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On upper bounds for the pseudo-achromatic index [1]

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Abstract

In this paper, we give another approach to the pseudo-achromatic index and the achromatic index of a graph and study upper bounds for them. We have obtained the following best possible upper bounds:

(i) $\Psi'(G) \leqslant \Psi'_S(G) \leqslant \lfloor (e(G) + \chi'(G))/2 \rfloor$; and

(ii) $\Psi'(G) \leqslant \Psi'_S(G) \leqslant \max_{1 \leqslant k \leqslant \lfloor p/2 \rfloor} \min\{\lfloor p\Delta(G)/2k \rfloor, 2k(\Delta(G) - 1) + 1\}$.

Using these bounds, the pseudo-achromatic indices of graphs of certain types are obtained which generalize the results of Bouchet (1978), Chiang and Fu (1995), Geller and Kronk (1974) and Jamison (1989) for achromatic indices to pseudo-achromatic indices.

1. Introduction

Let $G = (V,E)$ be a simple graph and $p(G)$, $e(G)$ and $\Delta(G)$ the order, size and maximum degree of G, respectively. A collection $D = \{E_1, E_2, \ldots, E_n\}$ of nonempty subsets of E is a *decomposition* of G if E is a disjoint union of $E_1, E_2, \ldots, E_n$. If every set in the decomposition D of G is a matching, then we say that D is a *proper decomposition* of G. The *decomposition graph* $D(G)$ is defined as follows: (i) $V(D(G)) = D$; and (ii) for $i \neq j$, $\{E_i, E_j\} \in E(D(G))$ if and only if $V(\langle E_i \rangle_G) \cap V(\langle E_j \rangle_G) \neq \phi$ where $\langle E_i \rangle_G$ is the edge-induced subgraph of G. If $D(G)$ is a complete graph, then we say the decomposition D of G is *complete*.

By a *proper edge k-coloring* of a graph we mean a proper decomposition of G into $D = \{E_1, E_2, \ldots, E_k\}$. A *pseudo-achromatic edge k-coloring* of G is a complete decomposition of G into $D = \{E_1, E_2, \ldots, E_k\}$. In either case, the set E_i, $i = 1, 2, \ldots, k$, is called a *color class* of D and the vertex set of $\langle E_i \rangle_G$ is called the *support* of E_i.

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The *chromatic index* $\chi'(G)$ of G can be defined as the smallest n such that G has a proper pseudo-achromatic edge n-coloring of G. The *pseudo-achromatic index* $\Psi'_S(G)$ of G is the largest number m such that G has a pseudo-achromatic edge m-coloring of G, and the *achromatic index* $\Psi'(G)$ is the largest number ℓ such that G has a proper pseudo-achromatic edge ℓ-coloring of G.

Example 1. Consider the graph C_4. Fig. 1 shows that $\Psi'(C_4) = 2$ and $\Psi'_S(C_4) = 3$.

The achromatic number, the vertex version of achromatic index, was discussed in [7–9] and generalized to pseudo-achromatic number in [2, 3]. The generalization is meaningful. For example, in the personnel attribution, we are concerned if there are related people between each pair of different departments as communication representatives of these two departments and we donot care if there are related people in the same department. In this situation, the concept of pseudo-achromatic number is much more meaningful. The achromatic index of complete graphs and complete multipartite graphs has been studied in [4, 6, 10]. In particular, Bouchet [4] proved a remarkable result.

Theorem 1.1. *Suppose q is odd and $p = q^2 + q + 1$. Then $\Psi'(K_p) = pq$ if and only if a projective plane of order q exists. Indeed, if $\Psi'(K_p) = pq$, then the supports of the color classes in any proper pseudo-achromatic edge $\Psi'(K_p)$-coloring form the lines of a projective plane with the vertices of K_p as points.*

Jamison [10] and Chiang and Fu [5] gave the best possible upper bounds for the achromatic indices of the complete graph and the regular complete multipartite graph, respectively. Chiang and Fu [5] also determined the achromatic numbers of regular complete multipartite graphs of certain kinds.

Theorem 1.2. *Let q be an odd order of a projective plane. Let n and m be positive integers such that $n|(q+1)$ and $m = q(q+1)/n$, and $K_{n[m]}$ be the regular complete n-partite graph with m vertices in each class of the partition. Then $\Psi'(K_{n[m]}) = q(n-1)m$.*

Studying the upper bounds for the achromatic index of a general graph, we find that it is appropriate to consider the pseudo-achromatic index of a graph. In Section 2, we

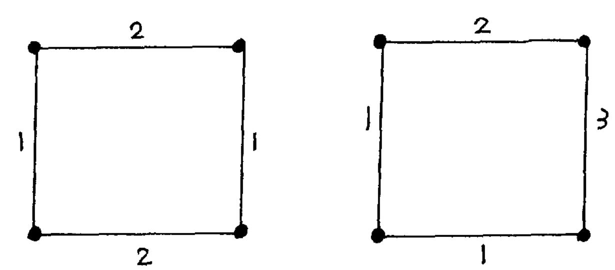

Fig. 1.

find the following upper bounds both of which are best possible.

(i) $\Psi'(G)\leqslant\Psi'_S(G)\leqslant\lfloor(e(G)+\chi'(G))/2\rfloor$; and

(ii) $\Psi'(G)\leqslant\Psi'_S(G)\leqslant\max_{1\leqslant k\leqslant\lfloor\frac{p}{2}\rfloor}\min\{\lfloor(p\Delta(G))/2k\rfloor,2k(\Delta(G)-1)+1\}$.

In Section 3, we show that the upper bound in (ii) actually gives the exact values of the achromatic indices of the graphs mentioned in Theorems 1.1 and 1.2. Furthermore, the pseudo-achromatic indices of these graphs are obtained.

2. The main results

The following lemmas are well known.

Lemma 2.1. *Let G be a graph. Then* $\chi'(G)\leqslant\Psi'(G)\leqslant\Psi'_S(G)$.

Lemma 2.2. *Let H be a subgraph of G. Then* $\Psi'(H)\leqslant\Psi'(G)$, *and* $\Psi'_S(H)\leqslant\Psi'_S(G)$.

Now we are ready to prove our first inequality.

Theorem 2.3. *For any graph G,* $\Psi'(G)\leqslant\Psi'_S(G)\leqslant\lfloor(e(G)+\chi'(G))/2\rfloor$.

Proof. It suffices to show that $\Psi'_S(G)\leqslant\lfloor(e(G)+\chi'(G))/2\rfloor$. Let $\chi'(G)=n$. Then there exists a proper complete decomposition $D=\{E_1,E_2,\dots,E_n\}$. Let $\Psi'_S(G)=m$ and $D'=\{F_1,F_2,\dots,F_m\}$ be a complete decomposition of G. It is clear that for $1\leqslant i\neq j\leqslant m$, $F_i\cup F_j\not\subseteq E_k$, $k=1,2,\dots,n$. Hence, at least $m-n$ sets of D' contains at least two edges from different sets in D. Thus $2(m-n)+n\leqslant e(G)$. So that $\Psi'_S(G)=m\leqslant\lfloor(e(G)+\chi'(G))/2\rfloor$. □

By considering the star graph $K_{1,q}$ (that is, the complete bipartite graph with bipartition (X,Y) of the vertex set such that $|X|=1$ and $|Y|=q$), it is not difficult to see that $\Psi'(K_{1,q})=\Psi'_S(K_{1,q})=q=\lfloor(q+q)/2\rfloor$ for every positive integer q. This shows that the upper bound in Theorem 2.3 is best possible. Moreover, we can use this bound to obtain a Nordhaus–Gaddum type theorem [1, 11] for the achromatic index and the pseudo-achromatic index.

Corollary 2.4. *For any graph G of order p, we have*

(i) $2\lfloor(p+1)/2\rfloor-1\leqslant\Psi'(G)+\Psi'(\bar{G})\leqslant\Psi'_S(G)+\Psi'_S(\bar{G})\leqslant\lfloor((p+4)(p-1))/4\rfloor$,

(ii) $0\leqslant\Psi'(G)\Psi'(\bar{G})\leqslant\Psi'_S(G)\Psi'_S(\bar{G})\leqslant\lfloor\frac{1}{4}\lfloor((p+4)(p-1))/4\rfloor^2\rfloor$.

And all the bounds are attainable.

Proof. Alavi and Behzad [1] proved that for an arbitrary graph G of order p the following inequalities hold:

(i) $2\lfloor(p+1)/2\rfloor-1\leqslant\chi'(G)+\chi'(\bar{G})\leqslant p+2\lfloor(p-2)/2\rfloor$,

(ii) $0\leqslant\chi'(G)\chi'(\bar{G})\leqslant(p-1)(2\lfloor p/2\rfloor-1)$

By these results and Lemma 2.2, we have

$$2\left\lfloor \frac{p+1}{2} \right\rfloor - 1 \leqslant \Psi'_S(G) + \Psi'_S(\bar{G}) \leqslant \frac{e(G)+\chi'(G)}{2} + \frac{e(\bar{G})+\chi'(\bar{G})}{2}$$
$$\leqslant \frac{p(p-1)}{4} + \frac{p+2\lfloor \frac{p-2}{2} \rfloor}{2} \leqslant \left\lfloor \frac{(p+4)(p-1)}{4} \right\rfloor.$$

While,

$$0 \leqslant \Psi'_S(G)\Psi'_S(\bar{G}) \leqslant \Psi'_S(G)\left(\left\lfloor \frac{(p+4)(p-1)}{4} \right\rfloor - \Psi'_S(G)\right) \leqslant \frac{1}{4}\left\lfloor \frac{(p+4)(p-1)}{4} \right\rfloor^2.$$

Hence (ii) holds. □

For (i) the equalities hold for P_2 and P_3. For (ii) the equality holds on the left-hand side for complete graphs while on the right-hand side equality holds for P_3.

Even though the upper bound obtained in Theorem 2.3 is the best possible, the difference between $\Psi'_S(G)$ and $\lfloor (e(G)+\chi'(G))/2 \rfloor$ can be bigger than any positive integer. For example, consider the double star $S_{p,q}$, $p \geqslant q$ as described in Fig. 2 (that is, $K_{1,p}$ and $K_{1,q}$ which share an edge). Then $\chi'(S_{p,q}) = p$, $e(G) = p + q - 1$ and $\Psi'(S_{p,q}) = \Psi'_S(S_{p,q}) = p$. Hence the difference between $\Psi'_S(G)$ and $\lfloor (e(G)+\chi'(G))/2 \rfloor$ is $\lfloor (q-1)/2 \rfloor$ and can be made arbitrarily large by suitable choices of p and q.

An easy upper bound on pseudo-achromatic index can be obtained by simple counting. Denote the degree of a vertex v by $d(v)$. Since any two color classes share a vertex, we must have

$$\binom{\Psi'_S(G)}{2} \leqslant \sum_{v \in V(G)} \binom{d(v)}{2}.$$

This is easy to compute if the degree sequence of G is known. For example, if G is r-regular, the right-hand side becomes $p(G)\binom{r}{2} = (r-1)e(G)$. The bound obtained also gives the value of the pseudo-achromatic index for friendship graphs (even if this is very easy to get otherwise). Another upper bound can be obtained in a different way.

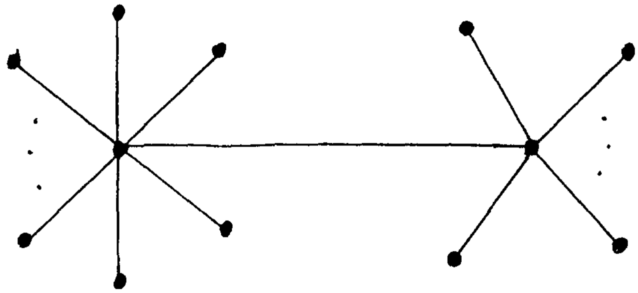

Fig. 2.

Theorem 2.5. *Let G be a graph of order p and size e with maximum degree Δ, $\Delta \geqslant 2$. Then* $\Psi'(G) \leqslant \Psi'_S(G) \leqslant \max_{1 \leqslant k \leqslant \lfloor p/2 \rfloor} \min\{\lfloor p\Delta/2k \rfloor, 2k(\Delta - 1) + 1\}$.

Proof. Let f be a complete edge $\Psi'_S(G)$-coloring of G. By Lemma 2.1 and Vizing's theorem, we know that $\Psi'_S(G) \geqslant \Delta(G) = \Delta$. Suppose that the smallest color class S of f consists of k edges. Then $1 \leqslant k \leqslant \lfloor \frac{e}{\Delta} \rfloor$.

Since $\langle S \rangle_G$ has at most $2k$ vertices and the degree of each vertex is at most Δ, the number of edges not in S but incident with some edges in S is at most $2k(\Delta - 1)$. Hence, $\Psi'_S(G) \leqslant 2k(\Delta - 1) + 1$. On the other hand, since each color class consists of at least k edges, the edge set $E(G)$ can be decomposed into at most $\lfloor p\Delta/2k \rfloor$ color classes. Therefore, $\Psi'_S(G) \leqslant \lfloor p\Delta/2k \rfloor$. Hence,

$$\Psi'_S(G) \leqslant \min\left\{\left\lfloor \frac{p\Delta}{2k} \right\rfloor, 2k(\Delta - 1) + 1\right\}$$

and

$$\Psi'_S(G) \leqslant \max_{1 \leqslant k \leqslant \lfloor \frac{e}{\Delta} \rfloor} \min\left\{\left\lfloor \frac{p\Delta}{2k} \right\rfloor, 2k(\Delta - 1) + 1\right\}.$$

Since $\lfloor p\Delta/2k \rfloor$ is nonincreasing as a function of k and $\lfloor p\Delta/2k \rfloor \leqslant 2k(\Delta - 1) + 1$ when $k \geqslant \lfloor p/2 \rfloor$, we have $\Psi'(G) \leqslant \Psi'_S(G) \leqslant \max_{1 \leqslant k \leqslant \lfloor \frac{p}{2} \rfloor} \min\{\lfloor p\Delta/2k \rfloor, 2k(\Delta - 1) + 1\}$. $\square$

To see that the upper bound in Theorem 2.5 is best possible, let us consider the graphs P_k and C_k, the path and the cycle of order k, respectively. We note here that $\Psi'(G) \leqslant \Psi'_S(G) \leqslant m(G)$, where $m(G) = \max\{n : \lceil (n-1)/2(\Delta(G) - 1) \rceil n \leqslant e(G)\}$, since it is easy to check that $m(G)$ is always larger than the upper bound in Theorem 2.5. This upper bound is appropriate in this case.

The following result is known [8].

Lemma 2.6. *Let* $m = \max\{n : \lceil (n-1)/2 \rceil n \leqslant k\}$. *Then,*

(i) *For* $k \geqslant 2$, $\Psi(P_k) = \begin{cases} m - 1 & \text{if } m \text{ is odd and } k = \lceil \frac{m-1}{2} \rceil m, \\ m & \text{otherwise.} \end{cases}$

(ii) *For* $k \geqslant 3$, $\Psi(C_k) = \begin{cases} m - 1 & \text{if } m \text{ is odd and } k = \lceil \frac{m-1}{2} \rceil m + 1, \\ m & \text{otherwise.} \end{cases}$

Using Theorem 2.5 and Lemma 2.6, we can get the following results.

Corollary 2.7. *For every* $k \geqslant 1$, *let* $m = \max\{n : \lceil (n-1)/2 \rceil n \leqslant k\}$. *Then,*

$$\Psi'_S(P_{k+1}) = \begin{cases} m - 1 & \text{if } m \text{ is odd and } k = \lceil \frac{m-1}{2} \rceil m, \\ m & \text{otherwise.} \end{cases}$$

Proof. Since the line graph $L(P_{k+1})$ of P_{k+1} is a path P_k, we have $\Psi'(P_{k+1}) = \Psi(P_k) = m - 1$ if m is odd and $k = \lceil (m-1)/2 \rceil m$, and $\Psi(P_k) = m$ otherwise.

By the upper bound we mentioned before Lemma 2.6, all we need to show is that there is no pseudo-achromatic edge m-coloring of P_{k+1} for the case when m is odd and $k = \lceil (m-1)/2 \rceil m$. Suppose to the contrary, $\Psi'_S(P_{k+1}) = m$. Let $m = 2h+1$. Then, either there is a color class of fewer than h edges, or all the color classes consist of exactly h edges. In either case, not all the color classes can meet. This contradicts the definition of pseudo-achromatic edge m-coloring. □

Corollary 2.8. *For every* $k \geqslant 3$, $\Psi'_S(C_k) = m$ *where* $m = \max\{n : \lceil (n-1)/2 \rceil n \leqslant k\}$.

Proof. Since the line graph of C_k is C_k, by Lemma 2.6(ii), we need only show that there is a pseudo-achromatic edge m-coloring of C_k for the case when m is odd and $k = \lceil (m-1)/2 \rceil m + 1$. In that case, there is an achromatic edge m-coloring for P_{k+1}. Identifying the first and the last vertices, we get a pseudo-achromatic edge m-coloring of C_k. □

Obviously, by Corollaries 2.7 and 2.8, the upper bound obtained in Thoerem 2.5 is best possible. Since the line graph of P_{k+1} is P_k and the line graph of C_k is C_k, we can get the following results about the pseudo-achromatic numbers which were previously obtained by Bories [5].

Corollary 2.9. *For every* $k \geqslant 2$, *let* $m = \max\{n : \lceil \frac{n-1}{2} \rceil n \leqslant k\}$. *Then*

$$\Psi_S(P_k) = \begin{cases} m-1 & \text{if } m \text{ is odd and } k = \lceil \frac{m-1}{2} \rceil m, \\ m & \text{otherwise.} \end{cases}$$

Corollary 2.10. *For every* $k \geqslant 3$, $\Psi_S(C_k) = m$ *where* $m = \max\{n : \lceil (n-1)/2 \rceil n \leqslant k\}$.

3. A sharp bound

In this section, along the approach developed by Jamison in [10], we expand the upper bound obtained in Theorem 2.5 into a form which reveals that the upper bound gives the exact value of the achromatic indices of the graphs mentioned in Theorems 1.1 and 1.2. As a consequence, we get the pseudo-achromatic indices of all these graphs.

Lemma 3.1. *Let* G *be a graph of order* p *with maximum degree* Δ. *Then for* $\bar{k} \geqslant 1$, *we have*

(i) $\Psi'_S(G) \leqslant 2(\Delta-1)\bar{k}+1$ *if* $4(\Delta-1)\bar{k}^2 + 2\bar{k} \leqslant p\Delta \leqslant 4(\Delta-1)\bar{k}^2 + (4\Delta-2)\bar{k}+2$; *and*

(ii) $\Psi'_S(G) \leqslant \lfloor p\Delta/2(\bar{k}+1) \rfloor$ *if* $4(\Delta-1)\bar{k}^2 + (4\Delta-2)\bar{k}+2 < p\Delta < 4(\Delta-1)(\bar{k}+1)^2 + 2(\bar{k}+1)$.

Proof. Let $g(x,y,z) = 2(x-1)z$, $h(x,y,z) = xy/2z$. Then it is clear that $g(x,y,z)$ is increasing in z and $h(x,y,z)$ is decreasing in z. Further,

$$\begin{aligned} g(\Delta,p,k)+1 \leqslant \lfloor h(\Delta,p,k)\rfloor &\Leftrightarrow g(\Delta,p,k)+1 \leqslant h(\Delta,p,k) \\ &\Leftrightarrow 2(\Delta-1)k+1 \leqslant \frac{p\Delta}{2k} \\ &\Leftrightarrow \Delta p \geqslant 4(\Delta-1)k^2+2k. \end{aligned}$$

Also,

$$\begin{aligned} g(\Delta,p,k)+1 \leqslant \lfloor h(\Delta,p,k+1)\rfloor &\Leftrightarrow g(\Delta,p,k)+1 \leqslant h(\Delta,p,k+1) \\ &\Leftrightarrow 2(\Delta-1)k+1 \leqslant p\Delta/2(k+1) \\ &\Leftrightarrow \Delta p \geqslant 4(\Delta-1)k^2+(4\Delta-2)k+2. \end{aligned}$$

So for fixed $\bar{k}$, if Δ and p satisfy that

$$4(\Delta-1)\bar{k}^2+2\bar{k} \leqslant p\Delta < 4(\Delta-1)(\bar{k}+1)^2+2(\bar{k}+1),$$

then we have

$$g(\Delta,p,k)+1 \leqslant \lfloor h(\Delta,p,k)\rfloor \quad \text{if } k \leqslant \bar{k},$$

and

$$g(\Delta,p,k)+1 \geqslant \lfloor h(\Delta,p,k)\rfloor \quad \text{if } k > \bar{k}.$$

Therefore $\max_{1\leqslant k\leqslant \lfloor p/\Delta\rfloor} \min\{\lfloor p\Delta/2k\rfloor, 2k(\Delta-1)+1\} = \max\{g(\Delta,p,\bar{k}), \lfloor h(\Delta,p,\bar{k}+1)\rfloor\}$. Since $4(\Delta-1)\bar{k}^2+(4\Delta-2)\bar{k}+2$ is located between $4(\Delta-1)\bar{k}^2+2\bar{k}$ and $4(\Delta-1)(\bar{k}+1)^2+2(\bar{k}+1)$ for every $\bar{k}\geqslant 1$, we have

$$\max_{1\leqslant k\leqslant \lfloor \frac{p}{2}\rfloor} \min\{\lfloor p\Delta/2k\rfloor, 2k(\Delta-1)+1\}$$

$$= \begin{cases} g(\Delta,p,\bar{k})+1 & \text{if } 4(\Delta-1)\bar{k}^2+2\bar{k} \leqslant p\Delta \\ & \quad \leqslant 4(\Delta-1)\bar{k}^2+(4\Delta-2)\bar{k}+2, \\ \lfloor h(\Delta,p,\bar{k}+1)\rfloor & \text{if } 4(\Delta-1)\bar{k}^2+(4\Delta-2)\bar{k}+2 < p\Delta \\ & \quad < 4(\Delta-1)(\bar{k}+1)^2+2(\bar{k}+1). \end{cases}$$

Hence the lemma holds. □

Theorem 3.2. *Suppose q is an odd order of a projective plane and $p = q^2+q+1$. Then $\Psi'_S(K_p) = pq$.*

Proof. Here, we have $\Delta = q^2+q$, $p = q^2+q+1$. It is easy to check that $p\Delta = (q^2+q)(q^2+q+1)$ lies between $4((q^2+q)-1)((q-1)/2)^2+(4(q^2+q)-2)\cdot((q-1)/2)+2$ and $4((q^2+q)-1)((q+1)/2)^2+2\cdot(q+1)/2$. Hence by Lemma 3.1 and Theorem

1.1, we have $\Psi'_S(K_p) \leqslant \Delta p/2(\frac{q-1}{2}+1) = pq = \Psi'(K_p)$. By Lemma 2.1, we have $\Psi'_S(K_p) = pq$. $\square$

Theorem 3.3. *Let q be an odd order of a projective plane. Let n and m be positive integers such that $n \mid (q+1)$ and $m = q(q+1)/n$, and $K_{n[m]}$ be the regular complete n-partite graph with m vertices in each class of the partition. Then $\Psi'_S(K_{n[m]}) = q(n-1)m$.*

Proof. Here we have $\Delta = (n-1)m = [(n-1)/n] \cdot q(q+1)$, $p = nm = q(q+1)$. It is easy to check that $p\Delta = [(n-1)/n] \cdot q^2(q+1)^2$ lies between $4([(n-1)/n] \cdot q(q+1) - 1)((q-1)/2)^2 + (4((n-1)/n) \cdot q(q+1) - 2)[(q-1)/2] + 2$ and $4([(n-1)/n] \cdot q(q+1) - 1)((q+1)/2)^2 + 2 \cdot [(q+1)/2]$. So we have $\Psi'_S(K_{n[m]}) \leqslant \Delta p/2([(q-1)/2]+1) = \{[(n-1)/n] \cdot q^2(q+1)^2/q+1\} = q(n-1)m = \Psi'(K_{n[m]})$. Hence $\Psi'_S(K_{n[m]}) = q(n-1)m$. $\square$

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