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PARTITION-OPTIMIZATION WITH SCHUR CONVEX SUM OBJECTIVE FUNCTIONS*

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Abstract. We study optimization problems over partitions of the finite set $N = \{1, \ldots, n\}$, where each element i in the partitioned set N is associated with a real number θ^i and the objective associated with a partition $\pi = (\pi_1, \ldots, \pi_p)$ has the form $F(\pi) = f(\theta_\pi)$, where $\theta_\pi = (\sum_{i\in\pi_1} \theta^i, \ldots, \sum_{i\in\pi_p} \theta^i)$. When F is to be either maximized or minimized, we obtain conditions that allow for simple constructions of partitions that are uniformly optimal for all Schur convex functions f.

Key words. partitions, optimization, Schur-convexity

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1. Introduction.

We consider partitions of the finite set $N = \{1, \ldots, n\}$ into nonempty parts. When a corresponding partition π has p parts, we refer to it as a *p-partition* and denote it by $\pi = (\pi_1, \ldots, \pi_p)$; also, we refer to the vector $(|\pi_1|, \ldots, |\pi_p|)$ as the *shape* of the partition π.

Throughout, we assume that each element i in the partitioned set N is associated with a real number θ^i and, by possibly permuting the elements of N, we may assume that $\theta^1 \le \theta^2 \le \cdots \le \theta^n$. A partition is called *consecutive* if (after the possible permutation of N) the elements in each part are consecutive integers.

We consider optimization problems (maximization and minimization) over families of partitions where the objective value $F(\pi)$ associated with a partition π is given through a real-valued function f that is defined on R^p and $F(\pi) = f\big(\sum_{i\in\pi_1} \theta^i, \ldots, \sum_{i\in\pi_\pi} \theta^i\big)$; such partitioning problems are called *sum partitioning problems.* Of particular interest are *constrained shape*, *bounded-shape*, and *single-shape* problems, where the underlying sets of partitions are defined, respectively, by restrictions, bounds, and specification on the shape of partitions. For many applications of partitioning problems see, for example, [1, 2, 3, 4].

An important tool for studying optimization problems is the identification of properties that are satisfied by optimal solutions. In particular, determining the existence of optimal solutions with a particular property allows one to restrict the search for an optimal solution to a smaller class of feasible solutions, namely, those that satisfy the property. For partitioning problems, consecutiveness is a particularly valuable property, as the number of p-partitions with prescribed shape is exponential in n, while the number of consecutive p-partitions is $p!$. Conditions on the function f that suffice for the optimality of consecutive partitions have been studied extensively in the literature. Hwang and Rothblum [3] introduced a class of functions called *asymmetric Schur convex functions*, unifying classical (quasi) convexity and Schur

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convexity; asymmetric Schur convexity was shown in Gao, Hwang, Li, and Rothblum [1] to be sufficient for optimality of consecutive partitions, generalizing many earlier results.

The goal of the current paper is to study bounded-shape partitioning problems where the function f is Schur convex and the objective is to either maximize F or to minimize it. We identify conditions that allow for explicit solution of such problems without the need to scan through all consecutive partitions. Under these conditions, optimality turns out to be invariant of the particular (Schur convex) function f. It follows that, depending on whether the objective function is to be maximized or minimized, the vector associated with an invariant optimal partition must majorize or be majorized by the vectors associated with all other feasible partitions (see section 2 for formal definitions). For bounded-shape maximization problems, we explicitly construct an invariant consecutive optimal partition when the ranking of the coordinates of the lower bounds on the part-sizes is consistent with that of the upper bounds and, in addition, the θ^i's have the uniform sign; further, we demonstrate that if either of these two conditions is dropped, an invariant optimal partition need not exist. For bounded-shape minimization problems, we explicitly construct an invariant solution when all the θ^i's are 1, that is, when the vector associated with a partition is the shape of the partition; further, we show via an example that this restriction cannot be relaxed. Our proof for minimization problems first identifies a vector which is majorized by all vectors that satisfy prescribed lower and upper bounds and have a prescribed coordinate-sum. We then show that when the bounds and the prescribed coordinate-sum are integers, the majorized vector can be rounded up/down to an integer vector that is majorized by all corresponding integer vectors. Results of Veinott [7] concern the construction of majorized vectors in a more general context of network flows, and his proofs depend on yet unpublished results in [8]. The proofs we derive herein are self-contained and simpler.

2. Preliminaries. Throughout, we let n be a positive integer and $N \equiv \{1, \ldots, n\}$. A *partition* (of N) is an ordered collection of sets $\pi = (\pi_1, \ldots, \pi_p)$, where $\pi_1, \ldots, \pi_p$ are disjoint *nonempty* subsets of N whose union is N. In this case we refer to p as the *size* of π and to the sets $\pi_1, \ldots, \pi_p$ as the *parts of* π. Also, if the number of elements in the parts of the partition $\pi = (\pi_1, \ldots, \pi_p)$ are $n_1, \ldots, n_p$, respectively, we refer to $(n_1, \ldots, n_p)$ as the *shape* of π; of course, in this case $\sum_{j=1}^{p} n_j = |N| = n$. We sometimes refer to *p-partitions* or to $(n_1, \ldots, n_p)$-*partitions* as partitions of size p or of shape $(n_1, \ldots, n_p)$, respectively. A partition is called *consecutive* if its parts consist of consecutive integers, that is, if there is an enumeration of its parts, say, $\pi_{j_1}, \ldots, \pi_{j_p}$, such that for $t = 1, \ldots, p$ and corresponding positive integers $n_{j_1}, \ldots, n_{j_p}$, $\pi_{j_t} = \left\{\sum_{s=1}^{t-1} n_{j_s} + 1, \ldots, \sum_{s=1}^{t} n_{j_s}\right\}$.

We assume that each element i in the given partitioned set N is associated with a real number θ^i and, without loss of generality,

$$\theta^1 \leq \theta^2 \leq \cdots \leq \theta^n. \tag{2.1}$$

We denote by θ the vector $(\theta^1, \ldots, \theta^n) \in R^n$. Also, for a subset $S \subseteq \{1, \ldots, n\}$ we define the *S-summation scalar* θ_S by $\theta_S \equiv \sum_{i \in S} \theta^i$. For a p-partition $\pi = (\pi_1, \ldots, \pi_p)$ we define the π-*summation-vector* θ_π by $\theta_\pi \equiv (\theta_{\pi_1}, \ldots, \theta_{\pi_p}) \in R^p$.

Throughout this paper we let p be a fixed positive integer. Given a real-valued function F over a set Π of p-partitions, we consider the problem of maximizing F over

Π. The problem is called *sum-partitioning* if there is a function $f: R^p \to R$ such that

$$F(\pi) = f(\theta_\pi) \quad \text{for each } p\text{-partition } \pi. \tag{2.2}$$

We refer to *single-shape*, *bounded-shape* and *constrained-shape* problems as partitioning problems with Π as the set of partitions with a prescribed shape, with a shape that satisfies the prescribed lower and upper bound and with a shape in a prescribed set, respectively. For constrained-shape problems the set of partitions is defined through a set Γ of positive integer p-vectors with the coordinate-sum n. For bounded-shape problems, Γ is defined by two positive integer p-vectors L and U satisfying $L \le U$ and $\sum_{j=1}^p L_j \le |N| \le \sum_{j=1}^p U_j$; we then write $\Gamma^{(L,U)}$ for Γ and $\Pi^{(L,U)}$ for the corresponding set of partitions. Finally, for single-shape problems, Γ is defined by a single positive integer p-vector $(n_1, \dots, n_p)$ satisfying $\sum_{j=1}^p n_j = |N|$; we then write $\Gamma^{(n_1,\dots,n_p)}$ for Γ and $\Pi^{(n_1,\dots,n_p)}$ for the corresponding set of partitions.

For a vector $x \in R^n$ and $k = 1, \dots, n$, let $x_{[k]}$ be the kth largest coordinate of x. We say that a vector $a \in R^p$ *majorizes* a vector $b \in R^p$, written $a \gg b$, if

$$\sum_{i=1}^k a_{[i]} \ge \sum_{i=1}^k b_{[i]} \quad \text{for all} \quad k = 1, \dots, p \tag{2.3}$$

and

$$\sum_{i=1}^p a_{[i]} = \sum_{i=1}^p b_{[i]}; \tag{2.4}$$

we note that (2.3) and (2.4) are, respectively, equivalent to

$$\max_{|I|=k} \sum_{i\in I} a_i \ge \max_{|I|=k} \sum_{i \in I} b_i \quad \text{for all} \quad k = 1, \dots, p \tag{2.3$'$}$$

and

$$\sum_{i=1}^p a_i = \sum_{i=1}^p b_i. \tag{2.4$'$}$$

We say that a *strictly majorizes* b if a majorizes b but does not majorize a.

A real-valued function f on a subset B of R^p is called *Schur convex* if $f(a) \ge f(b)$ for all a, $b \in B$ satisfying $a \gg b$, that is, if f is order-preserving with respect to the partial order majorization. The function f is called *strictly Schur convex* if it is Schur convex and $f(a) > f(b)$ for all a, $b \in B$ for which a strictly majorizes b. For example, a real-valued function f on R^p with $f(x) = \sum_{j=1}^p g(x_j)$, where g is a (strictly) convex real-valued function on R, is known to be (strictly) Schur convex (see [6]); such functions are called *separable (strictly) Schur convex.* We say that f is (strictly) Schur concave if -f is (strictly) Schur convex.

We say that a p-vector z is a *majorizing* vector in a finite set $\Lambda \subseteq R^p$ if $z \in \Lambda$ and z majorizes every vector in Λ; we say that z is a *minorizing* vector in Λ if $z \in \Lambda$ and z is majorized by every vector in Λ. Since majorization is a partial order that does not provide comparisons for all pairs of vectors, majorizing and minorizing vectors need not exist.

For $j = 1, \dots, p-1$, let $f^{(j)}$ be the real-valued function on R^p with $f^{(j)}(x) = \max_{\{I \subseteq \{1,\dots,p\}:|I|=j\}} \sum_{u\in I} x_u$ for each $x \in R^p$ (these functions are convex as the maximum of linear functions). The characterization of majorization through (2.3′)–(2.4′) shows that a finite set $\Lambda \subseteq R^p$ contains a majorizing/minorizing vector if and only if the functions $f^{(1)}, \dots, f^{(p-1)}$ are simultaneously maximized/minimized over Λ and, in addition, all vectors in Λ have a common coordinate-sum.

3. Maximization problems with *f* Schur convex. In this section we focus on maximization problems where the function f is Schur convex.

Let Π be a set of partitions. We say that a partition π^* is *shape-majorizing in* Π if $\pi^* \in \Pi$ and the shape of π^* majorizes the shape of every other partition in Π; when Π is defined as the set of partitions with its shape in a prescribed set Γ, π^* is shape-majorizing if and only if its shape is a majorizing vector in Γ. The next result shows that if Γ has a majorizing vector, a shape-majorizing partition exists.

PROPOSITION 3.1. *Suppose* Γ *is a set of positive integer p-vectors with coordinate-sum n and* Π *is the set of partitions with its shape in* Γ*. If* $(n_1, \dots, n_p)$ *is a majorizing vector in* Γ*, then there exists a consecutive shape-majorizing partition in* Π.

Proof. The conclusion of the lemma follows from the existence of consecutive partitions with any prescribed shape (in fact, the consecutive partitions with prescribed shape are in one-to-one correspondence with the permutations over $\{1, \dots, p\}$). ◻

We say that θ is *sign-uniform* if it is either nonpositive or nonnegative. The next result shows that this condition together with the assumptions of Proposition 3.1 facilitate a uniform solution for sum-partitioning problems under all Schur convex functions f. This is accomplished by first determining a majorizing shape and then assigning the elements to parts greedily (where greedily has different meanings for the case where $\theta \le 0$ and for the case where $\theta \ge 0$).

THEOREM 3.2. *Suppose f is Schur convex,* Γ *is a set of positive integer p-vectors with the coordinate-sum n,* $(n_1, \dots, n_p)$ *is a majorizing vector in* Γ *with* $n_1 \le \cdots \le n_p$*, and* Π *is the (constrained-shape) set of partitions with its shape in* Γ.

(i) *If* $\theta \le 0$*, then the (consecutive) p-partition* π^- *with* $\pi_j^- = \{n - \sum_{u=1}^{j} n_u + 1, \dots, n - \sum_{u=1}^{j-1} n_u\}$ *for* $j = 1, \dots, p$ *is in* Π *and maximizes* $F(.)$ *over* Π.

(ii) *If* $\theta \ge 0$*, then the (consecutive) p-partition* π^+ *with* $\pi_j^+ = \{\sum_{u=1}^{j-1} n_u + 1, \dots, \sum_{u=1}^{j} n_u\}$ *for* $j = 1, \dots, p$ *is in* Π *and maximizes* $F(.)$ *over* Π.

Further, if f is strictly Schur convex, the inequalities of (2.1) *hold strictly, and the* θ^i*'s are nonzero, then* π^- *and* π^+ *are, respectively, the only optimal partitions.*

Proof. We first consider the case where $\theta \ge 0$. Since the shape of π^+ is $(n_1, \dots, n_p) \in \Gamma$, then π^+ is shape-majorizing in Π. Also, from $n_1 \le \cdots \le n_p$ we have that $|\pi_1^+| \le \cdots \le |\pi_p^+|$. These properties of π^+ ensure that for each $\pi \in \Pi$, $j \in \{1, \dots, p\}$ and enumeration $u_1, \dots, u_p$ of the elements $1, \dots, p$,

$$
\begin{aligned}
\sum_{s=1}^{j} |\pi_{u_s}| &\le \max_{\{I \subseteq \{1,\dots,p\}:|I|=j\}} \sum_{u \in I} |\pi_u| \le \max_{\{I \subseteq \{1,\dots,p\}:|I|=j\}} \sum_{u \in I} |\pi_u^+| \\
&= \sum_{u=p-j+1}^{p} |\pi_u^+| = \sum_{u=p-j+1}^{p} n_u. \qquad (3.1)
\end{aligned}
$$

We conclude from (3.1), (2.1), the nonnegativity of the θ^i's, and the definition of π^+ that

$$
\sum_{s=1}^{j} (\theta_\pi)_{u_s} = \sum_{i \in \pi_{u_1} \cup \cdots \cup \pi_{u_j}} \theta^i \le \sum_{i=n_1+\cdots+n_{p-j}+1}^{n} \theta^i = \sum_{u=p-j+1}^{p} (\theta_{\pi^+})_u, \qquad (3.2)
$$

with equality holding when $j = p$. Since π^+ is in Π, it also satisfies (3.2). Applying (3.2) to π^+ and to π, we conclude that

$$\max_{\{I\subseteq\{1,\dots,p\}:|I|=j\}} \sum_{u\in I}(\theta_\pi)_u \le \sum_{i=n_1+\cdots+n_{p-j}+1}^{n} \theta^i = \max_{\{I\subseteq\{1,\dots,p\}:|I|=j\}} \sum_{u\in I}(\theta_{\pi^+})_u \tag{3.3}$$

with equality holding when $j = p$. Thus, θ_{π^+} majorizes θ_π and, therefore, the Schur convexity of f implies that $F(\pi^+) = f(\theta_{\pi^+}) \ge f(\theta_\pi) = F(\pi)$.

Next, assume that $\theta \le 0$. Since the shape of π^- is $(n_1,\dots,n_p) \in \Gamma$, π^- is also shape-majorizing in Π. Also, from $n_1 \le \cdots \le n_p$ we have that $|\pi_1^-| \le \cdots \le |\pi_p^-|$. These properties of π^- ensure that for each $\pi \in \Pi$, $j \in \{1,\dots,p\}$ and enumeration $u_1,\dots,u_p$ of the elements $1,\dots,p$,

$$\begin{aligned}\sum_{s=j+1}^{p} |\pi_{u_s}| &\le \max_{\{I\subseteq\{1,\dots,p\}:|I|=p-j\}} \sum_{u\in I} |\pi_u| \le \max_{\{I\subseteq\{1,\dots,p\}:|I|=p-j\}} \sum_{u\in I} |\pi_u^-| \\ &= \sum_{u=j+1}^{p} |\pi_u^-| = \sum_{u=j+1}^{p} n_u,\end{aligned} \tag{3.4}$$

and, therefore,

$$\sum_{s=1}^{j} |\pi_{u_s}| = n - \sum_{s=j+1}^{p} |\pi_{u_s}| \ge n - \sum_{u=j+1}^{p} |\pi_u^-| = \sum_{u=1}^{j} n_u. \tag{3.5}$$

From (2.1), (3.5), the nonpositivity of the θ^i's, and the definition of π^-, we see that

$$\sum_{s=1}^{j} (\theta_\pi)_{u_s} = \sum_{i\in\pi_{u_1}\cup\cdots\cup\pi_{u_j}} \theta^i \le \sum_{i=n-(n_1+\cdots+n_j)+1}^{n} \theta^i = \sum_{u=1}^{j} (\theta_{\pi^-})_u, \tag{3.6}$$

with equality holding when $j = p$. Since π^- is in $\Pi^{(n_1,\dots,n_p)}$, it also satisfies (3.6). Applying (3.6) to π^- and to π, we conclude that

$$\max_{\{I\subseteq\{1,\dots,p\}:|I|=j\}} \sum_{u\in I}(\theta_\pi)_u \le \sum_{i=n-(n_1+\cdots+n_j)+1}^{n} \theta^i = \max_{\{I\subseteq\{1,\dots,p\}:|I|=j\}} \sum_{u\in I}(\theta_{\pi^-})_u \tag{3.7}$$

with equality holding when $j = p$. Thus, θ_{π^-} majorizes θ_π and, therefore, the Schur convexity of f implies that $F(\pi^-) = f(\theta_{\pi^-}) \ge f(\theta_\pi) = F(\pi)$.

Finally, if the inequalities of (2.1) hold strictly and the θ^i's are nonzero, then for each $\pi \ne \pi^+$, (3.4) implies that (3.5) holds as a strict inequality for at least one j; thus, θ_{π^+} strictly majorizes θ_π. Consequently, if f is strictly Schur convex, we have that $F(\pi^+) = f(\theta_{\pi^+}) > f(\theta_\pi) = F(\pi)$. A similar argument shows that if the inequalities of (2.1) hold strictly, the θ^i's are nonzero, and f is strictly Schur convex, then $F(\pi^-) = f(\theta_{\pi^-}) > f(\theta_\pi) = F(\pi)$. ∎

Solution of constrained-shape partitioning problems with f Schur convex, sign-uniform θ, and given majorizing shape. Let Γ be a set of positive integer p-vectors with coordinate-sum n and let $(n_1,\dots,n_p)$ be a majorizing vector in Γ with $n_1 \le \cdots \le n_p$. Also, assume the $\theta^1,\dots,\theta^n$ are given and satisfy (2.1). Of course, if either the θ^i's and/or the n_u's are not ranked a priori, one can sort them and renumber indices in time $O[n(lg\, n)]$ and/or $O[p(lg\, p)]$, respectively. Once the indices are renumbered, Theorem 3.2 provides an explicit solution of the partitioning problem when either $\theta \geq 0$ or $\theta \leq 0$; only the partial sums of the n_j's are needed, and these can be determined with p additions and the associated vector can be determined with, at most, n additions.

Next we explain how the "expensive" sorting of the θ^i's can be reduced. Suppose a sorting of $n_1, \ldots, n_p$ is executed if needed (requiring time $O[p(lg\, p)]$ comparisons), and an index-enumeration $j_1, \ldots, j_p$ satisfying $n_{j_1} \leq n_{j_2} \leq \cdots \leq n_{j_p}$ becomes available. It is then not necessary to fully sort $\theta^1, \ldots, \theta^n$ in order to determine the optimal partition; all that is needed is to determine the set of n_{j_1}-smallest coordinates of θ, the next n_{j_2}-smallest coordinates, and so on. This block-sorting can be executed with $O(pn)$ comparisons [5], yielding an improved complexity bound of $O(pn)$. If the data is given with (2.1) in force, Theorem 3.2 provides an explicit solution of the partitioning problem requiring only the sorting of $n_1, \ldots, n_p$; so, in this case the problem is solvable in time $O[p(lg\, p)]$.

Theorem 3.2 yields an explicit solution to partitioning problems when a majorizing shape within the set of allowable shapes Γ is available. Such a shape is trivially available when Γ contains a single shape, e.g., when either $\sum_{j=1}^p L_j = n$ or $\sum_{j=1}^p U_j = n$. Next we obtain a sufficient condition for the existence of a majorizing shape in nondegenerate bounded-shape problems; further, under this condition the majorizing shape is easily computable.

LEMMA 3.3. *Let L and U be positive integer p-vectors satisfying $L \leq U$ and $\sum_{j=1}^p L_j < n < \sum_{j=1}^p U_j$. Then there exists an index $j \in \{1, \ldots, p\}$ with $\sum_{u=1}^j L_u + \sum_{u=j+1}^p U_u = \sum_{u=1}^p U_u - \sum_{u=1}^j (U_u - L_u) \leq n$; further, if j^* is the first such index and $\mu^* \equiv n - \sum_{u=1}^{j^*-1} L_u - \sum_{u=j^*+1}^p U_u$, then $(n_1^*, \ldots, n_p^*) \equiv (L_1, \ldots, L_{j^*-1}, \mu^*, U_{j^*+1}, \ldots, U_p) \in \Gamma^{(L,U)}$, and*

$$\sum_{u=1}^k n_u^* = \max\left\{\sum_{u=1}^k L_u, n - \sum_{u=k+1}^p U_u\right\} \quad \text{for } k = 1, \ldots, p. \tag{3.8}$$

Moreover, if

$$L_1 \leq L_2 \leq \cdots \leq L_p \tag{3.9}$$

and

$$U_1 \leq U_2 \leq \cdots \leq U_p, \tag{3.10}$$

then $n_1^* \leq \cdots \leq n_p^*$ and $(n_1^*, \ldots, n_p^*)$ majorizes every vector in $\Gamma^{(L,U)}$.

Proof. The existence of an index $j \in \{1, \ldots, p\}$ with $\sum_{u=1}^j L_u + \sum_{u=j+1}^p U_u = \sum_{u=1}^p U_u - \sum_{u=1}^j (U_u - L_u) \leq n$ is immediate from the fact that $\sum_{u=1}^p U_u > n$ and $\sum_{u=1}^p U_u - \sum_{u=1}^p (U_u - L_u) = \sum_{u=1}^p L_u < n$. With j^* as the first such index and with the definition of μ^* and $(n_1^*, \ldots, n_p^*)$ as in the statement of the lemma, we clearly have that $L_{j^*} \leq \mu^* < U_{j^*}$ and $(n_1^*, \ldots, n_p^*) \in \Gamma^{(L,U)}$. Also, from the definition of j^* and n_j^*'s we have that

$$\sum_{u=1}^k n_u^* = \begin{cases} \sum_{u=1}^k L_u > n - \sum_{u=k+1}^p U_u & \text{if } k < j^*, \\ n - \sum_{u=k+1}^p u_u^* \geq \sum_{u=1}^k L_u & \text{if } k \geq j^*. \end{cases}$$

When either $k < j^*$ or $k \geq j^*$, we have that (3.8) holds.

Next, assume that (3.9) and (3.10) hold. To verify that the coordinates of $(n_1^*, \ldots, n_p^*)$ are nondecreasing, observe that if $t < j^*$, we have $n_t^* = L_t \leq L_{t+1} \leq n_{t+1}^*$, and if $t \geq j^*$, we have $n_t^* \leq U_t \leq U_{t+1} = n_{t+1}^*$. Next, let I be a subset of $\{1, \ldots, p\}$ and let $(n_1, \ldots, n_p)$ be a vector in $\Gamma^{(L,U)}$. The complement of I within $\{1, \ldots, p\}$ will be denoted I^c. Since $\sum_{u \in I} n_u \leq \sum_{u \in I} U_u$ and $n - \sum_{u \in I} n_u = \sum_{u \in I^c} n_u \geq \sum_{u \in I^c} L_u$, we have that

$$\sum_{u \in I} n_u \leq \min\left\{n - \sum_{u \in I^c} L_u, \sum_{u \in I} U_u\right\} \leq \min\left\{n - \sum_{u=1}^{p-|I|} L_u, \sum_{u=p-|I|+1}^{p} U_u\right\}, \tag{3.11}$$

where (3.9)–(3.10) are used for the second inequality in (3.11). Also, for each $j = 1, \ldots, p-1$, we get from (3.8) (with $k = p - j$) that

$$\begin{aligned} \sum_{u=p-j+1}^{p} n_u^* &= n - \sum_{u=1}^{p-j} n_u^* = n - \max\left\{\sum_{u=1}^{p-j} L_u, n - \sum_{u=p-j+1}^{p} U_u\right\} \\ &= \min\left\{n - \sum_{u=1}^{p-j} L_u, \sum_{u=p-j+1}^{p} U_u\right\}. \end{aligned} \tag{3.12}$$

Since $(n_1^*, \ldots, n_p^*) \in \Gamma^{(L,U)}$, (3.11) applies to $(n_1^*, \ldots, n_p^*)$. It follows from (3.11) applied to $(n_1, \ldots, n_p)$ and to $(n_1^*, \ldots, n_p^*)$ and from (3.12) that, for $j = 1, \ldots, p-1$,

$$\begin{aligned} \max_{\{I \subseteq \{1,\ldots,p\}:|I|=j\}} \sum_{u \in I} n_u &\leq \min\left\{n - \sum_{u=1}^{p-j} L_u, \sum_{u=p-j+1}^{p} U_u\right\} = \sum_{u=p-j+1}^{p} n_u^* \\ &= \max_{\{I \subseteq \{1,\ldots,p\}:|I|=j\}} \sum_{u \in I} n_u^*, \end{aligned}$$

verifying that $(n_1^*, \ldots, n_p^*)$ majorizes $(n_1, \ldots, n_p)$. ◻

Next we state two immediate conclusions from Theorem 3.2 and Lemma 3.3.

THEOREM 3.4. *Suppose f is Schur convex and L and U are positive integer p-vectors satisfying $L \leq U$, $\sum_{j=1}^p L_j < n < \sum_{j=1}^p U_j$,* (3.9), *and* (3.10). *Let $(n_1^*, \ldots, n_p^*)$ be as in Lemma* 3.3.

(i) *If $\theta \leq 0$, then the (consecutive) p-partition π^- with $\pi_j^- = \{n - \sum_{u=1}^j n_u^* + 1, \ldots, n - \sum_{u=1}^{j-1} n_u^*\}$ for $j = 1, \ldots, p$ is in $\Pi^{(L,U)}$ and maximizes $F(.)$ over $\Pi^{(L,U)}$.*

(ii) *If $\theta \geq 0$, then the (consecutive) p-partition π^+ with $\pi_j^+ = \{\sum_{u=1}^{j-1} n_u^* + 1, \ldots, \sum_{u=1}^j n_u^*\}$ for $j = 1, \ldots, p$ is in $\Pi^{(L,U)}$ and maximizes $F(.)$ over $\Pi^{(L,U)}$.*

Further, if f is strictly Schur convex, the inequalities of (2.1) *hold strictly, and the θ^i's are nonzero, then π^- and π^+ are, respectively, the only optimal partitions.*

Under the assumptions of Theorem 3.4, the solution method discussed following Theorem 3.2 applies; further, Lemma 3.3 shows that the computation of the majorizing-shape vector $(n_1^*, \ldots, n_p^*)$ is available with $O(p)$ arithmetic operations.

We say that two vectors, L and U, in R^p are *consistent* if there exists a permutation $(\{u_1\}, \ldots, \{u_p\})$ such that the vectors $(L_{u_1}, \ldots, L_{u_p})$ and $(U_{u_1}, \ldots, U_{u_p})$ satisfy (3.9)–(3.10). Corollary 3.4 implies that when f is Schur convex, L and U are consistent positive integer p-vectors satisfying $L \leq U$ and $\sum_{j=1}^p L_j < n < \sum_{j=1}^p U_j$, and θ is sign-uniform, there exists a majorizing vector in $\Gamma^{(L,U)}$ and a (consecutive, shape-majorizing) partition in $\Pi^{(L,U)}$ which is optimal uniformly under all Schur convex functions f. Further, such a partition is easily computable by first (jointly) sorting the L_u's and U_u's and then selecting either of the two partitions constructed in Theorem 3.2.

Two important cases for which the assumptions of Lemma 3.3 and Theorem 3.4 apply are as follows:

(i) single-shape problem, where the coordinates of a single prescribed shape, say, $(n_1, \dots, n_p)$, can be ranked and permuted to satisfy the monotonicity assumption (3.9)–(3.10) with $L = U = (n_1, \dots, n_p)$, and

(ii) uniform bounded shape problem, where L_u's and U_u's are, respectively, independent of u.

The next two examples demonstrate, respectively, that neither the consistency of L and U nor the sign-uniformity of θ can be removed from Corollary 3.5.

EXAMPLE I. *Suppose* $p = 3$, $n = 9$, $L_1 = 1$, $L_2 = L_3 = 2$, $U_1 = 5$, $U_2 = U_3 = 4$, *and* $\theta^i = 1$ *for* $i = 1, \dots, 9$. *With* $\Pi \equiv \Pi^{(L,U)}$, $\max_{\pi\in\Pi} \max_u (\theta_\pi)_u = 5$, *and the maximum is realized by exactly the partitions with shape* $(5, 2, 2)$. *However,* $\max_{\pi\in\Pi} \max_{u,v}[(\theta_\pi)_u + (\theta_\pi)_v] = 8$, *and the maximum is realized by exactly the partitions with shape* $(1, 4, 4)$. *Thus, there is no shape-majorizing partition in* $\Pi^{(L,U)}$. *It is easily noted that* $\Gamma^{(L,U)}$ *does not have a vector which majorizes all other vectors in the set.*

To see that no partition is optimal uniformly under all (separable) Schur convex functions f, *let* f_1 *and* f_2 *be the (separable, strictly Schur convex) functions with* $f_1(x) = \sum_{u=1}^{3} |x_u|^3$ *and* $f_2(x) = \sum_{u=1}^{3} |x_u - 4|^3$. *The shapes in* $\Gamma^{(L,U)}$ *are* $(5, 2, 2)$, $(4, 3, 2)$, $(4, 2, 3)$, $(3, 4, 2)$, $(3, 3, 3)$, $(3, 2, 4)$, $(2, 4, 3)$, $(2, 3, 4)$, *and* $(1, 4, 4)$*; the values of these vectors under* (f_1, f_2) *are, respectively,* $(141, 17)$, $(99, 9)$, $(99, 9)$, $(99, 9)$, $(81, 3)$, $(99, 9)$, $(99, 9)$, $(99, 9)$, *and* $(129, 27)$. *So, the optimal partitions with the objective defined by* f_1 *and* f_2 *are, respectively, those with shape* $(5, 2, 2)$ *and those with shape* $(1, 4, 4)$.

EXAMPLE II. *Suppose* $p = 3$, $n = 6$, $n_j = j$ *for* $j = 1, 2, 3$, $\theta^i = -1$ *for* $i = 1, 2, 3$, *and* $\theta^i = 1$ *for* $i = 4, 5, 6$. *With* $\Pi \equiv \Pi^{(1,2,3)}$, $\max_{\pi\in\Pi} \max_u (\theta_\pi)_u = 3$, *and the maximum is realized by the partitions with* $\pi_3 = \{4, 5, 6\}$ *and only by those. However,* $\max_{\pi\in\Pi} \max_{u,v}[(\theta_\pi)_u + (\theta_\pi)_v] = 3$, *and the maximum is realized by the partition with* $\pi_3 = \{1, 2, 3\}$ *and only by them. Thus, there is no partition* π' *in* Π *with* $\theta_{\pi'}$ *majorizing each of the vectors associated with a partition* π *in* Π. *To see that no partition is optimal uniformly under all Schur convex functions* f, *let* f_1 *and* f_2 *be the (separable, strictly Schur convex) functions with* $f_1(x) = \sum_{u=1}^{3} |x_u + 3|^3$ *and* $f_2(x) = \sum_{u=1}^{3} |x_u - 3|^3$*; the optimal partitions with* f_1 *and* f_2 *are, respectively, precisely the partitions* π *with* $\pi_3 = \{4, 5, 6\}$ *and those with* $\pi_3 = \{1, 2, 3\}$.

4. Minimization problems with *f* Schur convex. In this section we focus on minimization problems where the function f is Schur convex. The main result of this section can be derived from more general results of Veinott [6, Theorem 2, p. 554] which depend on (yet unpublished) results of [8]; the proofs provided herein are self-contained and more elementary.

Let Π be a set of partitions. We say that a partition π^* is *shape-minorizing in* Π if $\pi^* \in \Pi$ and the shape of π^* is majorized by the shape of every other partition in Π; when Π is defined as the set of partitions with its shape in a prescribed set Γ, π^* is shape-minorizing if and only if its shape is a minorizing vector in Γ. The next result shows that if Γ has a minorizing vector, a shape-minorizing partition exists.

PROPOSITION 4.1. *Suppose* Γ *is a set of positive integer* p*-vectors with a coordinate-sum* n *and* Π *is the set of partitions with its shape in* Γ*. If* $(n_1, \ldots, n_p)$ *is a minorizing vector in* Γ*, then there exists a consecutive shape-minorizing partition in* Π.

Proof. As for Proposition 3.1, the conclusion follows from the existence of consecutive partitions with any prescribed shape. ∎

The next result is in the spirit of Theorem 3.2 with minimization replacing maximization—it provides conditions for the existence of a uniform solution to constrained-shape partitioning problems under the assumptions of Proposition 4.1. But here, more restrictive conditions than sign-uniformity of θ are required.

THEOREM 4.2. *Suppose that* $\theta^i = 1$ *for each* i *(that is, the objective function is a function of the shape of a partition). Then any shape-minorizing partition is optimal (minimizing) uniformly under all Schur convex functions* f.

Proof. The assumptions of the theorem imply that for each partition π, θ_π is the shape of π, and the conclusion of the theorem follows from the definition of Schur convexity. ∎

The next example demonstrates that sign-uniformity of θ is not sufficient for the set of vectors associated with partitions having a prescribed shape to contain a minorizing vector, nor is it sufficient for the existence of a uniformly minimizing partition under all Schur convex functions. So, in general, the conclusions of Theorem 3.2 do not generalize when minorization replaces majorization. It is noted that the example concerns a single-shape problem.

EXAMPLE III. *Let* $n = 11$*,* $p = 3$*,* $n_1 = 2$*,* $n_2 = 4$*,* $n_3 = 5$*,* $\theta^i = 1$ *for* $i = 1, 2, 3, 4$*,* $\theta^i = 2$ *for* $i = 5, 6, 7, 8$*, and* $\theta^i = 6$ *for* $i = 9, 10, 11$*. Let* X *be the set of positive integer* 3*-vectors with coordinate-sum* 30*. All vectors associated with feasible partitions are in* X*. Now,* $x^1 \equiv (10, 10, 10)$ *is majorized by all vectors in* X *and* $x^2 \equiv (11, 10, 9)$ *is majorized by all vectors in* X *except for* x^1*. But neither* x^1 *nor* x^2 *is realizable by a feasible partition because neither* 9 *nor* 10 *nor* 11 *is the sum of two elements among* $\{1, 2, 6\}$*. Next we observe that* $x^3 = (11, 11, 8)$ *and* $x^4 = (12, 9, 9)$ *are majorized by all vectors in* $X \backslash \{x^1, x^2, x^3, x^4\}$*, but neither majorizes the other. Representing parts of partitions by the multiset of the* θ^i*'s, we observe that* $(11, 11, 8)$ *is realizable by the partition* $\pi^3 = (\{5, 9\}, \{1, 6, 7, 10\}, \{2, 3, 4, 8, 11\})$ *and* $(12, 9, 9)$ *is realizable by the partition* $\pi^4 = (\{10, 11\}, \{1, 2, 3, 9\}, \{4, 5, 6, 7, 8\})$.

For $t > 0$*, let* $f_t : R^3 \to R$ *be given by* $f_t(x) = \sum_{j=1}^{3} |x_j - 10 - t|^3$ *for each* $x \in R^3$*. These functions are separable and strictly Schur convex; further, for all sufficiently small positive* t*,* $f_t(x^3) > f_t(x^4)$*, and the reverse inequality holds for all sufficiently large negative* t*. Since every vector in* $X \backslash \{x^1, x^2, x^3, x^4\}$ *majorizes either* x^3 *or* x^4*, the Schur convexity of the* f_t*'s implies that* π^4 *is optimal for all sufficiently small positive* t*, and* π^3 *is optimal for all sufficiently large negative* t*.*

We next show that every set of bounded shapes contains a minorizing shape, without the restriction concerning the consistency of the lower bound and the upper bound. Of course, Example III demonstrates that shape-minorization does not yield uniform optimality as does shape-majorization with sign-uniform θ. Our first step considers noninteger vectors.

THEOREM 4.3. *Let* L *and* U *be* p*-vectors satisfying* $L \le U$ *and* $\sum_{j=1}^{p} L_j < n < \sum_{j=1}^{p} U_j$*, respectively. For every real* $\beta > 0$ *define* $x(\beta)$ *as the* p*-vector with*

$$x(\beta)_j \equiv \begin{cases} L_j & \text{if } \beta \le L_j, \\ \beta & \text{if } L_j < \beta < U_j, \\ U_j & \text{if } \beta \ge U_j. \end{cases} \tag{4.1}$$

Then $x(.)$ *is nondecreasing and continuous, and* $\{x(\beta) \in R^p : \sum_{j=1}^p x(\beta)_j = n\}$ *contains a single vector, say,* x^**, which is majorized by every vector in* $\{x \in R^p : L \leq x \leq U$ *and* $\sum_{j=1}^p x_j = n\}$.

Proof. The fact that $x(.)$ is nondecreasing and continuous is immediate from (4.1). Further, since $\sum_{j=1}^p x(\beta)_j = \sum_{j=1}^p L_j < n$ for $\beta \leq \min_j L_j$, and $\sum_{j=1}^p x(\beta)_j = \sum_{j=1}^p U_j > n$ for $\beta \geq \max_j U_j$, continuity arguments assure that $\sum_{j=1}^p x(\beta)_j = n$ for some $\min_j L_j < \beta < \max_j U_j$. Since $x(.)$ is nondecreasing, $\sum_{j=1}^p x(\beta)_j = \sum_{j=1}^p x(\beta')_j$ if and only if $x(\beta) = x(\beta')$. So, $\{x(\beta) \in R^p : \sum_{j=1}^p x(\beta)_j = n\}$ contains a single element, say, x^*. We note that $\{\beta \in R : x(\beta) = x^*\}$ is a nonempty closed interval which is nondegenerate when $\{j = 1, \ldots, p : L_j < x_j^* < U_j\} = \varnothing\}$.

Let $N_- \equiv \{j = 1, \ldots, p : x_j^* = L_j\}$, $N_0 \equiv \{j = 1, \ldots, p : L_j < x_j^* < U_j\}$, $N_+ = \{j = 1, \ldots, p : x_j^* = U_j > L_j\}$, $v_- \equiv |N_-|$, $v_0 \equiv |N_0|$, and $v_+ \equiv |N_+|$. Of course, $v_- + v_0 + v_+ = p$. Select β^* such that $x(\beta^*) = x^*(\beta^*$ is unique when $N_0 \neq \varnothing)$. We then have that $x_j^* = L_j \geq \beta^*$ for $j \in N_-$, $x_j^* = \beta^*$ for $j \in N_0$, and $x_j^* = U_j \leq \beta^*$ for $j \in N_+$. It follows that by possibly permuting indices, we can assume that x^*'s coordinates are nonincreasing, all elements in N_- precede all elements in N_0, and all elements in N_0 precede all elements in N_+; in particular, $N_- = \{1, \ldots, v_-\}$, $N_0 = \{v_- + 1, \ldots, v_- + v_0\}$, and $N_+ = \{v_- + v_0 + 1, \ldots, p\}$.

Let $X \equiv \{x \in R^p : L \leq x \leq U$ and $\sum_{j=1}^p x_j = n\}$. Also, for $k = 1, \ldots, p$, let $W^k \equiv \{w \in R^p : 0 \leq w \leq 1$ and $\sum_{j=1}^p w_j = k\}$ (with 1 representing the vector $(1, \ldots, 1)^T$ in R^p), and let $h_k : X \to R$ with $h_k(x)$ for x in X being the sum of the k largest coordinates of x. We observe that the functions h_k have representations

$$h_k(x) = \sum_{u=1}^k x_{[u]} = \max_{[I]=k} \sum_{u \in I} x_u = \max_{w \in W^k} \sum_{u=1}^k w_u x_u = \max_{w \in W^k} w^T x. \tag{4.2}$$

The claim that $x^* \in X$ is majorized by all vectors x in X means that x^* minimizes each h_k over X. We consider three ranges for k.

$1 \leq k \leq v_-$: In this case for each $x \in X$,

$$h_k(x^*) = \sum_{u=1}^k x_{[u]}^* = \sum_{u=1}^k x_u^* = \sum_{u=1}^k L_u \leq \sum_{u=1}^k x_u \leq \sum_{u=1}^k x_{[u]} = h_k(x). \tag{4.3}$$

$p - v_+ \leq k \leq p$: In this case for each $x \in X$,

$$\begin{aligned} h_k(x^*) &= \sum_{u=1}^k x_{[u]}^* = \sum_{u=1}^k x_u^* = n - \sum_{u=k+1}^p x_u^* = n - \sum_{u=k+1}^p U_u \\ &\leq \sum_{u=1}^p x_u - \sum_{u=k+1}^p x_u \leq \sum_{u=1}^k x_u \leq \sum_{u=1}^k x_{[u]} = h_k(x). \end{aligned} \tag{4.4}$$

$v_- k < p - v_+$: We will construct a vector w^* in W^k that satisfies

$$w^T x^* \leq (w^*)^T x^* \leq (w^*)^T x \quad \text{for each} \quad x \in X \quad \text{and} \quad w \in W^k. \tag{4.5}$$

It will then follow from (4.2) that for every $x \in X$, $h_k(x^*) = \max_{w \in W^k} (w)^T x^* = (w^*)^T x^* \leq (w^*)^T x \leq h_k(x)$. (In fact, a variant of the classic minmax theorem of game theory ensures that the existence of such a vector w^* is necessary and sufficient for x^* to minimize h_k over X.) Specifically, let $\omega \equiv (k - v_-)/v_0$, and let w^* be the p-vector with

$$w_u^* \equiv \begin{cases} 1 & \text{for } \; u = 1, \dots, v_-, \\ \omega & \text{for } \; u = v_- + 1, \dots, v_- + v_0, \\ 0 & \text{for } \; u = v_- + v_0 + 1, \dots, p. \end{cases} \tag{4.6}$$

Since $v_- < k < p - v_+ = v_- + v_0$, we have that $v_0 = p - v_- - v_+ > 0$ and $0 < \omega < 1$; in particular, $w^* \in W^k$.

For $z \in R^p$ and $j = 0, 1, \dots, p$, let $\bar{z}_j = \sum_{u=1}^{j} z_u$; in particular, $\bar{x}_p = n$ and $\bar{w}_p = k$ for each $x \in X$ and $w \in W^k$. Further,

$$w^{*T} x = \sum_{u=1}^{p} w_u^* x_u = \sum_{u=1}^{p} w_u^* (\bar{x}_u - \bar{x}_{u-1}) = \sum_{u=1}^{p-1} (w_u^* - w_{u+1}^*) \bar{x}_u + w_p^* n \quad \text{for each } \; x \in X \tag{4.7}$$

and

$$w^T x^* = \sum_{u=1}^{p} w_u x_u^* = \sum_{u=1}^{p} (\bar{w}_u - \bar{w}_{u-1}) x_u^* = \sum_{u=1}^{p-1} \bar{w}_u (x_u^* - x_{u+1}^*) + k x_u^* \quad \text{for each } \; w \in W. \tag{4.8}$$

Applying (4.7) to x^* and to arbitrary $x \in X$, we observe that

$$\begin{aligned} (w^*)^T x^* - (w^*)^T x &= \sum_{u=1}^{p-1} (w_u^* - w_{u+1}^*)(\bar{x}_u^* - \bar{x}_u) \\ &= (1 - \omega)(\bar{x}_{v_-}^* - \bar{x}_{v_-}) + \omega(\bar{x}_{v_- + v_0}^* - \bar{x}_{v_- + v_0}) \end{aligned} \tag{4.9}$$

(the cases where $v_- = 0$ and/or $v_+ = 0$ require special attention). From (4.3) with $k = v_-$, we have that $\bar{x}_{v_-}^* \leq \bar{x}_{v_-}$, and from (4.4) with $k = v_- + v_0 = p - v_+$, we have that $\bar{x}_{v_- + v_0}^* \leq \bar{x}_{v_- + v_0}$; since $0 \leq \omega \leq 1$, we conclude from (4.9) that $(w^*)^T x^* \leq w^{*T} x$, establishing the right-hand side inequalities of (4.5). Next, by applying (4.8) to w^* and to arbitrary $w \in W^k$, we observe that

$$\begin{aligned} w^{*T} x^* - w^T x^* &= \sum_{u=1}^{p-1} (\bar{w}_u^* - \bar{w}_u)(x_u^* - x_{u+1}^*) \\ &= \sum_{u=1}^{v_-} (u - \bar{w}_u)(x_u^* - x_{u+1}^*) + \sum_{u=v_-+1}^{v_- + v_- - 1} (k - \bar{w}_u)(\beta^* - \beta^*) \\ &\quad + \sum_{u=v_- + v_0}^{p} (k - \bar{w}_u)(x_u^* - x_{u+1}^*) \end{aligned} \tag{4.10}$$

(here again, the cases where $v_- = 0$ and/or $v_+ = 0$ require special attention). Since $\bar{w}_u \leq u$ and $\bar{w}_u \leq k$ for each $w \in W^k$ and $u = 1, \dots, p$ and since $x_1^* \geq x_2^* \geq \dots \geq x_p^*$, we conclude from (4.10) that $(w^*)^T x^* \geq w^T x^*$ for every $w \in W^k$, completing the proof of (4.5). $\square$

In the next result, we use the notation $\| \; \|_\infty$ for the 1_∞ norm in R^p defined for $x \in R^p$ by $\|x\|_\infty = \max_{u \in \{1, \dots, p\}} x_u$.

THEOREM 4.4. *Let L and U be positive integer p-vectors satisfying $L \le U$ and $\sum_{j=1}^{p} L_j < n < \sum_{j=1}^{p} U_j$, and let x^* be as in Theorem 4.3. Then there exists an integer p-vector z^* with $\|z^* - x^*\|_\infty < 1$, and each such vector is majorized by every integer vector in $\left\{x \in R^p : L \le x \le U \text{ and } \sum_{j=1}^{p} x_j = n\right\}$.*

Proof. The conclusion of this theorem is trivial when x^* is integral, so assume that this is not the case. Let N_-, N_0, N_+, v_-, v_0, and v_+ be as in the proof of Theorem 4.3, and as in that proof assume that x^*'s coordinates are nonincreasing, all elements in N_- precede all elements in N_0, and all elements in N_0 precede all elements in N_+; in particular, $N_- = \{1, \dots, v_-\}$, $N_0 = \{v_- + 1, \dots, v_- + v_0\}$, and $N_+ = \{v_- + v_0 + 1, \dots, p\}$. The assertion that x^* is not integral means that $N_0 \ne \varnothing$ and the unique β^* with $x(\beta^*) = x^*$ is not integral.

Let $X \equiv \{x \in R^p : L \le x \le U \text{ and } \sum_{j=1}^{p} x_j = n\}$, let $\lfloor \beta^* \rfloor$ be the largest integer less than β^*, and let $\lceil \beta^* \rceil \equiv \lfloor \beta^* \rfloor + 1$. The integrality of L and U ensures that $L_u \le \lfloor \beta^* \rfloor < \beta^* < \lceil \beta^* \rceil \le U_u$ for $u \in N_0$. Further, we observe that $v_0 \beta^* = n - \sum_{u \in N_-} L_u - \sum_{u \in N_+} U_u$ is an integer and $v_0 \lfloor \beta^* \rfloor < v_0 \beta^* < v_0 \lceil \beta^* \rceil$, implying that $\mu \equiv v_0 \beta^* - v_0 \lfloor \beta^* \rfloor$ is an integer satisfying $1 \le \mu < v_0$ and $\mu \lceil \beta^* \rceil + (v_0 - \mu) \lfloor \beta^* \rfloor = v_0 \lfloor \beta^* \rfloor + \mu(\lceil \beta^* \rceil - \lfloor \beta^* \rfloor) = v_0 \lfloor \beta^* \rfloor + \mu = v_0 \beta^*$. It follows that the p-vector z^* with z_u^* for $u = 1, \dots, p$ given by

$$z_u^* \equiv \begin{cases} x_u^* & \text{if } u \in N_- \cup N_0, \\ \lceil \beta^* \rceil & \text{if } u = v_- + 1, \dots, v_- + \mu, \\ \lfloor \beta^* \rfloor & \text{if } u = v_- + \mu + 1, \dots, v_- + v_0 \end{cases} \tag{4.11}$$

is integral, is in X, and satisfies $\|z^* - x^*\|_\infty < 1$. We will show that z^* is majorized by any integer vector z in X by showing that $h_k(z) \ge h_k(z^*)$ for $k = 1, \dots, p$, where $h_k(.)$ is the function assigning to each p-vector the sum of its k largest coordinates (see the proof of Theorem 4.3).

Let z be an integer vector in X. For $u \in N_-, L_u \ge \beta^*$, and the integrality of L_u implies that $L_u \ge \lceil \beta^* \rceil$. Similarly, for $u \in N_+$, $U_u \le \beta^*$, and the integrality of U_u implies that $U_u \le \lfloor \beta^* \rfloor$. Consequently, z^*'s coordinates are nonincreasing and, therefore, $h_k(z^*) = \sum_{j=1}^{k} z_{[j]}^* = \sum_{j=1}^{k} z_j^*$ for $k = 1, \dots, p$. From Theorem 4.3, $h_k(z) \ge h_k(x^*) = h_k(z^*)$ for $1 \le k \le v_-$ and for $v_0 + v_+ \le k \le p$. Further, as Theorem 4.3 ensures that $h_{v_-+1}(z) \ge h_{v_-+1}(x^*) = h_{v_-}(x^*) + \beta^*$, the integrality of $h_{v_-+1}(z)$ and $h_{v_-}(x^*)$ implies that $h_{v_-+1}(z) \ge h_{v_-}(x^*) + \lceil \beta^* \rceil = h_{v_-+1}(z^*)$. To prepare for an inductive argument, assume that $h_k(z) \ge h_k(z^*)$ and $h_{k+1}(z) < h_{k+1}(z^*)$ for some $v_- + 1 \le k < v_0 + v_+ - 1$. Then $h_k(z^*) + z_{k+1}^* = h_{k+1}(z^*) > h_{k+1}(z) = h_k(z) + z_{[k+1]}$, implying that $z_{[k+1]} < h_k(z^*) + z_{k+1}^* - h_k(z) \le z_{k+1}^* \le \lceil \beta^* \rceil$. Since $z_{[k+1]}$ and $\lceil \beta^* \rceil$ are integral, we conclude that $z_{[k+1]} \le \lceil \beta^* \rceil - 1 = \lfloor \beta^* \rfloor$ and, therefore, $z_{[j]} \le \lfloor \beta^* \rfloor$ for $j = k+2, \dots, v_- + v_o$ (recall that the coordinates of z^* are nonincreasing). It follows that

$$\begin{aligned} h_{v_-+v_0}(z) &= h_{k+1}(z) + \sum_{u=k+2}^{v_-+v_0} z_{[u]} < h_{k+1}(z^*) + (v_- + v_0 - k - 1)\lfloor \beta^* \rfloor \\ &= \sum_{u=1}^{k+1} z_u^* + (v_- + v_0 - k - 1)\lfloor \beta^* \rfloor \le \sum_{u=1}^{v_-+v_0} z_u^* = h_{v_-+v_0}(x^*). \end{aligned}$$

This inequality contradicts the conclusion of Theorem 4.3, asserting that x^* is majorized by z, and thereby completes an inductive proof that $h_k(z) \ge h_k(z^*)$ for $k \in \{v_- + 1, \dots, v_0 + v_+\}$.

We finally observe that an integer vector z is in X and satisfies $\|z - x^*\|_\infty < 1$ if and only if $z_u = x^*_u$ for $u \in N_- \cup N_+$ (as each such x^*_u is integral), it has exactly μ of the v_0 coordinates z_u indexed by $u \in N_0$ equal $\lceil \beta^* \rceil$, and it has the remaining $v_0 - \mu$ coordinates indexed by $u \in N_0$ equal $\lfloor \beta^* \rfloor$. It follows that for each such z, a coordinate permutation of z^* exists, implying that $h_k(z) = h_k(z^*)$ for each $k = 1, \ldots, p$; in particular, such z, like z^*, is majorized by all integer vectors in X. □

REFERENCES

[1] B. Gao, F. K. Hwang, W. W-C. Li, and U. G. Rothblum, *Partition-polytopes over 1-dimensional points*, Math. Program., 85 (1999), pp. 335–362.
[2] F. K. Hwang, S. Onn, and U. G. Rothblum, *A polynomial time algorithm for shaped partition problems*, SIAM J. Optim., 10 (1999), pp. 70–81.
[3] F. K. Hwang, and U. G. Rothblum, *Directional-quasi-convexity, asymmetric Schur-convexity and optimality of consecutive partitions*, Math. Oper. Res., 21 (1996), pp. 540–554.
[4] F. K. Hwang, and U. G. Rothblum, *Partitions: Optimality and Clustering*, World Scientific, to appear.
[5] D. Knuth, *The Art of Computer Programming*, 2nd ed., Addison-Wesley, Reading, MA, 1981.
[6] A. W. Marshall, and I. Olkin, *Inequalities, Theory of Majorization and Its Applications*, Academic Press, New York, 1979.
[7] A. F. Veinott, Jr., *Least d-majorized network flows with inventory and statistical applications*, Management Sci., 17 (1971), pp. 547–567.
[8] A. F. Veinott, Jr., *On d-majorization and d-Schur convexity*, to appear.