

Self-induced transparency with transverse variations in resonant media by the power series approximation method

Bing-Chung Cheng,^{1,2,*} Hong-Yih Tseng,¹ and Sien Chi^{1,3}

¹*Institute of Electro-Optical Engineering, National Chiao-Tung University, Hsinchu, Taiwan 300, Republic of China*

²*Department of Physics, Fu Jen Catholic University, Hsinchuang, Taiwan 24205, Republic of China*

³*Department of Electrical Engineering, Yuan Ze University, Chung-Li, Taiwan, Republic of China*

(Received 31 March 2004; published 18 January 2005)

This paper presents the analytic solutions of self-induced transparency with transverse variations in resonant media by a self-consistent power series approximation method. We solve the Maxwell-Bloch equations by retaining all integral terms from the inhomogeneous broadening and the second-derivative terms with respect to distance and time. The lowest-order approximation is presented in detail. We discuss the characteristics of the distortionless pulses that propagate in an inhomogeneously broadening medium. It is shown that the group velocity and peak power of the self-induced transparency solitons could be represented in terms of the pulse width, the angle between phase velocity and group velocity, and the basic material parameters of the resonant medium. The chirping of the soliton is also discussed.

DOI: 10.1103/PhysRevE.71.016609

PACS number(s): 42.65.Tg, 42.50.Gy, 42.50.Md

I. INTRODUCTION

The self-induced transparency (SIT) soliton is a coherent pulse or pulse train propagating in a resonant two-level medium without loss and distortion if the pulse energy exceeds a critical value. Several types of these distortionless pulses were found [1–4]. Because of the coherent interaction, the group velocity of a SIT soliton depends on its pulse width and is slowed down with respect to the light speed in the host medium. The effect of the reduced group velocity of a SIT soliton in the inhomogeneously broadening two-level atoms embedded in a Kerr host medium was also studied [5]. It was found that an extra negative dispersion would be induced because of the reduction of the soliton's group velocity, which is not predicted by the theory under the slowly varying envelope approximation.

Although the research on SIT has been widely investigated, a lot of new results are continually being discovered. For example, the modification of SIT by using an additional control laser field could further reduce the group velocity of the SIT [6]. The dynamics of the SIT process are profoundly modified by the nonlinear optical interaction similar to the electromagnetically induced transparency (EIT) [7]. The presence of the control field allows SIT occurring under a much broader range of conditions and dramatically slows down the SIT soliton. Another interesting transparency is mixed electromagnetically and self-induced transparency (MIT) [8].

Most of these novel studies on the interaction between light and matter are limited to the homogeneously broadening resonant medium and assumed under the slowly varying envelope approximation (SVEA). For a practical resonant

medium, we should consider all components of the polarization and population difference contributed from the atoms with resonant frequencies in the whole range of detuning from the central transition frequency. This indicates the integral terms from inhomogeneous broadening should be taken into account in the Maxwell-Bloch equations. The simplest method for solving the inhomogeneously broadening case is the factorization ansatz. Such an ansatz also could be applied to V-type and Λ -type inhomogeneously broadening media [9,10]. However, in general the SVEA was still adopted in Maxwell's equations.

In this paper, we present a power series approximation method to deal with the inhomogeneous broadening case for obtaining the analytic solutions of distortionless pulses with transverse variations without using the SVEA.

II. MAXWELL-BLOCH EQUATIONS

From Maxwell's equations, the wave equation describing pulse propagation in a resonant medium can be written as

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \vec{E}(y, z, t) = - \frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} \vec{P}(y, z, t), \quad (1)$$

where \vec{E} is the electric field in the medium, \vec{P} is the polarization resulting from the two-level atoms, and c is the velocity of light in vacuum. The electric field and the macroscopic resonant polarization can be expressed as

$$\vec{E}(y, z, t) = \frac{1}{2} \hat{x} \{ \varepsilon(y, z, t) \exp[i\phi(y, z, t) + i(\omega t - \beta z)] + \text{c.c.} \}, \quad (2a)$$

$$\vec{P}(y, z, t) = \frac{1}{2} \hat{x} \{ [U(y, z, t) + iV(y, z, t)] \times \exp[i\phi(y, z, t) + i(\omega t - \beta z)] + \text{c.c.} \}, \quad (2b)$$

where $\varepsilon(y, z, t)$ is the envelope function, $\phi(y, z, t)$ is the

*Corresponding author. Present address: Department of Physics, Fu Jen Catholic University, 510 Chung Cheng Rd., Hsinchuang, Taiwan 24205, Republic of China. Email address: phys2011@mails.fju.edu.tw

phase function, β is the propagation constant, ω is the carrier frequency near the central transition frequency ω_0 , $U(y, z, t)$ corresponds to the dispersion induced by the resonant atoms, and $V(y, z, t)$ corresponds to the absorption caused by the resonant atoms. Note that in contrast with the previous study, we consider the transverse variation indicated by $\partial^2/\partial y^2$ in Eqs. (1). For a steady-state propagation, we assume that \vec{E} and \vec{P} depend on the moving-frame coordinate $\tau = t - y/V_y - z/V_z$, where V_z and V_y are the group velocities of the pulse in the \hat{z} direction and the \hat{y} direction, respectively.

For a two-level medium, the Bloch vector (u, v, w) relates to the macroscopic polarization and population difference as follows:

$$(U, V, W) = \int_{-\infty}^{\infty} (u, v, w) g(\Delta) d\Delta.$$

This equation describes the components of the polarization and population difference contributed from the atoms with resonant frequencies in the whole range of Δ , where Δ denotes the transition frequency detuning from central transi-

tion frequency and $g(\Delta)$ is the normalized distribution of detuning from inhomogeneous broadening. Here the quantity $W = \mu(N_1 - N_2)$ is the macroscopic population difference multiplied by the transition matrix element μ between the ground state (N_1) and upper state (N_2) of the two-level system. After neglecting the atomic relaxation times, we can express the Bloch equations as

$$\dot{u} = -(\Delta - \dot{\phi})v, \quad (3a)$$

$$\dot{v} = (\Delta - \dot{\phi})u + \frac{\mu}{\hbar} \varepsilon w, \quad (3b)$$

$$\dot{w} = -\frac{\mu}{\hbar} \varepsilon v, \quad (3c)$$

where $\dot{\phi}$ represents the chirping of the pulse and the derivative expression (\cdot) denotes $\partial/\partial\tau$ hereafter. Substituting Eqs. (2) into Eq. (1) and retaining both the second-derivative terms and all integral terms from inhomogeneous broadening in Eq. (1), we obtain

$$\left(\frac{1}{V_z^2} + \frac{1}{V_y^2} - \frac{1}{c^2} \right) (\ddot{\varepsilon} - \dot{\phi}^2 \varepsilon) + (k_0^2 - \beta^2) \varepsilon + 2 \left(\frac{k_0}{c} - \frac{\beta}{V_z} \right) \dot{\phi} \varepsilon = \frac{4\pi}{c^2} N \mu \int [\ddot{u} - u(\dot{\phi} + \omega)^2 - 2\dot{v}(\dot{\phi} + \omega) - v\ddot{\phi}] g(\Delta) d\Delta, \quad (4a)$$

$$\left(\frac{1}{V_z^2} + \frac{1}{V_y^2} - \frac{1}{c^2} \right) (\dot{\phi} \varepsilon + 2\dot{\phi}^2 \varepsilon) - 2 \left(\frac{k_0}{c} - \frac{\beta}{V_z} \right) \dot{\varepsilon} = \frac{4\pi}{c^2} N \mu \int [\ddot{v} - v(\dot{\phi} + \omega)^2 + 2\dot{u}(\dot{\phi} + \omega) + u\ddot{\phi}] g(\Delta) d\Delta, \quad (4b)$$

where $k_0 = \omega/c$. Using Eqs. (3), we have

$$\ddot{u} = -(\Delta - \dot{\phi})\dot{v} + \ddot{\phi}v = -(\Delta - \dot{\phi})^2 u - \frac{\mu}{\hbar} (\Delta - \dot{\phi}) \varepsilon w + \ddot{\phi}v, \quad (5a)$$

$$\ddot{v} = -(\Delta - \dot{\phi})^2 v - \dot{\phi}u + \frac{\mu}{\hbar} \dot{\varepsilon} w - \left(\frac{\mu}{\hbar} \right)^2 \varepsilon^2 v. \quad (5b)$$

Furthermore, by introducing the abbreviations

$$s_1 = \frac{4\pi}{c^2} N \mu, \quad s_2 = \frac{4\pi}{c^2} N \frac{\mu^2}{\hbar}, \quad s_3 = \frac{4\pi}{c^2} N \frac{\mu^3}{\hbar},$$

$$\gamma_1 = \frac{1}{V_z^2} + \frac{1}{V_y^2} - \frac{1}{c^2}, \quad \gamma_2 = k_0^2 - \beta^2, \quad \gamma_3 = 2 \left(\frac{k_0}{c} - \frac{\beta}{V_z} \right),$$

the wave equations now could be rearranged as

$$\begin{aligned} & \gamma_1 (\ddot{\varepsilon} - \dot{\phi}^2 \varepsilon) + \gamma_2 \varepsilon + \gamma_3 \dot{\phi} \varepsilon \\ &= -s_2 (2\omega + \dot{\phi}) \varepsilon \int w g(\Delta) d\Delta - s_2 \varepsilon \int \Delta w g(\Delta) d\Delta \\ & \quad - s_1 \omega^2 \int u g(\Delta) d\Delta - s_1 2\omega \int \Delta u g(\Delta) d\Delta \\ & \quad - s_1 \int \Delta^2 u g(\Delta) d\Delta, \end{aligned} \quad (6a)$$

$$\begin{aligned} & \gamma_1 (\ddot{\phi} \varepsilon + 2\dot{\phi} \dot{\varepsilon}) - \gamma_3 \dot{\varepsilon} \\ &= s_2 \dot{\varepsilon} \int w g(\Delta) d\Delta - (s_3 \varepsilon^2 + s_1 \omega^2) \int v g(\Delta) d\Delta \\ & \quad - s_1 2\omega \int \Delta v g(\Delta) d\Delta - s_1 \int \Delta^2 v g(\Delta) d\Delta. \end{aligned} \quad (6b)$$

Equations (3) and (6) are the Maxwell-Bloch equations for ultrashort pulses propagating in inhomogeneously broadened two-level media. The distortionless solution will be obtained without using the SVEA.

III. POWER SERIES APPROXIMATION METHOD AND SOLUTIONS

To realize a self-consistent integration of Eqs. (3)–(6), we should adopt a sufficiently general form for $w(\varepsilon)$. At first, we assume the form of the difference of population is

$$w = \sum_{\ell=0}^{\infty} w'_{\ell}(\Delta)(\kappa_d \varepsilon / \omega)^{2\ell},$$

where the dimensionless ratio $\kappa_d \varepsilon / \omega$ is chosen as the expansion parameter. The parameter κ_d is defined as $2\mu/\hbar$, where μ is the common dipole matrix element. There is a simple argument that shows the expansion parameter $\kappa_d \varepsilon / \omega$ must be much smaller than unity: the eigenenergies of the atom are of the order of $\hbar\omega$, and the perturbing atom-field interaction energy is $\vec{\mu} \cdot \vec{E} \approx \mu\varepsilon \approx \hbar\kappa_d \varepsilon$. The total Hamiltonian of the system is $\mathcal{H}_{\text{total}} = \mathcal{H}_0 - \vec{\mu} \cdot \vec{E}$. Thus one is forced to assume that $\kappa_d \varepsilon \ll \omega$ if the interaction energy is to be significantly weaker than the unperturbed energy [11]. Moreover, κ_d and ω are constants and independent of variable ε as the medium is chosen. Thus the expansion is modified as

$$w = -1 + \sum_{\ell=1}^{\infty} w_{\ell}(\Delta) \varepsilon^{2\ell}, \quad (7)$$

where the population difference satisfies $w = -1$ when $\varepsilon = 0$. Here the coefficients of the expansion, $w_{\ell}(\Delta) \equiv w'_{\ell}(\Delta) \times (\kappa_d / \omega)^{\ell}$, are functions of Δ to be determined by self-consistent requirements. Substituting Eq. (7) into Eq. (3c), we have

$$v = -\frac{\hbar}{\mu} \sum_{\ell=1}^{\infty} 2\ell w_{\ell} \varepsilon^{2\ell-2} \dot{\varepsilon}. \quad (8)$$

Substituting Eqs. (7) and (8) into Eq. (6b) and then integrating Eq. (6b) with multiplying an integral factor ε , we can obtain an equation describing the relation between $\dot{\phi}$ and ε ,

$$\begin{aligned} \dot{\phi} = & \frac{\gamma_3}{2\gamma_1} + \frac{1}{\gamma_1} \sum_{\ell=1}^{\infty} \left(\frac{s_2}{2\ell+2} + s_3 \frac{\hbar}{\mu} \frac{2\ell}{2\ell+2} \right) \langle w_{\ell} \rangle \varepsilon^{2\ell} \\ & + \frac{1}{\gamma_1 \mu} s_1 \sum_{\ell=2}^{\infty} [\omega^2 \langle w_{\ell} \rangle + 2\omega \langle \Delta w_{\ell} \rangle + \langle \Delta^2 w_{\ell} \rangle] \varepsilon^{2\ell-2}, \quad (9) \end{aligned}$$

where $\langle w_{\ell} \rangle \equiv \int_{-\infty}^{\infty} w_{\ell}(\Delta) g(\Delta) d\Delta$ and a similar form for $\langle \Delta w_{\ell} \rangle$ and $\langle \Delta^2 w_{\ell} \rangle$. From Eqs. (3a), (8), and (9) we have

$$\dot{u} = -(\Delta - \dot{\phi})v = \left[(\Delta - \dot{\phi}) \frac{\hbar}{\mu} \sum_{\ell=1}^{\infty} 2\ell w_{\ell} \varepsilon^{2\ell-2} \right] \dot{\varepsilon}. \quad (10)$$

According to Eq. (9), the quantity in the bracket is a polynomial of ε . Thus we can straightforwardly assume a functional relation for $u(\varepsilon)$,

$$u = \sum_{\ell=1}^{\infty} u_{\ell}(\Delta) \varepsilon^{2\ell+1}, \quad (11)$$

where $u_{\ell}(\Delta)$ are to be determined and related to the coefficients $w_{\ell}(\Delta)$. Now Eq. (6a) can be written only in terms of $\dot{\varepsilon}$

and power series of ε . We express this relation by the assumption of

$$\dot{\varepsilon} = \sum_{\ell=1}^{\infty} \varepsilon_{\ell}(\Delta) \varepsilon^{2\ell-1}, \quad (12)$$

where $\varepsilon_{\ell}(\Delta)$ are coefficients related to $w_{\ell}(\Delta)$. Integration of Eq. (12) yields

$$\varepsilon^2 = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \varepsilon_{\ell} \varepsilon^{2\ell}. \quad (13)$$

Consequently, the temporal evolution originally described by Eqs. (6) is reduced to an integration for ε as long as we determine the functional form of $w(\varepsilon)$.

From the above argument, we retain the terms up to ε^4 in the series of $w(\varepsilon)$,

$$w = -1 + w_1(\Delta) \varepsilon^2 + w_2(\Delta) \varepsilon^4. \quad (14)$$

According to Eqs. (7)–(13), we obtain

$$v = -\frac{\hbar}{\mu} 2w_1(\Delta) \dot{\varepsilon} - \frac{\hbar}{\mu} 4w_2(\Delta) \varepsilon^2 \dot{\varepsilon}, \quad (15a)$$

$$\dot{\phi} = K \varepsilon^2, \quad (15b)$$

$$u = \frac{2\hbar}{\mu} \Delta w_1 \varepsilon - \frac{2\hbar}{3\mu} (K w_1 - 2\Delta w_2) \varepsilon^3, \quad (15c)$$

$$\dot{\varepsilon}^2 = \frac{1}{\tau_0^2} \varepsilon^2 - \frac{1}{(\varepsilon_0 \tau_0)^2} \varepsilon^4, \quad (15d)$$

where

$$\begin{aligned} K = & \frac{1}{\gamma_1} \left(\frac{s_2}{4} + \frac{s_3 \hbar}{2\mu} \right) \langle w_1 \rangle \\ & + \frac{s_1 \hbar}{\gamma_1 \mu} [\omega^2 \langle w_2 \rangle + 2\omega \langle \Delta w_2 \rangle + \langle \Delta^2 w_2 \rangle], \quad (16a) \end{aligned}$$

$$\frac{s_2 - \gamma_3}{2\gamma_1} = s [\omega^2 \langle w_1 \rangle + 2\omega \langle \Delta w_1 \rangle + \langle \Delta^2 w_1 \rangle], \quad (16b)$$

$$\frac{1}{\tau_0^2} = \frac{2s_2}{\gamma_1} \omega - \frac{\gamma_2}{\gamma_1} - s [\omega^2 \langle \Delta w_1 \rangle + 2\omega \langle \Delta^2 w_1 \rangle + \langle \Delta^3 w_1 \rangle], \quad (16c)$$

$$\begin{aligned} \frac{1}{\varepsilon_0^2 \tau_0^2} = & \left(\frac{\gamma_3}{2\gamma_1} - \frac{s_2}{2\gamma_1} \right) K - \frac{s}{3} [\omega^2 \langle w_1 \rangle + 2\omega \langle \Delta w_1 \rangle + \langle \Delta^2 w_1 \rangle] K \\ & + \frac{s_2}{\gamma_1} \omega \langle w_1 \rangle + \frac{s_2}{2\gamma_1} \langle \Delta w_1 \rangle \\ & + \frac{2}{3} s [\omega^2 \langle \Delta w_2 \rangle + 2\omega \langle \Delta^2 w_2 \rangle + \langle \Delta^3 w_2 \rangle], \quad (16d) \end{aligned}$$

and $s = (s_1 / \gamma_1) (\hbar / \mu)$. Substituting Eqs. (15) into Eq. (3b) and then neglecting the higher-order terms, we obtain

$$w_1 = \frac{1}{2} \left(\frac{\mu}{\hbar} \right)^2 \tau_0^2 m_1, \quad (17a)$$

$$w_2 = \frac{3}{2} \left(\frac{\mu}{\hbar} \right)^2 \tau_0^4 \left[\frac{1}{\tau_0^2 \varepsilon_0^2} + \frac{2}{3} \Delta K - \frac{1}{4} \left(\frac{\mu}{\hbar} \right)^2 \right] m_2, \quad (17b)$$

where

$$m_1 = \frac{1}{1 + \Delta^2 \tau_0^2} \text{ and } m_2 = \frac{1}{9 + \Delta^2 \tau_0^2} m_1.$$

Clearly Eqs. (14)–(17) provide an implicit set of solution in terms of τ_0 for SIT parameters. The soliton solution is determined by Eqs. (15b) and (15d),

$$\vec{E}(\tau) = \hat{x} \varepsilon_0 \operatorname{sech} \left(\frac{\tau}{\tau_0} \right) \exp \left[i K \tau_0 \varepsilon_0^2 \tanh \left(\frac{\tau}{\tau_0} \right) \right] e^{i(\omega t - \beta z)}, \quad (18)$$

where ε_0 indicates the peak value and τ_0 represents the pulse width. Because in Eqs. (6) we retain all integral terms and the second-derivative terms with respect to distance and time, Eqs. (14)–(18) are valid for an ultrashort pulse with transverse variations in an inhomogeneously broadening two-level system. Notice that although we solve this problem without considering the dispersion and nonlinearity of a host medium, the contribution of the dispersion and nonlinearity can be added in the coefficients of ε^2 and ε^4 in Eq. (15d). Therefore, our method could be easily extended for the SIT in a resonant system embedded in a dispersive and Kerr host medium.

IV. RESULTS AND DISCUSSION

In this section, we discuss the solutions obtained in both Lorentzian and Gaussian line shapes. For convenience, we define

$$V_y = V_z \tan \alpha, \quad \frac{1}{\tau_c^2} = \frac{N \mu^2}{\hbar} \pi \omega,$$

$$\tilde{V} = \frac{c}{V_z}, \quad \tilde{\omega} = \frac{\beta}{k_0},$$

where α is the angle between the group velocity and phase velocity.

Case (i): Lorentzian line shape.

First, the Lorentzian line shape is assumed,

$$g(\Delta) = \frac{\omega_a}{2\pi \Delta^2 + (\omega_a/2)^2}, \quad (19)$$

where ω_a is the full width at half maximum (FWHM) of $g(\Delta)$. From Eqs. (16), we have

$$\tilde{V}^2 \sec^2 \alpha = \frac{b + \sqrt{b^2 + 4\tau_0^2 \omega^2 a^2 \sec^2 \alpha}}{2}, \quad (20a)$$

$$\tilde{\omega}^2 = \frac{2a^2 \sec^2 \alpha}{b + \sqrt{b^2 + 4\tau_0^2 \omega^2 a^2 \sec^2 \alpha}}, \quad (20b)$$

where

$$b = 1 + 8 \frac{\tau_0^2}{\tau_c^2} - \tau_0^2 \omega^2 - 8 \frac{\tau_0^4}{\tau_c^2} \left(\frac{1}{\tau_0^2 + 2\tau_0/\omega_a} \right), \quad (21a)$$

$$a = 1 - \frac{2}{\tau_c^2 \omega^2} + 2 \frac{\tau_0^2}{\tau_c^2} \left(\frac{1}{1 + \omega_a \tau_0/2} \right) + 2 \frac{\tau_0^2}{\tau_c^2 \omega^2} \left(\frac{1}{\tau_0^2 + 2\tau_0/\omega_a} \right), \quad (21b)$$

and

$$\varepsilon_0^2 = \left(\frac{2\hbar}{\mu} \right)^2 \frac{\omega^2}{(\omega^2 \tau_0^2 - 9)(\omega_a \tau_0 + 8)} \left(8 + \frac{3\omega_a}{\omega^2 \tau_0} + \omega_a \tau_0 \right). \quad (22)$$

For the pulses without transverse variations, i.e., $\alpha=0$, we have

$$\tilde{V}^2 = \frac{1}{2 + \tau_0 \omega_a} \left[1 + \frac{\tau_0^2}{\tau_c^2} (8 - \omega^2 \tau_c^2) + \frac{1}{2} (1 - \omega^2 \tau_0^2) \tau_0 \omega_a \right] + \frac{1}{2} (1 + \omega^2 \tau_0^2) \left[1 + \left(\frac{1}{2 + \omega_a \tau_0} \right)^2 \left(\frac{8\tau_0}{\omega^2 \tau_c^2} \right)^2 \right]^{1/2}, \quad (23)$$

and a relationship between the maximum frequency shift $K\varepsilon_0^2$ and the group velocity is determined via

$$\begin{aligned} \tau_0 [3\omega_a + \tau_0 \omega^2 (8 + \tau_0 \omega_a)] \left(\frac{1}{\varepsilon_0^2} \right) \\ = \left(\frac{\mu \tau_0}{2\hbar} \right)^2 (\tau_0 \omega_a + 8) (\tau_0^2 \omega^2 - 9) \\ + [\omega \tau_c^2 (2 + \tau_0 \omega_a) (6 + \tau_0 \omega_a) (\tilde{V}^2 - 1) - 4\tau_0^3 \omega \omega_a] K. \end{aligned} \quad (24)$$

Case (ii): Gaussian line shape.

Next, the Gaussian line shape is assumed,

$$g(\Delta) = \frac{1}{\omega_b} \sqrt{\frac{\ln(2)}{\pi}} \exp \left[-\frac{\ln(2) \Delta^2}{\omega_b^2} \right], \quad (25)$$

where ω_b is the full width at half maximum (FWHM) of $g(\Delta)$. We have

$$\langle m_1 \rangle = -\frac{\sqrt{\ln(2)} \pi}{\omega_b \tau_0} \exp \left[\frac{\ln(2)}{\omega_b^2 \tau_0^2} \right] \left\{ \operatorname{Erf} \left[\frac{\sqrt{\ln(2)}}{\omega_b \tau_0} \right] - 1 \right\}, \quad (26)$$

$$\begin{aligned} \langle \Delta^2 m_1 \rangle = \frac{\sqrt{\ln(2)} \pi}{\tau_0^3 \omega_b} \left\{ \frac{\omega_b \tau_0}{\sqrt{\ln(2)} \pi} + \exp \left[\frac{\ln(2)}{\omega_b^2 \tau_0^2} \right] \right. \\ \left. \times \left[\operatorname{Erf} \left(\frac{\sqrt{\ln(2)}}{\omega_b \tau_0} \right) - 1 \right] \right\}, \end{aligned} \quad (27)$$

$$\begin{aligned} \langle m_2 \rangle = \frac{\sqrt{\ln(2)} \pi}{24 \tau_0 \omega_b} \exp \left[\frac{\ln(2)}{\omega_b^2 \tau_0^2} \right] \left\{ 3 - \exp \left[\frac{8 \ln(2)}{\omega_b^2 \tau_0^2} \right] \right. \\ \left. - 3 \operatorname{Erf} \left[\frac{\sqrt{\ln(2)}}{\omega_b \tau_0} \right] + \exp \left[\frac{8 \ln(2)}{\omega_b^2 \tau_0^2} \right] \operatorname{Erf} \left[\frac{3\sqrt{\ln(2)}}{\omega_b \tau_0} \right] \right\}, \end{aligned} \quad (28)$$

$$\langle \Delta^2 m_2 \rangle = -\frac{\sqrt{\ln(2)}\pi}{8\omega_b\tau_0^3} \exp\left[\frac{\ln(2)}{\omega_b^2\tau_0^2}\right] \left\{ 1 - 3 \exp\left[\frac{8\ln(2)}{\omega_b^2\tau_0^2}\right] - \operatorname{Erf}\left[\frac{\sqrt{\ln(2)}}{\omega_b\tau_0}\right] + 3 \exp\left[\frac{8\ln(2)}{\omega_b^2\tau_0^2}\right] \operatorname{Erf}\left[\frac{3\sqrt{\ln(2)}}{\omega_b\tau_0}\right] \right\}, \quad (29)$$

$$\langle \Delta m_1 \rangle = \langle \Delta m_2 \rangle = 0. \quad (30)$$

Again, from Eqs. (16), we have

$$\tilde{V}^2 \sec^2 \alpha = \frac{b + \sqrt{b^2 + 4\tau_0^2\omega^2 a^2 \sec^2 \alpha}}{2}, \quad (31a)$$

$$\tilde{\omega}^2 = \frac{2a^2 \sec^2 \alpha}{b + \sqrt{b^2 + 4\tau_0^2\omega^2 a^2 \sec^2 \alpha}}, \quad (31b)$$

where

$$b = 1 + 8\frac{\tau_0^2}{\tau_c^2} - \tau_0^2\omega^2 - 8\frac{\tau_0^4}{\tau_c^4} \langle \Delta^2 m_1 \rangle, \quad (32a)$$

$$a = 1 - \frac{2}{\tau_c^2\omega^2} + 2\frac{\tau_0^2}{\tau_c^2} \langle m_1 \rangle + 2\frac{\tau_0^2}{\tau_c^2\omega^2} \langle \Delta^2 m_1 \rangle, \quad (32b)$$

and

$$\varepsilon_0^2 = \frac{4\hbar^2(\omega^2 \langle m_2 \rangle + \langle \Delta^2 m_2 \rangle)}{\mu^2(\tau_0^2 \langle \Delta^2 m_2 \rangle + \omega^2 \tau_0^2 \langle m_2 \rangle - \langle m_1 \rangle)}. \quad (33)$$

For the pulses without transverse variations, i.e., $\alpha=0$, we have

$$\begin{aligned} \tilde{V}^2 = & \frac{\tau_c^2 - \omega^2 \tau_0^2 \tau_c^2 + 8\tau_0^2 - 8\langle \Delta^2 m_1 \rangle \tau_0^4}{2\tau_c^2} \\ & + \left[\left(\frac{\tau_c^2 - \omega^2 \tau_0^2 \tau_c^2 + 8\tau_0^2 - 8\langle \Delta^2 m_1 \rangle \tau_0^4}{2\tau_c^2} \right)^2 \right. \\ & \left. + \frac{\tau_0^2(\omega^2 \tau_c^2 + 2\omega^2 \tau_0^2 \langle m_1 \rangle + 2\tau_0^2 \langle \Delta^2 m_1 \rangle - 2)^2}{\omega^2 \tau_c^4} \right]^{1/2}, \end{aligned} \quad (34)$$

and the relationship between the maximum frequency shift $K\varepsilon_0^2$ and the group velocity is determined via

$$\begin{aligned} & \frac{6\tau_0^2(\langle \Delta^2 m_2 \rangle + \omega^2 \langle m_2 \rangle)}{\omega\tau_c^2} \left(\frac{1}{\varepsilon_0^2} \right) \\ & = \left(\tilde{V}^2 - 1 - \frac{8\tau_0^4 \langle \Delta^2 m_2 \rangle}{\tau_c^2} \right) K \\ & \quad - \frac{3\mu^2 \tau_0^2 [\langle m_1 \rangle - (\omega^2 \langle m_2 \rangle + \langle \Delta^2 m_2 \rangle) \tau_0^2]}{2\hbar^2 \omega \tau_c^2}. \end{aligned} \quad (35)$$

To compare the results of case (i) and case (ii), we made two numerical simulations and the results are shown in the following figures. Figure 1 shows the soliton's peak value ε_0 versus τ_0 . In addition, Fig. 2 shows the maximum frequency shift versus τ_0 , where ω_a in the Lorentzian line shape and ω_b

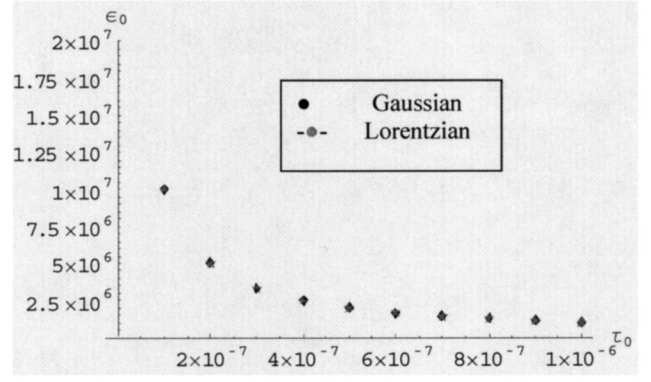


FIG. 1. Soliton peak value ε_0 vs τ_0 .

in the Gaussian line shape are assumed to be 1 THz.

These numerical results show that the parameters of the soliton solution are approximately the same for both Gaussian and Lorentzian line shapes. Notice that in contrast with the SIT theory under the SVEA, the coefficient of higher-order term leads to constraint on the maximum frequency shift.

Furthermore, if $g(\Delta)$ is a delta function, $\delta(\Delta)$, the inhomogeneous broadening becomes homogeneous broadening. Then $m_1=1$, $m_2=1/9$, and those average values such as $\langle \Delta m_1 \rangle$, $\langle \Delta m_2 \rangle$, $\langle \Delta^2 m_1 \rangle$, and $\langle \Delta^2 m_2 \rangle$ are all zeros. These results lead to

$$b = 1 + 8\frac{\tau_0^2}{\tau_c^2} - \tau_0^2\omega^2, \quad (36a)$$

$$a = 1 - \frac{2}{\tau_c^2\omega^2} + 2\frac{\tau_0^2}{\tau_c^2}, \quad (36b)$$

$$\varepsilon_0^2 = \frac{4\hbar^2\omega^2}{\mu^2(\omega^2\tau_0^2 - 9)}. \quad (37)$$

For the pulses without transverse variations, i.e., $\alpha=0$, we have

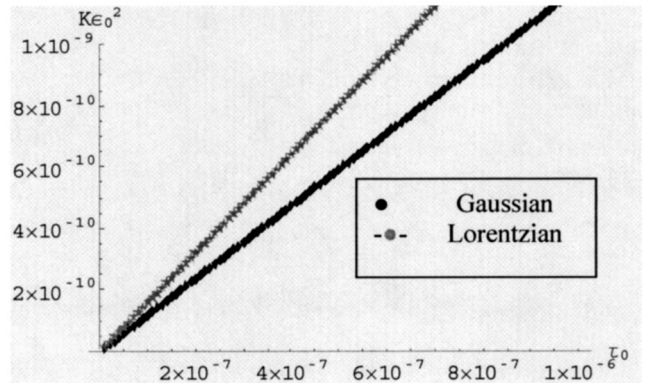


FIG. 2. Maximum frequency shift vs soliton's pulse width, where the needed parameters are $N=10^{25} \text{ m}^{-3}$, $\mu=1.4 \times 10^{-32} \text{ C m}$, and $V_z \approx c/3$.

$$\tilde{V}^2 = \frac{\tau_c^2 - \omega^2 \tau_0^2 \tau_c^2 + 8\tau_0^2}{2\tau_c^2} \left[\left(\frac{\tau_c^2 - \omega^2 \tau_0^2 \tau_c^2 + 8\tau_0^2}{2\tau_c^2} \right)^2 + \frac{\tau_0^2 (\omega^2 \tau_c^2 + 2\omega^2 \tau_0^2 - 2)^2}{\omega^2 \tau_c^4} \right]^{1/2}. \quad (38)$$

The relationship between the maximum frequency shift $K\varepsilon_0^2$ and the group velocity is determined via

$$\frac{6\tau_0^2 \omega^2}{\omega \tau_c^2} \left(\frac{1}{\varepsilon_0^2} \right) = (\tilde{V}^2 - 1)K - \frac{3\mu^2 \tau_0^2 (1 - \omega^2 \tau_0^2)}{2\hbar^2 \omega \tau_c^2}. \quad (39)$$

To compare our study with previous research, we consider the pulse with longer duration. For this consideration, we assume $(\omega\tau_0)^2 \gg 1$ and $(\omega\tau_c)^2 \gg 1$. These approximations result in

$$a = 1 + 2 \frac{\tau_0^2}{\tau_c^2}, \quad (40a)$$

$$b = 1 + 8 \frac{\tau_0^2}{\tau_c^2} - \tau_0^2 \omega^2 \quad (40b)$$

for Eqs. (20). Considering the pulses without transverse variations, we have

$$\tilde{V} = 1 + 2 \frac{\tau_0^2}{\tau_c^2}. \quad (41a)$$

$$K\varepsilon_0^2 = \frac{3}{2} \frac{1}{\tau_0^2 \omega} \left(1 - \frac{\tau_0^2}{\tau_c^2} \right), \quad (41b)$$

where $K\varepsilon_0^2$ indicates the maximum frequency shift. Obviously, we have $\tau_0 = \tau_c$ and $\tilde{V} = 3$ for an unchirped pulse without transverse variation [11]. In comparison, for an unchirped pulse with transverse variation, we solve Eqs. (16c) and (16d) and obtain

$$\frac{1}{V_z} \sec^2 \alpha - \frac{1}{c^2} = \frac{4}{V_z} \frac{\beta}{\omega} - \frac{4}{c^2}, \quad (42)$$

which is exactly the condition obtained in Ref. [12].

V. CONCLUSION

We have found the analytic solutions of the self-induced transparency with transverse variations in inhomogeneously broadening media with both Lorentzian and Gaussian line shapes by the self-consistent power series approximation method. The Maxwell-Bloch equations can be solved even if both the integral terms from inhomogeneous broadening and the second-derivative terms with respect to distance and time are retained. The chirping, group velocity, and peak power of the SIT soliton can be represented by the pulse width and material parameters of the resonant medium. We compare the numerical results of Lorentzian and Gaussian line shapes and have found they are approximately the same.

-
- [1] S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967).
 [2] J. H. Eberly, Phys. Rev. Lett. **22**, 760 (1969).
 [3] M. D. Crisp, Phys. Rev. Lett. **22**, 820 (1969).
 [4] L. Matulic and J. H. Eberly, Phys. Rev. A **6**, 822 (1972).
 [5] S. Chi, T. Y. Wang, and S. Wen, Phys. Rev. A **47**, 3371 (1993).
 [6] Q. H. Park and R. W. Boyd, Phys. Rev. Lett. **86**, 2774 (2001).
 [7] S. E. Harris, Phys. Rev. Lett. **62**, 1033 (1989).

- [8] M. O. Scully, G. S. Agarwal, O. Kocharovskaya, V. V. Kozlov, and A. B. Matsko, Opt. Express **8**, 66 (2001).
 [9] A. Rahman and J. H. Eberly, Phys. Rev. A **58**, R805 (1998).
 [10] A. Rahman, Phys. Rev. A **60**, 4187 (1999).
 [11] R. A. Marth, D. A. Holmes, and J. H. Eberly, Phys. Rev. A **9**, 2733 (1974).
 [12] S. Chi and S. R. Barone, Bull. Am. Phys. Soc. **16**, 594 (1971).