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Power partial isometry index and ascent of a finite matrix

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We give a complete characterization of nonnegative integers j and k and a positive integer n for which there is an n-by-n matrix with its power partial isometry index equal to j and its ascent equal to k. Recall that the power partial isometry index $p(A)$ of a matrix A is the supremum, possibly infinity, of nonnegative integers j such that $I, A, A^2, \dots, A^j$ are all partial isometries while the ascent $a(A)$ of A is the smallest integer $k \geq 0$ for which $\ker A^k$ equals $\ker A^{k+1}$. It was known before that, for any matrix A, either $p(A) \leq \min\{a(A), n-1\}$ or $p(A) = \infty$. In this paper, we prove more precisely that there is an n-by-n matrix A such that $p(A) = j$ and $a(A) = k$ if and only if one of the following conditions holds: (a) $j = k \leq n-1$, (b) $j \leq k-1$ and $j+k \leq n-1$, or (c) $j \leq k-2$ and $j+k = n$. This answers a question we asked in a previous paper.

1. Introduction

Let A be an n-by-n complex matrix. The *power partial isometry index* $p(A)$ of A is, by definition, the supremum of the nonnegative integers j for which $I, A, A^2, \dots, A^j$ are all

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partial isometries. (A^0 is understood to be I, even for $A = 0$.) Recall that A is a *partial isometry* if $\|Ax\| = \|x\|$ for all vectors x of $\mathbb{C}^n$ which are in the orthogonal complement $(\ker A)^\perp$ of $\ker A$. The *ascent* $a(A)$ of A is the smallest nonnegative integer k for which $\ker A^k = \ker A^{k+1}$. The relation between these two parameters of A was first explored in [1]. In particular, it was shown in [1, Corollary 2.5] that $0 \le p(A) \le \min\{a(A), n-1\}$ or $p(A) = \infty$. We asked in [1, p. 339] whether such conditions on $p(A)$ and $a(A)$ guarantee their attainment by some n-by-n matrix A. In this paper, we show that this is not always the case. It turns out that the situation is more delicate than what we have expected. More precisely, we prove that, for nonnegative integers j and k and a positive integer n, there is an n-by-n matrix A with $p(A) = j$ and $a(A) = k$ if and only if one of the following three conditions holds: (a) $j = k \le n-1$, (b) $j \le k-1$ and $j+k \le n-1$, or (c) $j \le k-2$ and $j+k = n$. This settles the question completely. The proof of it depends on the special matrix representation, under unitary similarity, of a matrix A for which $A, A^2, \ldots, A^j$ are all partial isometries for a certain j, $1 \le j \le \infty$ (cf. [1, Theorems 2.2 and 2.4]). We will review the necessary ingredients from [1] in Section 2 below. Section 3 then gives the proof of our main result.

Partial isometries were first studied in [3] and their properties have since been summarized in [2, Chapter 15]. Power partial isometries were considered first in [4].

2. Preliminaries

We start with the following result from [1, Theorem 2.2].

Theorem 2.1. *Let A be an n-by-n nonzero matrix and $1 \le j \le a(A)$. Then $A, A^2, \ldots, A^j$ are partial isometries if and only if A is unitarily similar to a matrix of the form*

$$A' = \begin{bmatrix} 0 & A_1 & & & \\ & 0 & \ddots & & \\ & & \ddots & A_{j-1} & \\ & & & 0 & B \\ & & & & C \end{bmatrix} \quad \text{on } \mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_j} \oplus \mathbb{C}^m, \tag{1}$$

where the A_ℓ's satisfy $A_\ell^* A_\ell = I_{n_{\ell+1}}$ for $1 \le \ell \le j-1$, and B and C satisfy $B^*B + C^*C = I_m$. In this case, $n_\ell = \operatorname{nullity} A$ if $\ell = 1$, and $\operatorname{nullity} A^\ell - \operatorname{nullity} A^{\ell-1}$ if $2 \le \ell \le j$, and $m = \operatorname{rank} A^j$.

Here, for any $p \ge 1$, I_p denotes the p-by-p identity matrix, and, for any matrix B, $\operatorname{nullity} B$ means $\dim \ker B$.

A consequence of Theorem 2.1 is the next result from [1, Theorem 2.4].

Theorem 2.2. *Let A be an n-by-n matrix and $j > a(A)$. Then the following conditions are equivalent:*

(a) *$A, A^2, \ldots, A^j$ are partial isometries,*
(b) *A is unitarily similar to a matrix of the form $U \oplus J_{k_1} \oplus \cdots \oplus J_{k_m}$, where U is unitary and $a(A) = k_1 \geq \cdots \geq k_m \geq 1$, or*
(c) *A^ℓ is a partial isometry for all $\ell \geq 1$.*

Here J_q denotes the q-by-q *Jordan block*

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

An easy corollary of the preceding theorem is the following estimate for $p(A)$ from [1, Corollary 2.5].

Corollary 2.3. *If A is an n-by-n matrix, then $0 \leq p(A) \leq \min\{a(A), n-1\}$ or $p(A) = \infty$.*

In constructing the examples for our main result, we need the class of S_n-matrices. Recall that an n-by-n matrix A is said to be of *class* S_n if A is a contraction ($\|A\| \equiv \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\} \leq 1$), its eigenvalues all have moduli strictly less than 1, and $\mathrm{rank}(I_n - A^*A) = 1$. Such matrices are finite-dimensional versions of the *compressions of the shift* $S(\phi)$ studied first by Sarason [5], which later featured prominently in the Sz.-Nagy–Foiaş contraction theory [6]. A special example of S_n-matrices is the Jordan block J_n. In fact, many properties of J_n can be extended to those for the more general S_n-matrices. Part (a) of the following theorem from [1, Proposition 3.1] is one such instance.

Theorem 2.4. *Let A be a noninvertible S_n-matrix. Then*

(a) *$a(A)$ equals the algebraic multiplicity of the eigenvalue 0 of A,*
(b) *$p(A)$ equals $a(A)$ or ∞, and*
(c) *$p(A) = \infty$ if and only if A is unitarily similar to J_n.*

3. Main result

The following is the main theorem of this paper.

Theorem 3.1. *Let j and k be nonnegative integers and n be a positive integer. Then there is an n-by-n matrix A such that $p(A) = j$ and $a(A) = k$ if and only if one of the following conditions holds:*

(a) $j = k \leq n - 1$,
(b) $j \leq k - 1$ *and* $j + k \leq n - 1$, *or*
(c) $j \leq k - 2$ *and* $j + k = n$.

Note that if we allow j to be infinity, then, for any k, $1 \leq k \leq n$, there is an n-by-n matrix A, namely, $A = J_k \oplus 0_{n-k}$ with $p(A) = \infty$ and $a(A) = k$.

To prove Theorem 3.1, we need the next two lemmas.

Lemma 3.2. *If A is an n-by-n matrix, which is unitarily similar to a matrix A' as in (1) with $1 \leq j \leq a(A)$, then* (a) $p(A) = j + p(C)$, *and* (b) $a(A) = j + a(C)$.

Proof. For any $\ell \geq 0$, multiplying A' with itself $j + \ell$ times results in

$$A'^{\,j+\ell} = \begin{bmatrix} 0 & \cdots & 0 & (\prod_{p=1}^{j-1} A_p)BC^{\ell} \\ 0 & \cdots & 0 & (\prod_{p=2}^{j-1} A_p)BC^{\ell+1} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & BC^{j+\ell-1} \\ 0 & \cdots & 0 & C^{j+\ell} \end{bmatrix}. \tag{2}$$

(a) Note that $A'^{\,j+\ell}$ is a partial isometry if and only if $A'^{\,j+\ell^*} A'^{\,j+\ell}$ is an (orthogonal) projection (cf. [2, Problem 127]), and the latter is equivalent to

$$\left(\sum_{q=\ell}^{j+\ell-1} C^{q^*} B^* \left(\prod_{p=q-\ell+1}^{j-1} A_p \right)^* \left(\prod_{p=q-\ell+1}^{j-1} A_p \right) BC^q \right) + C^{j+\ell^*} C^{j+\ell} \tag{3}$$

being a projection. Making use of $A_p^* A_p = I_{n_{p+1}}$, $1 \leq p \leq j-1$, and $B^*B + C^*C = I_m$, we can simplify (3) to

$$\begin{aligned} &\left(\sum_{q=\ell}^{j+\ell-1} C^{q^*} B^* BC^q \right) + C^{j+\ell^*} C^{j+\ell} \\ &= \left(\sum_{q=\ell}^{j+\ell-2} C^{q^*} B^* BC^q \right) + C^{j+\ell-1^*} (B^*B + C^*C) C^{j+\ell-1} \\ &= \left(\sum_{q=\ell}^{j+\ell-2} C^{q^*} B^* BC^q \right) + C^{j+\ell-1^*} C^{j+\ell-1} \\ &= \left(\sum_{q=\ell}^{j+\ell-3} C^{q^*} B^* BC^q \right) + C^{j+\ell-2^*} (B^*B + C^*C) C^{j+\ell-2} \\ &= \cdots \\ &= C^{\ell^*} C^{\ell}. \end{aligned}$$

Thus ${C^\ell}^* C^\ell$ is a projection, which is equivalent to C^ℓ being a partial isometry. From these, we conclude that $p(A) = p(A') = j + p(C)$.

(b) For any $\ell \geq 0$, let s_ℓ (resp., t_ℓ) denote the geometric (resp., algebraic) multiplicity of the eigenvalue 0 of $A^{j+\ell}$, and let u_ℓ (resp., v_ℓ) be the corresponding multiplicities of 0 of C^ℓ. Obviously, we have $t_\ell = t_0$ for all $\ell \geq 0$ and $v_\ell = v_1$ for $\ell \geq 1$. We claim that $s_\ell = (\sum_{i=1}^{j} n_i) + u_\ell$ for $\ell \geq 0$. Indeed, let $x_1 \oplus \cdots \oplus x_j \oplus y$ in $\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_j} \oplus \mathbb{C}^m$ be any vector in $\ker {A'}^{j+\ell}$. From (2), we have $(\prod_{p=q-\ell+1}^{j-1} A_p) B C^q y = 0$, $\ell \leq q \leq j + \ell - 1$, and $C^{j+\ell} y = 0$. Since $A_p^* A_p = I_{n_{p+1}}$ for $1 \leq p \leq j - 1$, we obtain $BC^q y = 0$ for $\ell \leq q \leq j + \ell - 1$. Applying $B^*B + C^*C = I_m$ to the vector $C^{j+\ell-1} y$ yields that

$$C^{j+\ell-1} y = B^* \big(BC^{j+\ell-1} y \big) + C^* \big(CC^{j+\ell-1} y \big) = 0 + 0 = 0.$$

We may then apply $B^*B + C^*C = I_m$ again to $C^{j+\ell-2} y$ as above to obtain $C^{j+\ell-2} y = 0$. Repeating this process inductively, we finally reach $C^\ell y = 0$, that is, y is in $\ker C^\ell$. This shows that $\ker {A'}^{j+\ell}$ is contained in the subspace $\mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_j} \oplus \ker C^\ell$. Since the reversed containment is easily seen to be true, it follows that

$$s_\ell = \text{nullity } A^{j+\ell} = \text{nullity } {A'}^{j+\ell} = \left(\sum_{i=1}^{j} n_i \right) + u_\ell$$

for any $\ell \geq 0$ as claimed. Note that, for any matrix T, its ascent is equal to the smallest nonnegative integer k for which the geometric and algebraic multiplicities of the eigenvalue 0 of T^k coincide. Thus

$$u_{a(C)} = v_{a(C)} = \begin{cases} v_1 & \text{if } a(C) \geq 1, \\ 0 & \text{if } a(C) = 0, \end{cases} \quad \text{and} \quad u_{a(C)-1} < u_{a(C)} = v_1 \quad \text{if } a(C) \geq 1.$$

Therefore,

$$s_{a(C)} = \left(\sum_{i=1}^{j} n_i \right) + u_{a(C)} = \begin{cases} (\sum_{i=1}^{j} n_i) + v_1 = t_0 = t_{a(C)} & \text{if } a(C) \geq 1, \\ \sum_{i=1}^{j} n_i & \text{if } a(C) = 0, \end{cases}$$

where the third equality follows from the upper-triangular block structure of A', and

$$s_{a(C)-1} = \left(\sum_{i=1}^{j} n_i \right) + u_{a(C)-1} < \left(\sum_{i=1}^{j} n_i \right) + v_1 = t_0 = t_{a(C)-1} \quad \text{if } a(C) \geq 1.$$

This shows that $j + a(C)$ is the smallest integer k for which the geometric and algebraic multiplicities of the eigenvalue 0 of A^k are equal to each other. Thus $a(A) = j + a(C)$ follows. □

Lemma 3.3. *Let A be an n-by-n matrix with $p(A) < \infty$.*

(a) *If* $p(A) + a(A) > n$, *then* $p(A) = a(A)$.
(b) *If* $p(A) + a(A) = n$, *then* $p(A) = a(A)$ *or* $p(A) \leq a(A) - 2$.

Proof. By Theorem 2.1, A is unitarily similar to a matrix A' in (1) with $j = p(A)$. In particular, this implies that $n_1 \geq n_2 \geq \cdots \geq n_j \geq 1$ for if $n_j = \text{nullity } A^j - \text{nullity } A^{j-1} = 0$, then we would have $\ker A^j = \ker A^{j-1}$, which yields the contradictory $p(A) \leq a(A) \leq j - 1$ by Corollary 2.3.

(a) Assuming $p(A)+a(A) > n$, we first check that $n_j = 1$. Indeed, if otherwise $n_j \geq 2$, then $n_i \geq 2$ for all i, $1 \leq i \leq j$. Making use of Lemma 3.2, we have

$$\begin{aligned} n &= \left(\sum_{i=1}^{j} n_i\right) + m \geq 2j + \big(a(C) - p(C)\big) \\ &= 2p(A) + \big(a(A) - p(A)\big) = p(A) + a(A) > n, \end{aligned}$$

which is a contradiction. Thus $n_j = 1$. This means that B is a 1-by-m matrix. If $p(A) < a(A)$, then $p(C) < a(C)$ by Lemma 3.2 again. In particular, this says that $a(C) > 0$ or 0 is an eigenvalue of C. After a unitary similarity, we may assume that C is upper triangular with a zero first column. Let C be partitioned as

$$\begin{bmatrix} 0 & C_1 \\ 0 & C_2 \end{bmatrix},$$

where C_1 (resp., C_2) is a 1-by-$(m-1)$ (resp., $(m-1)$-by-$(m-1)$) matrix. We deduce from $B^*B + C^*C = I_m$ that $B = [e^{i\theta}\ 0\ \ldots\ 0]$ for some real θ and $C_1^*C_1 + C_2^*C_2 = I_{m-1}$. Thus A' is of the form

$$\begin{bmatrix} 0 & A_1 & & & & \\ & 0 & \ddots & & & \\ & & \ddots & A_{j-1} & & \\ & & & 0 & e^{i\theta}\ 0\ \ldots\ 0 & \\ & & & & 0 & C_1 \\ & & & & & C_2 \end{bmatrix} \quad \text{on } \mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_j} \oplus \mathbb{C} \oplus \mathbb{C}^{m-1}.$$

Theorem 2.1 then leads to the contradictory $p(A) = p(A') > j = p(A)$. We conclude from Corollary 2.3 that $p(A) = a(A)$.

(b) Assume that $p(A) + a(A) = n$ and $p(A) = a(A) - 1$. We consider two separate cases, both of which will lead to contradictions:

(i) $a(C) - p(C) \leq m - 1$. In this case, we proceed as in (a) to first prove that $n_j = 1$. Indeed, if $n_j \geq 2$, then

$$n - 1 = \left(\sum_{i=1}^{j} n_i\right) + m - 1 \geq 2j + \big(a(C) - p(C)\big)$$

$$= 2p(A) + \big(a(A) - p(A)\big) = a(A) + p(A) = n,$$

which is a contradiction. Hence $n_j = 1$ and B is a 1-by-m matrix. Then, since $p(A) < a(A)$, the second-half arguments in proving (a) yield that $p(A) = a(A)$, which contradicts our assumption of $p(A) = a(A) - 1$.

(ii) $a(C) - p(C) = m$. Note that this can happen only when $a(C) = m$ and $p(C) = 0$. Thus $m = a(C) - p(C) = a(A) - p(A) = 1$ by Lemma 3.2 and our assumption. This shows that C is a 1-by-1 matrix, say, $C = [c]$ with $a(C) = 1$ and $p(C) = 0$. The former condition $a(C) = 1$ yields that $c = 0$, which results in $p(C) = \infty$, contradicting the latter $p(C) = 0$.

We conclude that $p(A) + a(A) = n$ implies $p(A) \neq a(A) - 1$. □

Finally, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. The existence of an n-by-n matrix A with $p(A) = j$ and $a(A) = k$ implies, by Corollary 2.3, that $j \leq \min\{k, n-1\}$. Lemma 3.3 then yields that one of (a), (b) and (c) must hold.

For the converse, assume that (a) holds. If $j = k = 0$, then $A = (1/2)I_n$ will do. Otherwise, we have $1 \leq j = k \leq n-1$. Let A be a noninvertible S_n-matrix whose eigenvalue 0 has algebraic multiplicity k. Then Theorem 2.4 gives $p(A) = j = k = a(A)$.

Next assume that (b) holds. Let $A = A_1 \oplus A_2$, where A_1 (resp., A_2) is an S_{j+1}-matrix (resp., S_{n-j-1}-matrix) whose eigenvalue 0 has algebraic multiplicity j (resp., k). Then $a(A_1) = j$ and $a(A_2) = k$. Hence $a(A) = \max\{a(A_1), a(A_2)\} = k$. On the other hand, we also have $p(A_1) = a(A_1) = j$ and

$$p(A_2) = \begin{cases} a(A_2) = k & \text{if } k < n-j-1, \\ \infty & \text{if } k = n-j-1, \end{cases}$$

by Corollary 2.3 and Theorem 2.4. Thus $p(A) = \min\{p(A_1), p(A_2)\} = j$.

Finally, if (c) holds, then there are two cases to be considered:

(i) $j = k - 2$. In this case, let

$$A = \begin{bmatrix} 0 & I_2 & & & \\ & 0 & \ddots & & \\ & & \ddots & I_2 & \\ & & & 0 & B \\ & & & & C \end{bmatrix} \quad \text{on } \mathbb{C}^n = \underbrace{\mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2}_{j} \oplus \mathbb{C}^2,$$

where $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{smallmatrix}\right]$ and $C = \left[\begin{smallmatrix} 0 & 1/\sqrt{2} \\ 0 & 0 \end{smallmatrix}\right]$. Since $n = j + k = j + (j+2) = 2j + 2$, A is indeed an n-by-n matrix with $B^*B + C^*C = I_2$. We infer from Lemma 3.2(a) (resp., (b)) that

$$p(A) = j + p(C) = j + 0 = j \quad \big(\text{resp., } a(A) = j + a(C) = j + 2 = k\big).$$

(ii) $j \le k-3$. Let $m = k-j \ge 3$, and let

$$A = \begin{bmatrix} 0 & I_2 & & & \\ & 0 & \ddots & & \\ & & \ddots & I_2 & \\ & & & 0 & B \\ & & & & C \end{bmatrix} \quad \text{on } \mathbb{C}^n = \underbrace{\mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2}_{j} \oplus \mathbb{C}^m,$$

where

$$B = \begin{bmatrix} 1 & 0 & \overbrace{0 & \cdots & 0}^{m-3} & 0 \\ 0 & 1/\sqrt{2} & 0 & \cdots & 0 & 1/2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & -1/\sqrt{2} & \overbrace{0 & 0 & \cdots & 0}^{m-3} & 1/2 \\ & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & 0 & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & 0 & 0 \\ & & & & \ddots & 1 & 0 \\ & & & & & 0 & 1/\sqrt{2} \\ & & & & & & 0 \end{bmatrix}.$$

Since $n = j+k = 2j+m$, A is an n-by-n matrix with $B^*B + C^*C = I_m$. Again, we infer from Lemma 3.2(a) that

$$p(A) = j + p(C) = j + 0 = j,$$

where the second equality follows from the fact that C^*C is not a projection and hence C is not a partial isometry. On the other hand, Lemma 3.2(b) implies that

$$a(A) = j + a(C) = j + m = j + (k-j) = k,$$

where the second equality holds because C is similar to J_m. □

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