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Disjoint odd integer subsets having a constant odd sum

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Abstract

We prove that for positive k, n and m, the set $\{1, 3, \ldots, 2n-1\}$ of odd integers contains k disjoint subsets having a constant odd sum m if and only if $9(k-1) \leqslant m \leqslant 2n-1$, or $9k \leqslant m \leqslant n^2/k$ and $n^2 - mk \neq 2$.

1. Introduction

Ando et al. [1], proved that for positive integers n, m and k, the set $\{1, 2, \ldots, n\}$ of integers contains k disjoint subsets having a constant sum m if and only if $2k-1 \leqslant m \leqslant n(n+1)/(2k)$. In the same paper, they posed the following conjecture.

Conjecture 1.1. Let k, n and m be positive integers. Then the set $\{1, 3, \ldots, 2n-1\}$ of odd integers contains k disjoint subsets having a constant sum m if and only if one of the following two conditions hold:

(i) m is even, $4k \leqslant m \leqslant n^2/k$, $n^2 - mk \neq 2$. and either $m \neq 4n-2$ or $n \neq 4k$.

(ii) m is odd, and either $9(k-1) \leqslant m \leqslant 2n-1$, or $9k \leqslant m \leqslant n^2/k$ and $n^2 - mk \neq 2$.

They also mention that conjecture (i) has been proved by Enomoto and Kano [2]. In this paper, we prove conjecture (ii).

2. The main result

For convenience, we will use $A_1, A_2, \ldots, A_k$ to denote k mutually disjoint subsets and let $A = [a_{ij}]_{t \times k}$ be the array such that $A_j = \{a_{ij} \mid i = 1, 2, \ldots, t\}$ $j = 1, 2, \ldots, k$.

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$$\begin{bmatrix} 41 & 39 & 37 & 35 & 33 & 31 & 29 \\ 17 & 21 & 25 & 15 & 19 & 23 & 27 \\ 5 & 3 & 1 & 13 & 11 & 9 & 7 \end{bmatrix}$$

Fig. 1.

Moreover, we write A as $[A_1 \mid A_2 \mid \cdots \mid A_k]$. We note that $t = \max_{1 \leqslant i \leqslant k} |A_i|$ and some of the cells in A may be empty. The sum of the elements in A_i is denoted by $\sum(A_i)$. Figure 1 is an example of $k=7$ and $\sum(A_i)=63$, $1 \leqslant i \leqslant 7$.

The following result is very helpful in our proof.

Theorem 2.1. (*Enomoto and Kano* [2]). *Let n and k be positive integers and m be a positive even integer. Then the following two statements hold:*

(a) $\{1, 3, \ldots, 2n-1\}$ *contains k disjoint subsets with sum m if and only if* (i) $4k \leqslant m \leqslant n^2/k$; (ii) $n^2 - mk \neq 2$; *and* (iii) $n \neq 4k$ *or* $m \neq 16k-2$.

(b) $\{1, 3, \ldots, 2n-1\}$ *contains* $k+1$ *disjoint subsets* $A_1, \ldots, A_k$ *and B such that* $\sum(B) = m/2$ *and* $\sum(A_i) = m$ *for all i,* $1 \leqslant i \leqslant k$ *if and only if* (iv) *either* $m \equiv 0 \pmod 4$ *and* $4k+8 \leqslant m \leqslant 2n^2/(2k+1)$ *or* $m \equiv 2 \pmod 4$ *and* $4k+2 \leqslant m \leqslant 2n^2/(2k+1)$; (v) $n^2 - (2k+1)m/2 \neq 2$; *and* (vi) $n \neq 4k+3$ *or* $m \neq 16k+14$.

With the above theorem, we can obtain the following propositions.

Proposition 2.2. $\{1, 3, \ldots, 2l-1\}$ *contains j disjoint subsets with sum* $4k$ *if* $4j \leqslant 4k \leqslant l^2/j$.

Proof. Since $l^2 - 4jk \neq 2$ and $4k \neq 16j - 2$, by Theorem 2.1 (a), we conclude the proof. □

Proposition 2.3. $\{1, 3, \ldots, 2l-1\}$ *contains disjoint subsets* $A_1, A_2, \ldots, A_j$ *such that* $\sum(A_i) = 4k$ *for each* $i = 1, 2, \ldots, j-1$, *and* $\sum(A_j) = 6k$, *if* (i) $4j+4 \leqslant 4k \leqslant 2l^2/(2j+1)$; *and* (ii) $l^2 - (4j+2)k \neq 2$.

Proof. Since $4k \neq 16j+14$, if $4j+8 \leqslant 4k$, then by Theorem 2.1. (b) we conclude the proof by letting the sums of subsets be $\langle 4k, 4k, \ldots, 4k, 2k \rangle$ ($j+1$ – tuple) and combining the last two subsets to obtain a set with sum $6k$. Finally, if $4j+4 = 4k$, then $(4j+4)((2j+1)/2) \leqslant l^2$, i.e. $l \geqslant 2j+2$. In this case, let $A_1 = \{3, 4j+1\}$, $A_2 = \{5, 4j-1\}, \ldots, A_{j-1} = \{2j-1, 2j+5\}$ and $A_j = \{4j+3, 2j+3\}$, then we have the proof. □

Proposition 2.4. $\{1, 3, \ldots, 2l-1\}$ *contains* $j+1$ *disjoint subsets* $A_1, A_2, \ldots, A_{j+1}$ *such that* $\sum(A_i) = 4k$ *for each* $1 \leqslant i \leqslant j$ *and* $\sum(A_{j+1}) = 5k$, *if* (i) *k is even.* (ii) $4j+5 \leqslant 4k \leqslant 4l^2/(4j+5)$, *and* (iii) $l^2 - (4j+5)k \neq 2$.

Proof. If $l \leqslant k$, then $(4j+5)k \leqslant l^2 \leqslant k^2$. Hence $8j+12 \leqslant 2k$. By Proposition 2.3, $\{1, 3, \ldots, 2l-1\}$ contains $2j+2$ disjoint subsets $A_1, A_2, \ldots, A_{2j+1}, A_{2j+2}$ such that $\sum(A_i) = 4 \cdot (k/2)$ for each $1 \leqslant i \leqslant 2j+1$, and $\sum(A_{2j+2}) = 6 \cdot (k/2)$. Thus, by combining $A_1, A_2; A_3, A_4; \ldots, A_{2j+1}, A_{2j+2}$ we have the proof of this case. If $l > 2k$, then we will show that $\{1, 3, \ldots, 4k-1\}$ contains desired $j+1$ subsets as $(2k)^2 - (4j+5)k = (4k-4j-5)k \geqslant 3k > 2$. Hence we may assume that $k < l \leqslant 2k$. Furthermore, $k=2$ is a trivial case, without loss of generality, let $k \geqslant 4$. Now consider the following three cases:

(1) $[(2l-1)-(2k+1)]/2+1 \leqslant j$. Let $A_1 = \{2l-1, 4k-2l+1\}$, $A_2 = \{2l-3, 4k-2l+3\}, \ldots, A_t = \{2k+1, 2k-1\}$. Then $A_{t+1}, A_{t+2}, \ldots, A_{j+1}$ can be obtained as in the case $l \leqslant k$.

(2) $[(2l-1)-(2k+1)]/2+1 = j+1$. Let $A_1 = \{2l-1, 4k-2l+1\}$, $A_2 = \{2l-3, 4k-2l+3\}, \ldots, A_j = \{2k-3, 2k+3\}$ and $A'_{j+1} = \{2k-1, 2k+1\}$. By Theorem 2.1 (a) $\{1, 3, \ldots, 4k-2l-1\}$ contain a subset A''_{j+1} s.t. $\sum(A''_{j+1}) = k$ Let $A_{j+1} = A'_{j+1} \cup A''_{j+1}$ then we conclude this case.

(3) $[(2l-1)-(2k+1)]/2+1 > j+1$. First, if $3k-1 \geqslant 2l-1$, i.e. $4k-2l+1 \geqslant k+1$, let $A_1 = \{2l-1, 4k-2l+1\}$, $A_2 = \{2l-3, 4k-2l+3\}, \ldots, A_j = \{2l+1-2j, 4k-2l+2j-1\}$, $A_{j+1} = \{1, k-1, 2k-1, 2k+1\}$, we have $j+1$ disjoint subsets we need. Secondly, if $3k-1 < 2l-2j+1 \leqslant 2l-1$, the $j+1$ disjoint subsets will be $A_1 = \{2l-1, 4k-2l+1\}$, $A_2 = \{2l-3, 4k-2l+3\}, \ldots, A_j = \{2l-2j+1, 4k-2l+2j-1\}$, $A_{j+1} = \{2k+1, 3k-1\}$. Finally, if $2l-2j+1 \leqslant 3k-1 < 2l-1$, then there exists an index $1 \leqslant i \leqslant j$ in the above decomposition such that $A_i = (k+1, 3k-1\}$. By replacing this A_i with $\{2k-3, 2k+3\}$, we have the proof of this case. □

Proposition 2.5. $\{1, 3, \ldots, 2l-1\}$ *contains* $j+1$ *disjoint subsets with* j *of them sum to* $4k$, *and one sums to* $3k$ *if* (i) k *is even,* (ii) $4j+7 \leqslant 4k \leqslant 4l^2/(4j+3)$, *and* (iii) $l^2-(4j+3)k \neq 2$.

Proof. Similar to the proof of Proposition 2.4. □

Now we are ready to prove the main theorem. First, we consider the necessary condition.

Proposition 2.6. (Necessity) *Let n, m and k be positive integers and m is odd. If the odd integer set* $\{1, 3, \ldots, 2n-1\}$ *contains k disjoint subsets having a constant sum m, then* (i) *either* $9(k-1) \leqslant m \leqslant 2n-1$ *or* $9k \leqslant m \leqslant n^2/k$ *and* (ii) $n^2 - mk \neq 2$.

Proof. Suppose that $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets $A_1, A_2, \ldots, A_k$ having constant sum m. If $m \leqslant 2n-1$, since m is odd, only one subset A_i could have one element, the other $k-1$ subsets each contains at least 3 elements. Thus $m(k-1) = \sum_{j=1, j \neq i}^{k} \sum(A_j) \geqslant 1+3+\cdots+(6k-7)$, which implies that $m \geqslant 9k-9$. Hence $9(k-1) \leqslant m \leqslant 2n-1$. Or, if $m > 2n-1$, then each A_i, $i = 1, 2, \ldots, k$, contains at least three elements. Therefore, $mk \geqslant 1+3+\cdots+(6k-1) = (3k)^2$. On the other hand,

$mk \leqslant 1+3+\cdots+(2n-1)=n^2$. Hence $9k \leqslant m \leqslant n^2/k$. The result $n^2-mk \neq 2$ is easy to see. □

The sufficiency of the main theorem is more complicated. For clearness, we will consider separate cases in the following three propositions.

Proposition 2.7. *If m is odd, $9k-9 \leqslant m \leqslant 2n-1$ and $n^2-mk \neq 2$, then $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets having constant sum m.*

Proof. By direct construction. Arrays A_o and A_e are for k is odd and even, respectively.

$$A_o = \begin{bmatrix} A_1 & A_2 & A_{k-1/2} & A_{k+1/2} & A_{k+3/2} & A_{k-1} \quad A_k \\ m-3k+3, & m-3k+1,\ldots, & m-4k+6, & m-4k+2, & m-4k,\ldots, & m-5k+5, \quad m \\ 2k-1 & 2k+3,\ldots, & 4k-7, & 2k+1, & 2k+5,\ldots, & 4k-5, \\ k-2, & k-4,\ldots, & 1, & 2k-3, & 2k-5,\ldots, & k, \end{bmatrix}$$

$$A_e = \begin{bmatrix} A_1 & A_2 & A_{k/2-1} & A_{k/2} & A_{k/2+1} & A_{k-1} \quad A_k \\ m-3k+2, & m-3k,\ldots, & m-4k+6, & m-4k+4, & m-4k+2,\ldots, & m-5k+6, \quad m \\ 2k+1 & 2k+5,\ldots, & 4k-7, & 2k-1, & 2k+3,\ldots, & 4k-5, \\ k-3, & k-5,\ldots, & 1, & 2k-3, & 2k-5,\ldots, & k-1, \end{bmatrix}$$

Note that we often use the same constructions $A_o - A_k$ and $A_e - A_k$ in the proofs of Propositions 2.8 and 2.9, respectively.

Proposition 2.8. *If m and k are odd positive integers, $9k \leqslant m \leqslant n^2/k$, and $n^2-mk \neq 2$, then $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets having a constant sum m.*

Proof. First, consider $m \geqslant 25k+4$; let l be the positive integer such that $(5k+l-1)^2 < mk \leqslant (5k+l)^2$. If $(5k+l)^2-mk \neq 2$, we assume $n=5k+l$; otherwise, let $n=5k+l+1$. In any case, let $A_{j+1} \supseteq \{2(n-2k)+2j+1, 2n-2j-1\}, j=0, 1, 2, \ldots, k-1$; then we reduce the case $m \geqslant 25k+4$ to $n'=3k+l$ or $n'=3k+l+1$ and $m'=m-16k-4l$ or $m'=m-16k-4l-4$, which satisfy $9k \leqslant m' \leqslant n'^2/k$ and $n'^2-m'k \neq 2$ (by direct checking) and the odd integer set will be $\{1, 3, \ldots, 2n'-1\}$. As to the m, $25k \leqslant m \leqslant 25k+3$, we will give a direct construction which can be found in Fig. 5. Thus if we can prove the case when $9k \leqslant m \leqslant 25k$, we have the proof of this proposition.

(1) $9k \leqslant m \leqslant 6n-9k$. Since $0 \leqslant m-9k \leqslant 6(n-3k)$, there exist three integers x, y and z such that $2(x+y+z)=m-9k$ and $0 \leqslant z \leqslant y \leqslant x \leqslant n-3k$. Thus by the array shown in Fig. 2, we have the proof of this case.

$$\begin{bmatrix} 2x+6k-1, 2x+6k-3, \ldots, & 2x+5k+2, 2x+5k, \ldots, & 2x+4k+1 \\ 2y+2k+3, 2y+2k-7, \ldots, & 2y+4k-3, 2y+2k+1, \ldots, & 2y+4k-1 \\ 2z+k-2, 2z+k-4, \ldots, & 2z+1, 2z+2k-1, \ldots, & 2z+k \end{bmatrix}$$

Fig. 2.

$$\begin{bmatrix} 2n-1, 2n-3, \ldots, & 2n-k+2, 2n-k, 2n-k-2, \ldots, & 2n-2k+1 \\ 2n-4k+3, 2n-4k+7, \ldots, & 2n-2k-3, 2n-4k+1, 2n-4k+5, \ldots, & 2n-2k-1 \\ 2n-5k, 2n-5k-2, \ldots, & 2n-6k+3, 2n-6k+1, 2n-4k-1, \ldots, & 2n-5k+2 \\ & \boxed{2k} & \end{bmatrix}$$

Fig. 3.

(2) $6n-9k<m<25k$. If $m=6n-9k+2$, then $(6n-9k+2)k \leqslant n^2$, i.e. $2k \leqslant (n-3k)^2$ and $(n-3k)^2-2k \neq 2$, so by Theorem 2.1(a) $\{1, 3, \ldots, 2n-6k-1\}$ contain a subset having a constant sum $2k$. By Fig. 3 we have proved the case.

There are two other situations to consider.

Case 1: $m-(6n-9k)=4j$, $j \geqslant 1$. If $j \geqslant k$ then we can derive a contradiction from $m<25k$ and $n^2 \geqslant mk$. Thus $j<k$. Consider the array

$$B' = \begin{bmatrix} 2n-1+2j, 2n-3+2j, \ldots, & 2n-k+2+2j, 2n-k+2j, \ldots, & 2n-2k+1+2j \\ 2n-4k+3+2j, 2n-4k+7+2j, \ldots, & 2n-2k-3+2j, 2n-4k+1+2j, \ldots, & 2n-2k-1+2j \\ 2n-5k-2, 2n-5k-4, \ldots, & 2n-6k+1, 2n-4k-1, \ldots, & 2n-5k \end{bmatrix}.$$

As can be seen in B', $2n-1+2j$, $2n-3+2j, \ldots, 2n+1$ are in first row, but $2n-4k-1+2j$, $2n-4k-3+2j, \ldots$, $2n-4k+1$ are not in any row. Change a part of the first row of B' by letting $2n-1+2j=(4k)+(2n-4k-1+2j)$, $2n-3+2j=(4k)+(2n-4k-3+2j), \ldots, 2n+1=(4k)+(2n-4k+1)$. We obtain B'' in Fig. 4.

Since $\{1, 3, \ldots, 2(n-3k)-1\}$ contains j disjoint subsets having a constant sum $4k$ (by Proposition 2.2), we have the proof of this case.

Case 2: $m-(6n-9k)=4j+2$, $j \geqslant 1$.

The array A' is obtained from B' by adding two to each cell of the third row of B', and the A'' can be obtained similar to Fig. 4 except a part of the first row will be

$$B'' = \begin{bmatrix} 4k & 4k & \cdots & 4k & & \\ 2n-4k-1+2j & 2n-4k-3+2j & & 2n-4k+1 & 2n-1, \ldots, & 2n-2k+1 \\ 2n-4k+3+2j & \cdots & & \cdots & & 2n-2k-1+2j \\ 2n-5k-2 & \cdots & B' & \cdots & & 2n-5k \end{bmatrix}$$

Fig. 4.

$$B_1=\begin{bmatrix} 10k-1, 10k-3,\dots, & 9k+2, 9k, 9k-2,\dots, & 8k+1 \\ 6k+1,\ 6k+3,\dots, & 7k-2, 7k, 7k+2,\dots, & 8k-1 \\ 6k-1,\ 6k-3,\dots, & 5k+2, 5k, 5k-2,\dots, & 4k+1 \\ 2k+3,\ 2k+7,\dots, & 4k-3, 2k+1, 2k+5,\dots, & 4k-1 \\ k-2,\ k-4,\dots, & 1,\ 2k-1, 2k-3,\dots, & k \end{bmatrix}$$

$$B_2=\begin{bmatrix} 10k+1, 10k-1,\dots, & 9k+4, 9k+2, 9k,\dots, & 8k+3 \\ 6k+1,\ 6k+3,\dots, & 7k-2, 7k, 7k+2,\dots, & 8k-1 \\ 6k-1,\ 6k-3,\dots, & 5k+2, 5k, 5k-2,\dots, & 4k+1 \\ 2k+3,\ 2k+7,\dots, & 4k-3, 2k+1, 2k+5,\dots, & 4k-1 \\ k-2,\ k-4,\dots, & 1, 2k-1, 2k-3,\dots, & k \end{bmatrix}$$

Fig. 5.

$\langle 4k, 4k, \dots, 4k, 6k\rangle$($j$-tuple) and this can be obtained by Proposition 2.3. Thus we conclude the proof of this proposition by the given direct constructions B_1, and B_2 (Fig. 5) for $m=25k$ and $25k+2$, respectively. □.

Finally, we consider the case when k is even.

Proposition 2.9. *Let n, m and k be a positive integer, a positive odd integer and a positive even integer, respectively. Then the odd integer set* $\{1, 3, \dots, 2n-1\}$ *contains k disjoint subsets having a constant sum m, if* $9k+1 \leqslant m \leqslant n^2/k$ *and* $n^2-mk \neq 2$.

Proof. Similar to the proof of the Proposition 2.8, we consider $m \leqslant 25k+3$, and the case $m=25k-1, 25k+1, 25k+3$ will be obtained by direct constructions (Fig. 7). Hence let $m<25k-1$.

First, if $m \leqslant 6n-9k+5$, let C_1, C_2 and C_3 be three arrays where the column sums of these arrays are $9k+1$, $9k+3$ and $9k+5$, respectively.

$$C_1:\ \begin{bmatrix} 6k+1, & 6k-1,\dots, & 5k+3, & 5k-1, & 5k-3,\dots, & 4k+1 \\ 2k+1, & 2k+5,\dots, & 4k-3, & 2k+3, & 2k+7,\dots, & 4k-1 \\ k-1, & k-3,\dots, & 1, & 2k-1, & 2k-3,\dots, & k+1 \end{bmatrix}$$

$$C_2:\ \begin{bmatrix} 6k+1, & 6k-3,\dots, & 4k+5, & 6k-1, & 6k-5,\dots, & 4k+3 \\ 2k+1, & 2k+3,\dots, & 3k-1, & 3k+3, & 3k+5,\dots, & 4k+1 \\ k+1, & k+3,\dots, & 2k-1, & 1, & 3,\dots, & k-1 \end{bmatrix}$$

$$C_3:\ \begin{bmatrix} 6k+1, & 6k-1,\dots, & 5k+3, & 5k+1, & 5k-1,\dots, & 4k+3 \\ 2k+5, & 2k+9,\dots, & 4k+1, & 2k+3, & 2k+7,\dots, & 4k-1 \\ k-1, & k-3,\dots, & 1, & 2k+1, & 2k-1,\dots, & k+3 \end{bmatrix}.$$

Similar to the idea of Proposition 2.8 (1), if $m-(9k+2i-1)$ is a multiple of 6, then C_i will be the array to use, $i=1, 2, 3$. Thus we have the proof of this situation. Now consider $6n-9k+5<m \leqslant 25k-3$.

Case 1: $m-(6n-9k+5)=4j$.
Consider the array D.

$$D=\begin{bmatrix} 2n+1+2j, 2n-1+2j, \ldots, & 2n-k+3+2j, 2n-k+1+2j, 2n-k-1+2j, \ldots, & 2n-2k+3+2j \\ 2n-4k+5+2j, 2n-4k+9+2j, \ldots, & 2n-2k+1+2j, 2n-4k+3+2j, 2n-4k+7+2j, \ldots, & 2n-2k-1+2j \\ 2n-5k-1, 2n-5k-3, \ldots, & 2n-6k+1, 2n-4k+1, 2n-4k-1, \ldots, & 2n-5k+3 \end{bmatrix}.$$

It is easy to see that $2n+1+2j$, $2n-1+2j, \ldots, 2n+1$ appear in D but $2n-4k+1+2j$, $2n-4k-1+2j, \ldots, 2n-4k+3$, $2n-5k+1$ do not appear in D. By a similar technique as in the case 1 and 2 of Proposition 2.8, we replace a part of the first row in D and obtain D' in Fig. 6.

$$\begin{bmatrix} 4k & 4k & \cdots & 4k & 5k & & \\ 2n-4k+1+2j & 2n-4k-1+2j & & 2n-4k+3 & 2n-5k+1 & 2n-1, \ldots, & 2n-2k+3+2j \\ 2n-4k+5+2j & 2n-4k+9+2j & \cdots & & & \cdots & 2n-2k-1+2j \\ 2n-5k-1 & 2n-5k-3 & \cdots & D & & \cdots & 2n-5k+3 \end{bmatrix}$$

Fig. 6.

$m=25k-1$:

$$\begin{bmatrix} 10k-1, 10k-3, 10k-5, \ldots, & 9k+1, 9k-3, 9k-5, \ldots, & 8k+1, 8k-1 \\ 6k+1, 6k+3, 6k+5, \ldots, & 7k-1, 7k+1, 7k+3, \ldots, & 8k-3, 6k-1 \\ 9k-1, 6k-3, 6k-5, \ldots, & 4k+1, 5k-1, 5k-3, \ldots, & 4k+3, 5k+1 \\ 2k+5, 2k+9, \ldots, & 4k-3, 2k+3, 2k+7, \ldots, & 4k-5, 4k-1 \\ k-3, k-5, \ldots\ 3, & k+1, 2k-1, 2k-3, \ldots, & k+3, 2k+1 \end{bmatrix}$$

$m=25k+1$:

$$\begin{bmatrix} 10k+1, 10k-1, \ldots, 9k+3, & 9k-1, 9k-3, \ldots, & 8k+3, 8k+1 \\ 6k+1, 6k+3, \ldots, 7k-1, & 7k+1, 7k+3, \ldots, & 8k-3, 8k-1 \\ 6k-1, 6k-3, \ldots, 5k+1, & 5k-1, 5k-3, \ldots, & 4k+3, 4k+1 \\ 2k+1, 2k+5, \ldots, 4k-3, & 2k+3, 2k+7, \ldots, & 4k-5, 4k-1 \\ k-1, k-3, \ldots, 1 & 2k-1, 2k-3, \ldots, & k+3, k+1 \end{bmatrix}$$
$$B_1(9k-1) \qquad B_2(9k+1)$$

$m=25k+3$:

$$\begin{bmatrix} 10k+1, 10-1, \ldots, 9k+3, & 9k+1, 9k-1, \ldots, 8k+5, 8k+3 \\ 6k+1, 6k+3, \ldots, 7k-1, & 7k+3, 7k+5, \ldots, 8k-1, 8k+1 \\ B_2(9k+1) & B_1(9k-1) \end{bmatrix}$$

Fig. 7.

Since, by Proposition 2.4, $\{1, 3, \ldots, 2(n-3)-1\}$ contains $j+1$ disjoint subsets with j of them sum up to $4k$ and one to $5k$, we conclude the proof of this case.

Case 2: $m-(6n-9k+5)=4j-2$.

The proof is similar to case 1, except that we will use array E and apply Proposition 2.5 instead of Proposition 2.4.

$$E:\begin{bmatrix} 2n+1+2j, 2n-3+2j, \ldots, & 2n-2k+5+2j, 2n-1+2j, 2n-5+2j, \ldots, & 2n-2k+3+2j \\ 2n+1-4k+2j, 2n-4k+3+2j, \ldots, & 2n-3k-1+2j, 2n-3k+3+2j, 2n-3k+5+2j, \ldots, & 2n-2k+1+2j \\ 2n-5k+1, 2n-5k+3, \ldots, & 2n-4k-1, 2n-6k+1, 2n-6k+3, \ldots, & 2n-5k-1 \end{bmatrix}$$

For the case $m=25k-1$, $25k+1$, $25k+3$, we will use a direct construction for each m (Fig. 7). Thus we conclude the proof of this proposition. □

By Propositions 2.6–2.9 we have proved the following theorem.

Theorem 2.10. *Let k, n and m be positive integers and m is odd. Then the set $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets having a constant sum m if and only if $9(k-1) \leqslant m \leqslant 2n-1$, or $9k \leqslant m \leqslant n^2/k$ and $n^2-mk \neq 2$.*

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