# Improved Quantitative Measures of Robustness for Multivariable Systems

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*Abstract*— **Asymptotically stable linear systems subject to unstructured time varying perturbations are considered. Allowable perturbation bounds are obtained such that the perturbed systems remain stable. These bounds are derived iteratively by means of adjusting a sequence of Lyapunov matrices. In comparison with existing methods, less conservative quantitative measures of robustness are obtained.**

## I. Nomenclature

| | |
|---|---|
| $\mathbb{R}^n$ | Real vector space of dimension $n$. |
| $\mu$, $\nu$ | QRBM (Quantitative Robustness-Bound Measure). |
| $A'$ | Transposed matrix of $A$. |
| $\sigma_{\max}(A)$ | Maximum singular value of matrix $A$. |
| $\sigma_{\min}(A)$ | Minimum singular value of matrix $A$. |
| $A^{-1}$ | Inverse matrix of an invertible matrix $A$. |
| $A^{1/2}$ | Square root of positive–definite matrix $A$. |
| $\lVert v \rVert$ | Euclidean norm of vector $v$. |
| $P > 0$ | Square symmetric matrix $P$ being positive-definite. |
| $P \geq 0$ | Square symmetric matrix $P$ being positive-semidefinite. |
| $P > Q$ | Square symmetric matrices $P$ and $Q$ that satisfy $P - Q > 0$. |
| $P \geq Q$ | Square symmetric matrices $P$ and $Q$ that satisfy $P - Q \geq 0$. |

## II. Introduction

Recently, the aspect of developing explicit upper bound on the perturbation of linear systems, such that the perturbed systems remain stable, has received much attention. Starting with Patel *et al.* [1], Patel and Toda [2], and Lee [3], considerable effort has been given on the reduction of conservatism in quantitative measures of robustness with increasingly complicated ways of defining the structure of perturbations [4]–[7]. In these literatures, perturbations are broadly categorized as being unstructured or structured. For unstructurally perturbed systems, while it is assumed that only the norm bound of the perturbation is available, robustness measures can be derived by use of the Lyapunov theory [1], [2]. Since the robustness measures derived for the unstructured cases are generally used to propose robustness bounds for the structured cases, it is desirable to always derive less conservative robustness measures for unstructurally perturbed systems.

In this correspondence, new robustness measures for unstructurally perturbed systems are derived. By way of integrating matrices specified within a pair of Lyapunov equations, less conservative Quantitative Robustness-Bound Measure (QRBM) can be achieved.

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## III. Problem Formulation

We consider systems described by the differential equation

$$x(t)/dt = Ax(t) + f(x(t), t) \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is the stable nominal matrix, and $f(x(t), t)$ is an unstructured perturbing function with the property of $f(0, t) = 0$ for all time $t$. It is assumed that, while an exact expression of the perturbation cannot be written explicitly, an estimate of some bound on the perturbation is available. The problem which we investigate in this correspondence is to derive a less conservative quantitative bound on the perturbing function $f(x(t), t)$ such that the perturbed system described by (1) remains stable.

For unstructurally perturbed systems described by (1), the Lyapunov-based method of deriving QRBM's has been considered as well established for a long time [1], [2]. Existing results are the following two theorems where sufficient conditions for the stability of the system described by (1) were expressed as upper norm-bounds of the perturbing function $f(x(t), t)$.

*Theorem 1:* (Theorem 1 of [2]) The perturbed system described by (1) is stable if

$$\lVert f(z, t) \rVert / \lVert z \rVert \leq \mu_Q \equiv \sigma_{\min}(Q)/\sigma_{\max}(P) \tag{2}$$

for all $(z, t) \in \mathbb{R}^{n+1}$ with nonzero $z$, where $Q \in \mathbb{R}^{n \times n}$ is some symmetric positive–definite matrix, and $P \in \mathbb{R}^{n \times n}$ is the symmetric positive–definite matrix that fulfils the Lyapunov equation

$$A'P + PA = -2Q. \tag{3}$$

*Theorem 2:* (Lemma 2 of [2]) The bound $\mu_Q$ defined in (2) is maximum (i.e., the least conservative result of $\mu_Q$ denoted as $\mu_I$) when the matrix $Q = I$ is assigned in the Lyapunov equation (3), where $I$ is the $n \times n$ identity matrix.

It can be shown that these theorems generally produce QRBM's that are very conservative. In the following, another Lyapunov-based method of deriving QRBM's is proposed. The basic idea comes from the generally accepted fact that, more often than not, several Lyapunov functions are better than one. Since the results of the existing two theorems are not flexible enough for mingling Lyapunov functions, it is necessary for us to rederive the problem specified in Theorem 1.

*Theorem 3:* The perturbed system described by (1) is stable if

$$\lVert f(z, t) \rVert / \lVert z \rVert \leq \mu_C \equiv \sigma_{\min}(Q^{1/2})/\sigma_{\max}(Q^{-1/2}P) \tag{4}$$

for all $(z, t) \in \mathbb{R}^{n+1}$ with nonzero $z$, where matrices $P$ and $Q$ fulfil the Lyapunov equation (3).

*Proof:* Followed from the Lyapunov equation (3), a Lyapunov function of the stable nominal system matrix $A$ is given by

$$V(x) = x'Px. \tag{5}$$

Employing the quadratic function $V(x)$ on the perturbed system described by (1), a sufficient condition for justifying the stability of the perturbed system is given by

$$\begin{aligned} 0 \geq dV(x)/dt &= (dx/dt)'Px + x'P(dx/dt) \\ &= x'(A'P + PA)x + 2f'Px. \end{aligned} \tag{6}$$

Using Lyapunov equation (3), the sufficient condition for stability becomes

$$f'Px \leq x'Qx. \tag{7}$$

Making a slight modification, condition (7) is equivalently given by

$$(Q^{-1/2}Pf)'(Q^{1/2}x) \le (Q^{1/2}x)'(Q^{1/2}x) \tag{8}$$

and a sufficient condition for stability can be given as

$$\|Q^{-1/2}Pf\| \le \|Q^{1/2}x\|. \tag{9}$$

Knowing that

$$\|Q^{-1/2}Pf\| \le \sigma_{\max}(Q^{-1/2}P)\|f\| \tag{10}$$

and

$$\|Q^{1/2}x\| \ge \sigma_{\min}(Q^{1/2})\|x\| \tag{11}$$

condition (9) is sufficiently justified by

$$\sigma_{\max}(Q^{-1/2}P)\|f\| \le \sigma_{\min}(Q^{1/2})\|x\| \tag{12}$$

which complete the proof. □

We note that Theorem 1 was derived without the modification stage from (7) to (8) given in the proof of Theorem 3. Thus, it is implicitly assumed in Theorem 1 that matrices $P$ and $Q$ of the Lyapunov equation (3) are structurally unrelated. This is certainly not realistic, and Theorem 3 can be looked at as the rectified version of Theorem 1 in this particular point of view. Comparing the proposed bound given in (4) with the existing bound given in (2), since

$$\begin{aligned}\sigma_{\min}(Q^{1/2})/\sigma_{\max}(Q^{-1/2}P) &= \sigma_{\min}(Q^{1/2})\sigma_{\min}(P^{-1}Q^{1/2})\\ &\ge \sigma_{\min}(Q^{1/2})\sigma_{\min}(Q^{1/2})\sigma_{\min}(P^{-1})\\ &= \sigma_{\min}(Q)/\sigma_{\max}(P)\end{aligned} \tag{13}$$

the proposed bound ($\mu_C$) is always better than the existing bound ($\mu_Q$) for all possible choices of matrix $Q$ in the Lyapunov equation (3). In this correspondence, the following two facts are used:

i) $\sigma_{\min}(A+B) \ge \sigma_{\min}(A) + \sigma_{\min}(B)$, where $A \ge 0$, $B \ge 0$;

ii) $\sigma_{\min}(AB) \ge \sigma_{\min}(A)\sigma_{\min}(B)$.

More discussion of singular values and their properties can be found in various texts [8].

Heuristically, there is a particular choice of matrix $Q$ which will bring forth possibly the least conservative bound. The structural relation between matrices $P$ and $Q$ of the Lyapunov equation (3) can not be resolved, however, without first having the Lyapunov equation solved. This is a common difficulty confronting the robustness analyses by ways of the Lyapunov-based methods. Thus, we are obliged to create an iterative process such that a proper choice of matrix $Q$ for the Lyapunov equation (3) may be acquired with less conservative QRBM ($\mu_C$).

## IV. Reducing Conservatism in QRBM

Let $A$ be a stable system matrix, so will $A'$ be stable. The following Lyapunov equations specify two symmetric positive–definite matrices $P_1$ and $P_2$:

$$A'P_1 + P_1A + 2Q_1 = 0 \tag{14}$$

$$P_2^{-1}A' + AP_2^{-1} + 2Q_2 = 0 \tag{15}$$

where matrices $Q_1$ and $Q_2$ are symmetric positive–definite. Simultaneously, we have the following alternative expressions of (14) and (15):

$$P_1^{-1}A' + AP_1^{-1} + 2P_1^{-1}Q_1P_1^{-1} = 0 \tag{16}$$

$$A'P_2 + P_2A + 2P_2Q_2P_2 = 0. \tag{17}$$

Employing Theorem 3, the Lyapunov equations (14) and (17) provide us with two QRBM's ($\mu_C$) for the perturbed system described by (1). These are

$$\mu_1 = \sigma_{\min}(Q_1^{1/2})\sigma_{\min}(P_1^{-1}Q_1^{1/2}) \tag{18}$$

$$\mu_2 = \sigma_{\min}(Q_2^{1/2})\sigma_{\min}(P_2Q_2^{1/2}). \tag{19}$$

Without loss of generality, it is assumed in the followings that matrices $Q_1$ and $Q_2$ are always normalized such that

$$\sigma_{\min}(Q_1) = \sigma_{\min}(Q_2) = 1. \tag{20}$$

Thus, in matrix sense, relations given in (20) are expressed by

$$Q_1 \ge I \quad \text{and} \quad Q_2 \ge I \tag{21}$$

and relations given in (18) and (19) are expressed by

$$P_1^{-1}Q_1P_1^{-1} \ge \mu_1^2 I \tag{22}$$

$$P_2Q_2P_2 \ge \mu_2^2 I. \tag{23}$$

We note that (14) and (17) can be interpolated to make new Lyapunov equations, so can (15) and (16) in dual manner.

*Lemma 1:* Given Lyapunov equations (14) and (15), the following interpolated Lyapunov equations are fulfilled for all interpolating parameters $a \in (0, 1/\mu_2^2)$ and $b \in (0, 1/\mu_1^2)$:

$$A'X_1 + X_1A + 2Y_1 = 0 \tag{24}$$

$$X_2^{-1}A' + AX_2^{-1} + 2Y_2 = 0 \tag{25}$$

where

$$X_1 = (1 - a\mu_2^2)P_1 + aP_2, \tag{26}$$

$$Y_1 = (1 - a\mu_2^2)Q_1 + aP_2Q_2P_2, \tag{27}$$

$$X_2^{-1} = (1 - b\mu_1^2)P_2^{-1} + bP_1^{-1}, \tag{28}$$

$$Y_2 = (1 - b\mu_1^2)Q_2 + bP_1^{-1}Q_1P_1^{-1}. \tag{29}$$

*Proof:* Equation (24) is the direct interpolated result of (14) and (17). Equation (25) is the direct interpolated result of (15) and (16). □

Employing Theorem 3, the interpolated Lyapunov equations (24) and (25) provide us with two QRBM's ($\mu_C$) for the perturbed system described by (1). These are

$$\nu_1 = \sigma_{\min}(Y_1^{1/2})\sigma_{\min}(X_1^{-1}Y_1^{1/2}) \tag{30}$$

$$\nu_2 = \sigma_{\min}(Y_2^{1/2})\sigma_{\min}(X_2Y_2^{1/2}). \tag{31}$$

The following lemmas provide some qualitative properties that are useful for producing iterative interpolating procedures from which a proper choice of matrix $Q$ for the Lyapunov equation (3) can be acquired with less conservative QRBM ($\mu_C$).

*Lemma 2:* Given Lyapunov equations (14) and (15) with their QRBM's $\mu_1$ and $\mu_2$ given in (18) and (19), the interpolated Lyapunov equations (24) and (25) defined in Lemma 1 produce new QRBM's $\nu_1$ and $\nu_2$ given in (30) and (31) such that, for all interpolating parameters $a \in (0, 1/\mu_2^2)$ and $b \in (0, 1/\mu_1^2)$

$$\min\{\nu_1, \nu_2\} \ge \min\{\mu_1, \mu_2\}. \tag{32}$$

*Proof:* We shall prove that $\nu_1 \geq \min\{\mu_1, \mu_2\}$, and the other relation showing that $\nu_2 \geq \min\{\mu_1, \mu_2\}$ can be proved in a similar way.

By use of relations given in (27), (20), and (19), we obtain

$$\begin{aligned}\sigma_{\min}(Y_1) &= \sigma_{\min}((1-a\mu_2^2)Q_1 + aP_2Q_2P_2)\\ &\geq (1-a\mu_2^2)\sigma_{\min}(Q_1) + a\sigma_{\min}(P_2Q_2P_2)\\ &= 1 - a\mu_2^2 + a\mu_2^2 = 1. \qquad (33a)\end{aligned}$$

Thus, the interpolated Lyapunov equation (24) produce the QRBM $\nu_1$ given in (30) such that

$$\nu_1 = \sigma_{\min}(Y_1^{1/2})\sigma_{\min}(X_1^{-1}Y_1^{1/2}) \geq \sigma_{\min}(X_1^{-1}Y_1^{1/2}). \qquad (34)$$

Defining $\xi = \min\{\mu_1, \mu_2\}$, then $\nu_1 \geq \xi$ is sufficiently justified if

$$\sigma_{\min}(X_1^{-1}Y_1^{1/2}) \geq \xi \qquad (35)$$

which is rewritten, in matrix sense, by

$$X_1^{-1}Y_1X_1^{-1} \geq \xi^2 I \qquad (36)$$

or

$$Y_1 \geq \xi^2 X_1^2. \qquad (37)$$

By definitions of matrices $X_1$ and $Y_1$ given in (26) and (27), condition (37) becomes

$$(1-a\mu_2^2)Q_1 + aP_2Q_2P_2 \geq \xi^2[(1-a\mu_2^2)P_1 + aP_2]^2. \qquad (38)$$

Pre- and post-multiplied by the matrix $P_1^{-1}$, we have

$$\begin{aligned}&(1-a\mu_2^2)P_1^{-1}Q_1P_1^{-1} + aP_1^{-1}P_2Q_2P_2P_1^{-1}\\ &\quad\geq \xi^2[(1-a\mu_2^2)I + aP_1^{-1}P_2][(1-a\mu_2^2)I + aP_2P_1^{-1}]. \qquad (39)\end{aligned}$$

By use of the matrix relation given in (21) and (22), condition (39) is sufficiently justified if

$$\begin{aligned}&(1-a\mu_2^2)\mu_1^2 I + aP_1^{-1}P_2P_2P_1^{-1}\\ &\quad\geq \xi^2[(1-a\mu_2^2)I + aP_1^{-1}P_2][(1-a\mu_2^2)I + aP_2P_1^{-1}] \qquad (40)\end{aligned}$$

which is rearranged to be

$$\begin{aligned}&(1-a\mu_2^2)[\mu_1^2 - (1-a\mu_2^2)\xi^2]I + a(1-a\xi^2)P_1^{-1}P_2P_2P_1^{-1}\\ &\qquad\geq a(1-a\mu_2^2)\xi^2[P_1^{-1}P_2 + P_2P_1^{-1}]. \qquad (41)\end{aligned}$$

Since it is known that $(1-a\xi^2) \geq (1-a\mu_2^2)$, condition (41) is sufficiently justified if

$$\begin{aligned}&[\mu_1^2 - (1-a\mu_2^2)\xi^2]I - a\xi^2(P_1^{-1}P_2 + P_2P_1^{-1})\\ &\qquad + aP_1^{-1}P_2P_2P_1^{-1} \geq 0 \qquad (42)\end{aligned}$$

which is equivalently expressed by

$$\begin{aligned}&a(P_1^{-1}P_2 - \xi^2 I)(P_2P_1^{-1} - \xi^2 I)\\ &\qquad + [\mu_1^2 - (1-a\mu_2^2)\xi^2 - a\xi^4]I \geq 0. \qquad (43a)\end{aligned}$$

Thus, $\nu_1 \geq \xi$ is sufficiently justified if

$$\mu_1^2 - (1-a\mu_2^2)\xi^2 - a\xi^4 \geq 0 \qquad (43b)$$

which is always fulfilled, for all $a \in (0, 1/\mu_2^2)$, since

$$\mu_1^2 - (1-a\mu_2^2)\xi^2 - a\xi^4 = (\mu_1^2 - \xi^2) + a\xi^2(\mu_2^2 - \xi^2) \geq 0. \qquad (44)$$

□

*Lemma 3:* Given Lyapunov equations (14) and (15) with their QRBM's $\mu_1$ and $\mu_2$ given in (18) and (19), the interpolated Lyapunov equations (24) and (25) defined in Lemma 1 produce new QRBM's $\nu_1$ and $\nu_2$ given in (30) and (31). There are interpolating parameters $a \in [a_{\min}, 1/\mu_2^2)$ and $b \in [b_{\min}, 1/\mu_1^2)$ such that

$$\max\{\nu_1, \nu_2\} \geq \max\{\mu_1, \mu_2\} \qquad (45)$$

if the following conditions are fulfilled:

$$P_2(Q_2 - I)P_2 + (P_2 - \mu_2^2P_1)^2 > \mu_2^2(\mu_2^2P_1^2 - Q_1) \qquad (46)$$

$$P_1^{-1}(Q_1 - I)P_1^{-1} + (P_1^{-1} - \mu_1^2P_2^{-1})^2 > \mu_1^2(\mu_1^2P_2^{-2} - Q_2). \qquad (47)$$

where $a_{\min} \in [0, 1/\mu_2^2)$ is the smallest value that fulfils

$$a_{\min}\{P_2(Q_2 - I)P_2 + (P_2 - \mu_2^2P_1)^2\} \geq \mu_2^2P_1^2 - Q_1 \qquad (48)$$

and $b_{\min} \in [0, 1/\mu_1^2)$ is the smallest value that fulfils

$$b_{\min}\{P_1^{-1}(Q_1 - I)P_1^{-1} + (P_1^{-1} - \mu_1^2P_2^{-1})^2\} \geq \mu_1^2P_2^{-2} - Q_2. \qquad (49)$$

*Proof:* Lemma 2 implies the result for the case of $\mu_1 = \mu_2$. We shall prove that, given $\mu_2 > \mu_1$, conditions (46) and (47) lead to an improved QRBM with $\nu_1 \geq \mu_2$. In a similar way, the other relation, showing that $\nu_2 \geq \mu_1$, can be proved for the case of $\mu_1 > \mu_2$.

Given $\mu_2 > \mu_1$, by use of the matrix relation given in (23), it can be shown that

$$\mu_1^2P_2^{-2} - Q_2 < \mu_2^2P_2^{-2} - Q_2 \leq 0. \qquad (50)$$

Thus, condition (47) is automatically fulfilled and is redundant.

Let condition (48) be fulfilled; there are values of $a \in (a_{\min}, 1/\mu_2^2)$ such that

$$a\{P_2(Q_2 - I)P_2 + (P_2 - \mu_2^2P_1)^2\} \geq \mu_2^2P_1^2 - Q_1. \qquad (51)$$

Given matrix relations (21), since $Q_2 \geq I$, we have

$$\begin{aligned}0 &\leq (1-a\mu_2^2)\{a\{P_2(Q_2 - I)P_2 + (P_2 - \mu_2^2P_1)^2\} + Q_1 - \mu_2^2P_1^2\}\\ &\quad + a^2\mu_2^2P_2(Q_2 - I)P_2,\\ &= (1-a\mu_2^2)\{a(P_2 - \mu_2^2P_1)^2 + Q_1 - \mu_2^2P_1^2\} + aP_2(Q_2 - I)P_2,\\ &= -a^2\mu_2^2P_2^2 - a\mu_2^2(1-a\mu_2^2)(P_2P_1 + P_1P_2) - \mu_2^2(1-a\mu_2^2)^2P_1^2\\ &\quad + (1-a\mu_2^2)Q_1 + aP_2Q_2P_2,\\ &= (1-a\mu_2^2)Q_1 + aP_2Q_2P_2 - \mu_2^2[(1-a\mu_2^2)P_1 + aP_2]^2. \qquad (52)\end{aligned}$$

By definitions of matrices $X_1$ and $Y_1$ given in (26) and (27), relation (52) becomes

$$Y_1 \geq \mu_2^2X_1^2 \qquad (53)$$

or

$$\sigma_{\min}(X_1^{-1}Y_1X_1^{-1}) \geq \mu_2^2. \qquad (54)$$

On the other hand, it has been shown in the proof of Lemma 2 that

$$\sigma_{\min}(Y_1) \geq 1. \qquad (33b)$$

Thus, followed from the definition of the interpolated QRBM given in (30), relations (33) and (54) are combined to provide

$$\nu_1^2 = \sigma_{\min}(X_1^{-1}Y_1X_1^{-1})\sigma_{\min}(Y_1) \geq \mu_2^2. \qquad (55)$$

□

Utilizing these lemmas, the following procedure is devised to interpolate Lyapunov equations and to extract the improved QRBM ($\mu_C$).

*Algorithm 1:*

1) Assigning $Q_1 = Q_2 = I$.
2) Equate Lyapunov equations (14) and (15), and acquire the QRBMSs $\mu_1$ and $\mu_2$ from (18) and (19).
3) Compute $a_{\min}$ and $b_{\min}$ from relations (48) and (49) and make the following adjustments: if $a_{\min} < 0$, we make $a_{\min} = 0$; and if $a_{\min} > 1/\mu_2^2$, we make $a_{\min} = 1/\mu_2^2$; if $b_{\min} < 0$, we make $b_{\min} = 0$; and if $b_{\min} > 1/\mu_1^2$, we make $b_{\min} = 1/\mu_1^2$.
4) Selecting $a = 0.5(1/\mu_2^2 + a_{\min})$ and $b = 0.5(1/\mu_1^2 + b_{\min})$, the interpolated matrices $Y_1$ and $Y_2$ are obtained from (27) and (29).
5) Normalize matrices $Y_1$ and $Y_2$ to form new matrices $Q_1$ and $Q_2$ that fulfil (20), and repeat from Step 2 until a convergent condition is detected.

We note that, in each iteration, two robustness measures ($\mu_1$ and $\mu_2$) are obtained in Step 2 of the proposed algorithm. To make appropriate use of Lemmas 2 and 3, the qualitative property of interpolating Lyapunov equations is investigated in Step 3. There are only two cases:

i) If $a_{\min} = 0$ and $b_{\min} \in [0, 1/\mu_1^2)$, or if $b_{\min} = 0$ and $a_{\min} \in [0, 1/\mu_2^2)$, then both Lemma 2 and Lemma 3 are applied in Step 4 such that the magnitudes of both $\max\{\mu_1, \mu_2\}$ and $\min\{\mu_1, \mu_2\}$ are improved in the next iteration.

ii) If $a_{\min} = 0$ and $b_{\min} = 1/\mu_1^2$, or if $b_{\min} = 0$ and $a_{\min} = 1/\mu_2^2$, then only Lemma 2 is applied in Step 4 such that the magnitude of $\min\{\mu_1, \mu_2\}$ is improved in the next iteration. Since Lemma 3 is not applicable in this case, the process attempts to preserve the magnitude of $\max\{\mu_1, \mu_2\}$. Nevertheless, it is still possible that the magnitude of $\max\{\mu_1, \mu_2\}$ is improved in the next iteration.

Obviously, the iterative process persistently improves the magnitude of $\max\{\mu_1, \mu_2\}$ until it is found that the magnitude of $\max\{\mu_1, \mu_2\}$ is trivially affected by further interpolations.

*Example 1:* Consider the system given in [2], the nominal matrix is given by

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}.$$

The bound $\mu_I$ in Theorem 2 is known to be 0.3820. By use of the proposed procedure, the result is

$$\mu_C = 0.4495$$

where matrices in the Lyapunov equation (3) are given by

$$Q = \begin{bmatrix} 5.2361 & 2.6180 \\ 2.6180 & 2.6180 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2.1817 & 1.3090 \\ 1.3090 & 3.0544 \end{bmatrix}.$$

An improvement of 18% in QRBM is observed.

*Example 2:* Consider the stabilized STOL aircraft given in [2], the nominal matrix is

$$A = \begin{bmatrix} -0.201 & 0.755 & 0.351 & -0.075 & 0.033 \\ -0.149 & -0.696 & -0.160 & 0.110 & -0.048 \\ 0.081 & 0.004 & -0.189 & -0.003 & 0.001 \\ -0.173 & 0.802 & 0.251 & -0.804 & 0.056 \\ 0.092 & -0.467 & -0.127 & 0.075 & -1.162 \end{bmatrix}.$$

The bound $\mu_I$ in Theorem 2 is known to be 0.0774. By use of the proposed procedure, the result is

$$\mu_C = 0.0929$$

where matrices in the Lyapunov equation (3) are given by

$$Q = \begin{bmatrix} 2.2744 & -1.7412 & -1.7071 & 1.0771 & -1.0905 \\ -1.7412 & 23.9420 & 7.2540 & -11.8002 & 10.3020 \\ -1.7071 & 7.2540 & 4.7158 & -3.1945 & 2.5491 \\ 1.0771 & -11.8002 & -3.1945 & 12.3753 & -0.8221 \\ -1.0905 & 10.3020 & 2.5491 & -0.8221 & 27.1638 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 9.2069 & 2.1312 & -0.6771 & 0.0540 & -0.4574 \\ 2.1312 & 26.6291 & 8.3456 & -6.0982 & 4.6249 \\ -0.6771 & 8.3456 & 14.9825 & -0.8588 & 0.7625 \\ 0.0540 & -6.0982 & -0.8588 & 14.6391 & 0.8907 \\ -0.4574 & 4.6249 & 0.7625 & 0.8907 & 23.2163 \end{bmatrix}.$$

An improvement of 20% in QRBM is observed.

## V. Concluding Remarks

The main theme of this correspondence is to derive a less conservative QRBM for the unstructurally perturbed systems. In this regard, interpolated Lyapunov equations have been examined, and an iterative computational procedure has been proposed. The two examples show that, with a handful of iterations, our method achieves a considerable improvement over old results.

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