34})$	1
$\tilde{x}_{35}$	0.3	$\mu_{35}(\tilde{x}_{35})$	1

$$P(tp+|ta+, er-, im+, fm-) > 0.6,$$

$$P(su+|ta+, er-, im+, fm-) > 0.8. \tag{4.7}$$

LINGO 8.0 solves the above program in approximately 5 s, obtaining the following results:

$$\alpha = 30.2648,$$

$$P(tp+|ta+, er-, im+, fm-) = 0.6480,$$

$$P(su+|ta+, er-, im+, fm-) = 0.8,$$

$$P(tr+|ta+, er-, im+, fm-) = 0.3155,$$

$$P(qd+|ta+, er-, im+, fm-) = 0.8510,$$

$$P(tq+|ta+, er-, im+, fm-) = 0.1489.$$

Table 9 lists the details.

5. Conclusions

This study develops a nonlinear programming model for dealing with constrained abductive reasoning on Bayesian networks. This model can be built on any exact propagation methods in Bayesian networks. The present study involves some fuzzy parameters and certain extra constraints. Optimization techniques, including piecewise linearization, are adopted to solve this nonlinear programming model and obtain optimal solutions to the abductive reasoning problems. Since the constraints in this model are extremely nonlinear, and numerous nonseparable terms are involved, local optima are obtained at the present stage. To enhance the solution quality, some global optimization techniques [11,12,15] can be further used for extended studies. Simultaneously, various reasoning-related constraints are considered, including boundary constraints, dependency and disjunctive constraints. Compared to traditional methods that deal with constraints by dummy auxiliary nodes [5,7], this optimization model of abduction avoids network restructuring. All extra information related to reasoning is considered to be additional constraints in the proposed nonlinear program. We hope that the approach presented here contributes to the field of probabilistic reasoning with constraints and fuzzy information.

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