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Embedding cycles in IEH graphs

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Abstract

We embed cycles into IEH graphs. First, IEH graphs are proved to be Hamiltonian except when they are of size $2^n - 1$ for all $n \geqslant 2$. Next, we show that for an IEH graph of size N, an arbitrary cycle of even length N_e where $3 < N_e < N$ is found. We also find an arbitrary cycle of odd length N_o where $2 < N_o < N$ if and only if a node of this graph has at least one forward 2-Inter-Cube (IC) edges. These results help describe the whole cycle structure in IEH graphs. © 1997 Elsevier Science B.V.

Keywords: Hypercubes; Embedding; Hamiltonian cycles; Incrementally Extensible Hypercubes; Interconnection networks

1. Introduction

Hypercube graphs are one class of the most popular topologies for implementing massively parallel machines. It has many advantages: regularity, symmetry, low diameter, optimal fault tolerance, and so on [7]. However, the hypercube has one major drawback: that it is not incrementally extensible. The number of nodes for hypercubes must be a power of two, and thus considerably limit the choice of the number of nodes in the graphs. A few papers have so far been written to improve this drawback [2,5,8,9] but still have problems as described briefly in the following. Bhuyan and Agrawal [2] proposed *Generalized Hypercubes* that have two drawbacks: the network reduces to a complete graph when the number of nodes is prime and changes significantly when a new node is added. Katseff [5] proposed *Incomplete Hypercubes* that suffer from the problem of fault tolerance: failure of a single node will cause the entire network to be disconnected. Sen [8] proposed *Supercubes* that become more irregular as the size of the networks grows. Recently, Sur and Srimani [9] have proposed a new generalization class of hypercube graphs, *Incrementally Extensible Hypercube* (IEH) graphs. This topology can be defined for an arbitrary number of nodes and still preserves several advantages such as optimal fault tolerance, a low diameter, a simple routing algorithm, and almost regularity.

Many papers have addressed the problem of finding cycles in various network topologies [1,3,4,6,7]. However, finding cycles in IEH graphs has never been studied. In this paper, we focus on IEH graphs and obtain the following results. First, IEH graphs are proved to be *Hamiltonian* except when they are of size $2n - 1$ for all $n \geqslant 2$. Next, we show that for an IEH graph of size N, an arbitrary cycle of even length N_e where $3 < N_e < N$ is found. We also find an arbitrary cy-

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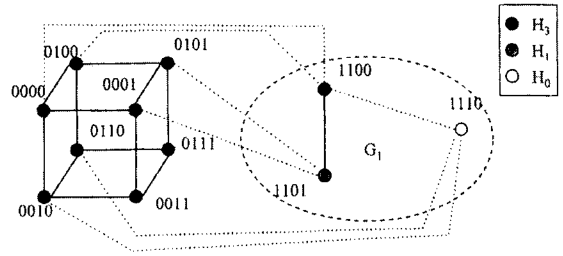

Fig. 1. IEH(11) graph.

cle of odd length N_o where $2 < N_o < N$ if and only if a node of this graph has at least one forward 2-*Inter-Cube* (IC) edges. These results help describe the whole cycle structure in IEH graphs.

The rest of this paper is organized as follows. In Section 2, we introduce basic terminology for hypercube graphs and IEH graphs. In Section 3, we show IEH graphs are Hamiltonian when their sizes are not equal to $2n - 1$ for all $n \geqslant 2$. In Section 4, we describe the cycle structure of IEH graphs. Finally, we give a conclusion in Section 5.

2. Preliminaries

A hypercube H_n is a graph $G(V, E)$, where V is the set of $2n$ nodes which are labeled as binary numbers of length n; E is the set of edges that connects two nodes if and only if they differ in exact one bit in their labels. An IEH graph, a generalized hypercube graph, is composed of several hypercubes of different sizes. These hypercubes are connected with *Inter-Cube* (IC) edges. Let IEH(N) be an IEH graph of N nodes. The graph is constructed by the following algorithm [9].

Algorithm CONSTR

1. Express N as a binary number $(c_n, \ldots, c_1, c_0)_2$ where $c_n = 1$. For each c_i, with $c_i \neq 0$, construct a hypercube H_i. The edges constructed in this step are called regular edges.
2. For all H_i, label each node with a dedicated binary number $11\ldots10b_{i-1}\ldots b_0$ where the number of leading 1s is $n - i$ and $b_{i-1}\ldots b_0$ is the label of this node in the regular hypercube of dimension i.
3. Find the minimum i such that $c_i = 1$, set $G_j = H_i$, and set $j = i$.
   $i = i + 1$.

While $i \leqslant n$

if $c_i \neq 0$ **then**
Connect the node $11\ldots1b_jb_{j-1}\ldots b_0$ in G_j to the following $i - j$ nodes in H_i:

$$\overbrace{11\ldots1}^{n-i}0\overbrace{11\ldots1}^{i-j-1}b_jb_{j-1}\ldots b_0,$$

$$\overbrace{11\ldots1}^{n-i}0\overbrace{01\ldots1}^{i-j-1}b_jb_{j-1}\ldots b_0,$$

$\ldots,$

$$\overbrace{11\ldots1}^{n-i}0\overbrace{11\ldots0}^{i-j-1}b_jb_{j-1}\ldots b_0.$$

Set $j = i$ and let G_i be the composed graph obtained in this step. /* G_i is the graph which is composed of the H_ks for $k \leqslant i$. */
endif

$i = i + 1$.
endwhile
Thus obtain the IEH(N) graph, G_n.

In Algorithm CONSTR, we observe two useful properties. First, G_i is the IEH($\sum_{k=0}^{i} c_k 2^k$) graph. Second, two nodes which are joined by IC edges differ in one or two bits of their labels. For illustration, Fig. 1 shows the IEH(11) graph. Note that solid lines represent regular edges and dot lines represent IC edges.

For convenience of discussion, we divide IC edges into two classes: *1-IC* edges and *2-IC* edges. A 1-IC edge connects nodes which differ in exactly one bit in their labels; and a 2-IC edge connects nodes which differ in exactly two bits. Let (u, v) be an IC edge with u in H_i and v in H_j for $i \neq j$. We call (u, v) a forward IC edge of u if $i < j$, and a backward one otherwise. Fig. 1 shows that (1100, 1110) is a forward 1-IC edge of node 1110 and that (0000, 1100) is a backward 2-IC edge of node 0000. Note that a node u which has forward 2-IC edges, connecting to some nodes in H_k for $k > i$, has exactly one forward 1-IC edge to a dedicated node in the same hypercube.

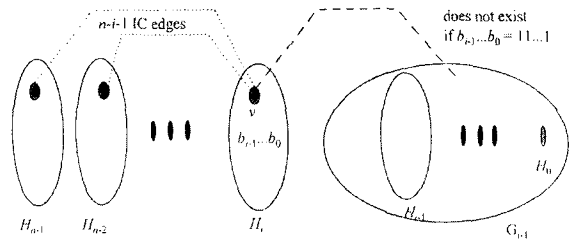

Fig. 2. IEH($2^n - 1$) graphs are not Hamiltonian for all $n \geqslant 2$.

3. Hamiltonian cycles in IEH graphs

In this section, we show IEH(N) graphs are Hamiltonian for all $N \neq 2^n - 1$ and $n \geqslant 2$. To prove this theorem, we need the following lemmas.

Lemma 1 (Saad–Schultz [7]). *A hypercube does not admit odd cycles.*

Lemma 2 (Saad–Schultz [7]). *A cycle of length l can be mapped into H_n when l is even between 4 and 2^n.*

Lemma 3. IEH($2^n - 1$) *graphs are not Hamiltonian for all $n \geqslant 2$.*

Proof. By Algorithm CONSTR, let

$$2^n - 1 = (1, 1, \ldots, 1)_2$$

where $(1, 1, \ldots, 1)_2$ is the binary representation of $2^n - 1$. We construct the IEH($2^n - 1$) graph which is a composite graph of H_{n-1}, H_{n-2}, ..., H_0. Since all H_is exist for $0 \leqslant i \leqslant n-1$, any IC edge connects two nodes when they are different in exactly one bit. Consider a node v in H_i (see Fig. 2). Observe that v has $n - i - 1$ forward IC edges, i regular hypercube edges, and one backward IC edge if and only if the last i significant bits of its label are not all 1. Thus, the IEH($2^n - 1$) graph is a subgraph of H_n induced by the vertices $V(H_n) \setminus \{(1, 1, \ldots, 1)\}$. By this and Lemma 1, the IEH($2^n - 1$) graph is not Hamiltonian. □

Fig. 3 shows the IEH(7) graph is not Hamiltonian since $7 = 2^3 - 1$. Except for the case of IEH(N) where $N = 2^n - 1$ and $n \geqslant 2$, we prove in the following theorem that IEH graphs are Hamiltonian.

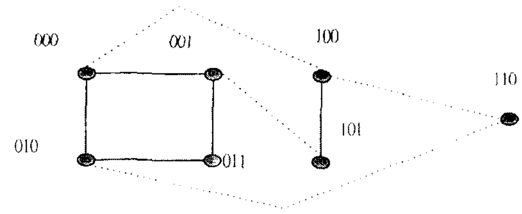

Fig. 3. IEH(7) is not Hamiltonian.

Theorem 4. IEH(N) *graphs are Hamiltonian for all $N \neq 2^n - 1$ and $n \geqslant 2$.*

Proof. For simplicity, we define that P_{uv}^G is a Hamiltonian path from u to v in a graph G. We consider two situations: (1) N is even and (2) N is odd.

Case 1: N is even, $N > 3$. Let $N = (c_n, \ldots, c_1, c_0)_2$, with $c_n = 1$. Because N is even, H_0 does not exist. Let H_j and H_i be two adjacent hypercubes. Without loss of generality, let $j < i$. Let v_j and u_j be adjacent nodes in H_j. Hence, there exists a Hamiltonian path from v_j to u_j in H_j by Lemma 2. By Algorithm CONSTR, v_j is connected to one node of H_i by a forward 1-IC edge. For the same reason, u_j also has a neighbor node in H_i. Let these two neighbor nodes of v_j and u_j in H_i be v_i and u_i, respectively. Note that u_i and v_i will differ in the same bit that u_j and v_j do. Since there exists a Hamiltonian path from u_i to v_i in H_i, we get a cycle

$$u_j—P_{u_j v_j}^{H_j}—v_j—v_i—P_{u_i v_i}^{H_i}—u_i—u_j.$$

Now, we prove G_k is Hamiltonian by induction on n, where n is the number of hypercubes of G_k. The base cases for $n = 1$ and 2 can be easily verified by Lemma 1 and the previous argument. Assume G_k is Hamiltonian and consists of l hypercubes. Let G_x be the graph composed of G_k and H_x where $x > k$. Let u and v be adjacent nodes in G_k and let u_x and v_x, respectively, denote their neighbor nodes in H_x. We then obtain a Hamiltonian cycle

$$u—P_{uv}^{G_k}—v—v_x—P_{v_x u_x}^{H_x}—u_x—u$$

in G_x, which consist of $l + 1$ hypercubes. This proves the induction and we have that IEH(N) is Hamiltonian.

Case 2: N is odd where $N \neq 2^n - 1$ and $n \geqslant 2$. Since $N \neq 2^n - 1$, there exists a c_j with $c_j = 0$, for $j \neq 0, n$. By Algorithm CONSTR, H_0 has at least

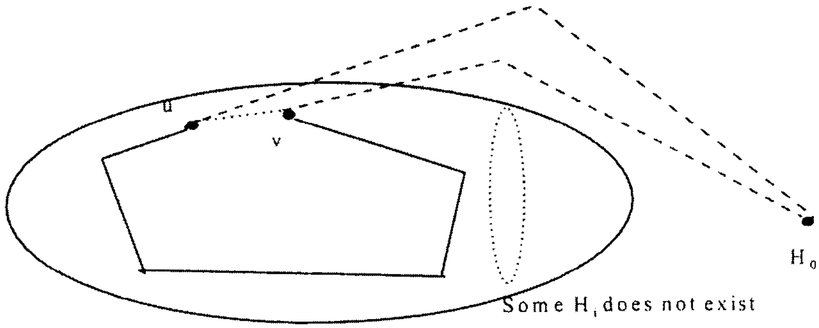

Fig. 4. IEH(N) graph is Hamiltonian for odd N and $N \neq 2^n - 1$ for all $n \geqslant 2$.

two forward IC edges connecting to some nodes in H_i with $c_i = 1$ and $c_k = 0$ for $j \leqslant k < i \leqslant n$. Let v be the neighbor node of H_0 in H_i connected by the 1-IC edge and u be one of the neighbor nodes in H_i connected by the 2-IC edges. Note that u and v are adjacent. Thus we have a cycle from u to v in $V(\mathrm{IEH}(N)) \setminus H_0$ by the previous case. By adding two edges, (H_0, u) and (H_0, v) and deleting (u, v), we find that IEH(N) graphs are Hamiltonian since we have a cycle

$$H_0—u—P_{uv}^{\mathrm{IEH}(N)-H_0}—v—H_0$$

as Fig. 4 shows. □

4. Cycles in IEH graphs

In this section, we describe the cycle structure of IEH graphs. In the following theorem, we show that there exists a cycle of even length N_e in an IEH(N) graph for $2 < N_e < N$.

Theorem 5. *Given an arbitrary even number N_e with $2 < N_e < N$, there exists a cycle of length N_e in an IEH(N) graph.*

Proof. Let $N = (c_n, \ldots, c_0)_2$, with $c_n = 1$, and $N_e = (d_n, \ldots, d_0)_2$. Since $N > N_e$, there exists some j such that $c_i = d_i = 1$, for $n + 1 > i > j$ and $c_j = 1 \neq d_j = 0$. Let $N_e = N_{e1} + N_{e2}$ where $N_{e1} = \sum_{i=j+1}^{n} d_i 2^i$ and $N_{e2} = \sum_{i=0}^{j-1} d_i 2^i$. Since $c_i = d_i$ for $n + 1 > i > j$, we find a cycle of length N_{e1} in the IEH(N_{e1}) graph, G_p, composed of H_n, $H_n - 1$, ..., $H_j + 1$, in the same way as we do in Case 1 of Theorem 4. By Lemma 2, we find a cycle C of length N_{e2} in H_j since $2^j > N_{e2}$. Let u_j and v_j be two neighbor nodes in C. Note that both u_j and v_j have neighbor nodes in H_n by 1-IC edges. Let u_n and v_n be these two nodes in H_n. Thus the following cycle of length N_e exists:

$$u_n—P_{u_n v_n}^{G_p}—v_n—v_j—P_{u_j v_j}^{G_p}—u_j—u_n. \quad \square$$

In hypercubes, there exist no odd cycles. However, it is interesting to note that there exist odd cycles in IEH(N) graphs for certain special integers N. The following theorem shows that for an odd integer N_o where $2 < N_o < N$, there exists a cycle of length N_o in an IEH(N) graph if and only if there exists one node in H_j which has at least one forward 2-IC edge.

Theorem 6. *Given an arbitrary odd number N_o with $2 < N_o < N$, there exists a cycle of length N_o in the IEH(N) graph if and only if there exists some node u in H_j that has at least one forward 2-IC edge.*

Proof. Assume that all nodes in the IEH(N) graph have no 2-IC edges. It is obvious that the IEH(N) graph is a subgraph of H_n for some n and $2^n \geqslant N$. By Lemma 1, there exist no odd cycles in these IEH(N) graphs.

For the converse part, recall that the two adjacent graphs G_j and H_i are connected by IC edges for $i > j$ in step 3 of Algorithm CONSTR. Thus, if some node x in H_j has forward 2-IC edges, then all nodes in H_k have 2-IC edges for all $k \leqslant j$. Without loss

Table 1

	If the graph is Hamiltonian	If the graph contains even cycles	If the graph contains odd cycles
IEH(N)	Yes, except when $N = 2^n - 1$	Yes	Yes, when it has 2-IC edges
IH(N)	Yes, only N is even	Yes	No

of generality, let H_0 have forward 2-IC edges, let u be the neighbor node by a forward 1-IC edge of H_0, and let v be the neighbor node by a forward 2-IC edge of H_0. By applying Theorem 5, we find a cycle C' of length $N_0 - 1$ through the edge (u, v). By adding H_0 and its IC edges and deleting the edge (u, v), we find a cycle of length N_0 in the IEH(N) graph. □

5. Conclusion

In this paper, we describe the whole cycle structure in IEH graphs. The main results are summarized in the second row of Table 1. The properties of Incomplete Hypercubes (IHs) are listed in the third row [5] to be compared with the above results in the second row. It is obvious that IEH graphs are superior to IHs in embedding cycles.

References

[1] V. Auletta, A.A. Rescigno and V. Scarano, Embedding graphs onto the supercube, IEEE Trans. Comput. 44 (4) (1995) 593-597.

[2] L. Bhuyan and D.P. Agrawal, Generalized hypercube and hyperbus structure for a computer network, IEEE Trans. Comput. 33 (3) (1984) 323-333.

[3] K. Day and A. Tripathi, Embedding of cycles in arrangement graphs, IEEE Trans. Comput. 42 (8) (1993) 1002-1006.

[4] A. Kanevsky and C. Feng, On the embedding of cycles in pancake graphs, Parallel Comput. 21 (1995) 923-936.

[5] H.P. Katseff, Incomplete Hypercubes, IEEE Trans. Comput. 37 (5) (1988) 604-607.

[6] S. Latifi and N. Bagherzadeh, On the clustered-star graph and its properties, Comput. System Sci. Engineering 11 (3) (1996) 145-159.

[7] Y. Saad and M.H. Schultz, Topological properties of hypercubes, IEEE Trans. Comput. 37 (7) (1988) 867-872.

[8] A. Sen, Supercube: An optimal fault tolerant network architecture, Acta Inform. 26 (1989) 741-748.

[9] S. Sur and P.K. Srimani, IEH graphs: A novel generalization of hypercube graphs, Acta Inform. 32 (1995) 597-609.