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W. L. Pearn , Y. C. Chang & Chien-Wei Wu
[a] Department of Industrial Engineering & Management , National Chiao Tung University , 1001 Ta Hsueh Road, Hsinchu, Taiwan
[b] Department of Industrial Engineering & Management , National Chiao Tung University , 1001 Ta Hsueh Road, Hsinchu, Taiwan E-mail:
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Quality-yield measure for production processes with very low fraction defective

W. L. PEARN*, Y. C. CHANG and CHIEN-WEI WU

Process yield is the most common criterion used in manufacturing industry for measuring process performance. A more advanced measurement formula, called the quality yield index, Y_q, is proposed to calculate the quality yield for arbitrary processes by taking customer loss into consideration. Y_q penalizes yield for the variation of the product characteristics from its target, which presents a measure of the average product loss. Quality yield could be expressed as the traditional yield minus the truncated expected relative loss within the specifications to quantify how well a process can reproduce product items satisfactory to the customers. The paper proposes a reliable approach for measuring quality yield by converting the estimate into a lower confidence bound for processes with a very low fraction of defectives. The lower confidence bound not only provides information about actual process performance that is tightly related to both the fraction of defective units and customer quality loss, but also is useful in making decisions for capability testing.

1. Introduction

During the last decade, numerous process capability indices, including C_p, C_{pk}, C_{pm} and C_{pmk} (Kane 1986, Chan *et al.* 1988, Pearn *et al.* 1992), have been proposed in manufacturing industries to provide numerical measures on process performance. Those indices are effective tools for process capability analysis and quality assurance. Two process characteristics including the process location in relation to its target value and the process spread (overall process variation) are used to establish the formula of those capability indices. The closer the process output is to the target value and the smaller the process spread, the more capable is the process. That is, the larger the process capability index, the more capable is the process. Because C_p and C_{pk} are independent of the target T, they can fail to account for process loss incurred by the departure from the target. For this reason, two more advanced indices, C_{pm} and C_{pmk}, were developed. Those indices have been defined explicitly as follows:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\}, \quad C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} \quad \text{and}$$

$$C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\},$$

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Department of Industrial Engineering & Management, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan.
*To whom correspondence should be addressed. E-mail: roller@cc.nctu.edu.tw

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where μ is the process mean, σ is the process standard deviation, and USL and LSL are the upper and the lower specification limits, respectively. The indices are designed to monitor the performance for normal and near-normal processes with symmetric tolerances. It has been assumed that the target $T = M = (USL + LSL)/2$ (which is quite common in practical situations) for the simplicity of the present discussions. It is essential that process capability indices must be applied under the condition that the process is in statistical control (stable).

The index C_p considers the overall process variability relative to the manufacturing tolerance, reflecting product quality consistency. Due to the simplicity of the design, C_p cannot reflect the tendency of process centring (targeting). The index C_{pk} takes the process mean into consideration but can fail to distinguish between on-target processes from off-target processes. The index C_{pm} takes the proximity of process mean from the target value into account, which is more sensitive to process departure than C_{pk}. Because C_{pm} is based on the average process loss relative to the manufacturing tolerance, it has been alternatively called the Taguchi index. The index C_{pmk} is constructed from combining the modifications to C_p that produced C_{pk} and C_{pm}, which inherits the merits of both indices.

In the literature, several authors have promoted the use of various process capability indices and examined with differing degrees of completeness. Examples include Chou and Owen (1989), Chou *et al.* (1990), Franklin and Wasserman (1992), Kushler and Hurley (1992), Kotz *et al.* (1993), Vännman and Kotz (1995), Vännman (1997), Kotz and Lovelace (1998), Hoffman (2001), Pearn and Shu (2003), and references therein. Kotz and Johnson (2002) presented a thorough review for the development of process capability indices in the past 10 years, and Spiring *et al.* (2003) consolidated the research papers in process capability analysis for 1990–2002. Applications of those indices include the manufacturing of semiconductor products (Hoskins *et al.* 1988), head/gimbals assembly for memory storage systems (Rado 1989), flip-chips and chip-on-board (Noguera and Nielsen 1992), rubber edge (Pearn and Kotz 1994–95), aluminium electrolytic capacitors (Pearn and Chen 1997), and couplers and wavelength division multiplexers (Wu and Pearn 2003). Other applications include performance measures on process with tool-wear problem (Spiring 1989), supplier selections (Tseng and Wu 1991, Chou 1994), capability measures for multiple manufacturing streams (Bothe 1999) and many others.

1.1. *Process yield*

An important measure for interpreting process capability is yield, defined as:

$$Y = \int_{LSL}^{USL} \mathrm{d}F(x),$$

where $F(x)$ is the cumulative distribution function of the measured characteristic X. The disadvantage of the yield measure is that it does not distinguish among the products that fall inside of the specification limits.

1.2. *Process loss*

To rectify this disadvantage, the quadratic loss function is considered to distinguish the products by increasing the penalty as the departure from the target increases. However, the quadratic loss function itself does not provide comparison with the specification limits and depends on the unit of the characteristic. To address these issues, Johnson (1992) developed the relative expected loss L_e for a symmetric case as follows:

$$L_e = \int_{-\infty}^{\infty} \left[\frac{(x-T)^2}{d^2} \right] \mathrm{d}F(x) = \frac{\sigma^2 + (\mu - T)^2}{d^2},$$

where $d = (USL - LSL)/2$ is the half specification width. This measure has a direct relationship with C_{pm} because $L_e = (3C_{pm})^{-2}$. The advantage of L_e over C_{pm} is that the estimator of the former has better statistical properties than that of the latter, as the former does not involve a reciprocal transformation of process mean and variance.

1.3. *Quality yield*

To incorporate the proportion conforming measure Y with loss function-based index L_e, Ng and Tsui (1992) proposed the quality yield (Q-yield) index, Y_q. In contrast to the yield index Y, Q-yield emphasizes the ability of the process clustering around the target, which therefore reflects the degree of the process targeting (centring) by considering only the relative loss within the specifications. By only taking the relative expected loss L_e within the specifications into account, Ng and Tsui defined the standardized quality as one minus the relative loss, and so the Q-yield, Y_q is defined as the expected value of the standardized quality within the specification:

$$Y_q = \int_{LSL}^{USL} \left[1 - \frac{(x-T)^2}{d^2} \right] \mathrm{d}F(x).$$

This Q-yield index differs from the expected relative worth index defined in Johnson (1992) by truncating the deviation outside the specifications. With this truncation, the Q-yield index will be between 0 and 1 and thus provides a standardized measure. In addition, by relating to the yield measure widely accepted in the manufacturing industry, it will be understood and accepted as a capability measure. Similar to the yield measure Y, an ideal Y_q is 1, which provides the user a clear guide about the standard. Similar to the yield Y, Y_q requires no normality assumption. While yield is the proportion of conforming products, Q-yield can be interpreted as the average degree of products reaching 'perfect' or 'on target'.

The present paper first rewrites the Q-yield as a representation of process yield and expected relative loss, focusing on production processes with a very low fraction of defectives. It then obtains a lower confidence bound on process capability index C_{pk} and an upper confidence bound on the expected relative loss to convert the estimated Q-yield into a reliable lower confidence bound, which is the main contribution of the present work. The paper is organized as follows. Section 2 presents the comparisons of yield and Q-yield, with some illustrative examples. Section 3 investigates the estimator of Q-yield. Since $Y - L_e$ provides a lower bound on the Q-yield, estimations of process yield Y and process loss L_e are also explored. Section 4 proposes a reliable method to obtain a lower confidence bound on Q-yield. Section 5 presents an application example of the amplified pressure sensor (APS). Section 6 demonstrates the proposed methodology by calculating the Q-yield for pressure sensor product. Conclusions are made in Section 7.

2. Comparisons of yield and Q-yield

Process yield is currently defined as the percentage of the processed product units passing the inspections. Units are inspected according to specification limits placed on various key product characteristics and sorted into two categories: passed (conforming) and rejected (defectives). Use of yield as a quality measure implies that each rejected unit costs the factory an additional amount (scrap or repair), while each passed unit costs the factory nothing additional. By inference, all passed units are equally acceptable to the next-in-line customer. A customer in this sense refers to any user of goods such as materials, components, subassemblies, assemblies or systems.

However, customers do notice unit-to-unit difference in these characteristics, especially when the variance is large and/or the mean is offset from the target. A more customer-oriented measure Y_q is then proposed to account for both the fraction of defectives and variation from target for the passed units. Penalty to the yield increases as the departure from the target T increases. When all conforming products are on target, then $Y_q = Y$. Figures 1a and b show two normally distributed processes, $N(\mu = T, \sigma = d/3)$ and $N(\mu = T + d/3, \sigma = d/6)$, respectively, with the quadratic loss function. The latter process has a higher yield but with a lower Q-yield since it has larger departure from the target value than the former. Furthermore, if the process characteristic X follows uniform distribution, $U(LSL, USL)$, then the yield is $Y = 1.00$ (100% conforming) and Q-yield is $Y_q = 0.665$ (66.5% perfect),

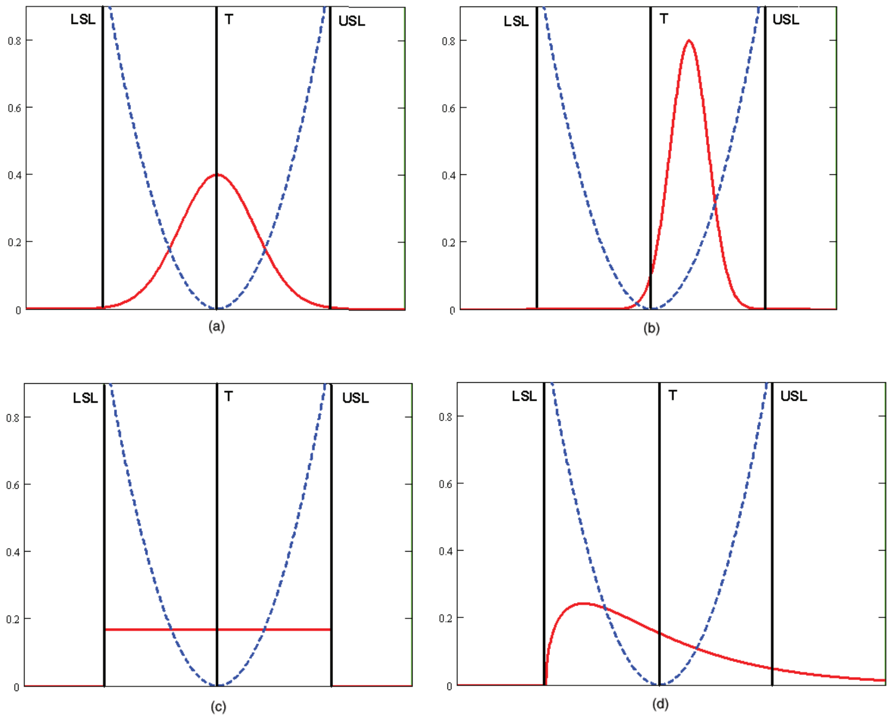

Figure 1. (a) Plots of process $N(T, d/3)$ with loss function, (b) Plots of process $N(T + d/3, d/6)$ with loss function, (c) Plots of process $U(LSL, USL)$ with loss function, (d) Plots of process $\chi^2(3)$ with loss function.

respectively. Obviously, this is a low-quality process. On the other hand, if X follows the chi-square distribution with three degrees of freedom, the yield would be $Y = 0.888$ (88.8% conforming) and Q-yield would be $Y_q = 0.62$ (62% perfect) (figures 1c, d).

To extend the applicability of the plot for normally distributed processes, we rewrite the definition of Y and Y_q as a function of $C_d = (\mu - T)/d$ and $C_v = \sigma/d$. Note that the subindex C_d measures the departure ratio, and the subindex C_v measures the process variation relative to the specification tolerances. The value of C_d (abscissa) considered is from -2 to 2 and hence μ is from $T - 2d$ to $T + 2d$. Moreover, C_v (ordinate) is from 0 to 1 to cover a wide range of σ. Therefore, using C_d as the x-axis and C_v as the y-axis, one can plot the surface of Y and Y_q with various $-2 \le C_d \le 2$ and $0 \le C_v \le 1$ (figure 2a and b, respectively). Figure 2c and d displays the cross-section plots of Y and Y_q versus $-2 \le C_d \le 2$ for various $C_v = 1/6$, 1/4, 1/3, 1/2, 1 (top to bottom in plot). Note that the plots of Y and Y_q are invariable irrespective of the specification limits. Processes with multiple characteristics with different characteristic specification limits can thus be plotted simultaneously on a single chart.

Therefore, high Q-yields are desirable and can be viewed as improved product quality from the customer's viewpoint. Q-yield is more flexible because it compares

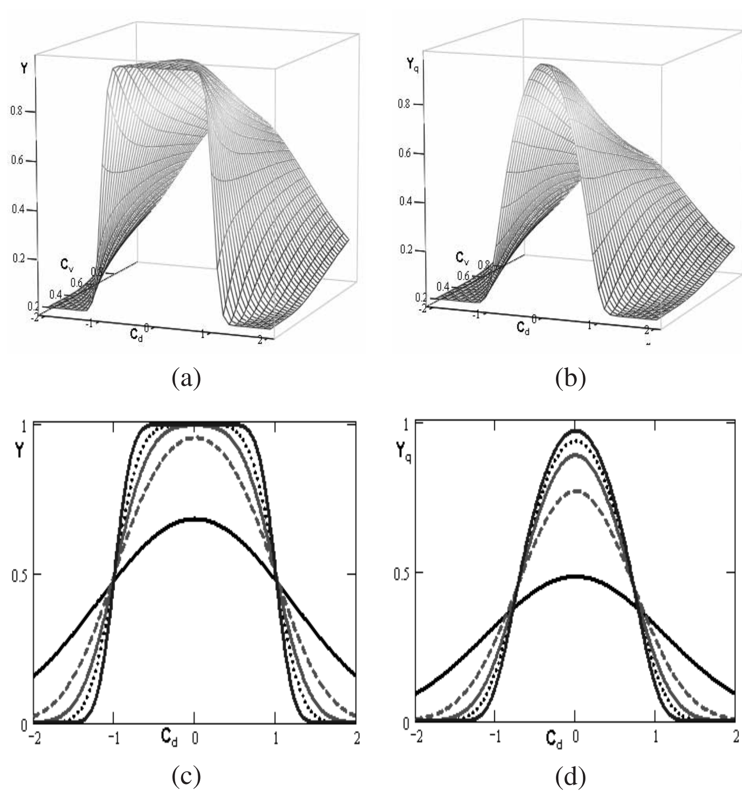

Figure 2. (a) Surface plot of Y versus $-2 \le C_d \le 2$ and $0 \le C_v \le 1$, (b) Surface plot of Y_q versus $-2 \le C_d \le 2$ and $0 \le C_v \le 1$, (c) Plots of Y versus $-2 \le C_d \le 2$ for various $C_v = 1/6$, 1/4, 1/3, 1/2, 1 (top to bottom), (d) Plots of Y_q versus $-2 \le C_d \le 2$ for various $C_v = 1/6$, 1/4, 1/3, 1/2, 1 (top to bottom).

Case	Y(%)	Y_q(%)	C_p	C_{pk}	C_{pm}	C_{pmk}
$N(T, d)$	68.27	48.39	0.33	0.33	0.33	0.33
$N(T, d/2)$	95.45	76.99	0.67	0.67	0.67	0.67
$N(T, d/3)$	99.73	88.94	1.00	1.00	1.00	1.00
$N(T, d/4)$	99.99	93.75	1.33	1.33	1.33	1.33
$N(T \pm d/3, d/2)$	90.50	69.13	0.67	0.44	0.55	0.37
$N(T \pm d/3, d/3)$	97.72	78.41	1.00	0.67	0.71	0.47
$N(T \pm d/3, d/4)$	99.62	82.70	1.33	0.89	0.80	0.53
$N(T \pm d/3, d/6)$	99.997	86.11	2.00	1.33	0.89	0.60

Table 1. Comparisons of yield, Q-yield, and C_p, C_{pk}, C_{pm}, and C_{pmk}.

the quality of different characteristics of a product on a single percentage scale and indicates how close a product comes to meeting 100% customer satisfaction. Comparing with the existing capability indices, note that those capability indices rely on the underlying assumption of normal distribution. Although new capability indices have been developed for non-normal distributions (e.g. the Clements 1989 and Johnson *et al.* 1994 methods). Those indices are more complicated to analyse and harder to interpret, and are sensitive to data peculiarities such as bimodality or truncation. Second, these indices do not explicitly account for the manufacturing cost or customer's loss. Capability indices are generally defined with respect to the specification limits rather than to the customer's functional limits. Table 1 summarizes values of those indices for some cases to illustrate the differences among Y, Y_q and C_p, C_{pk}, C_{pm}, C_{pmk}.

3. Estimation of Q-yield, Y_q

Ng and Tsui (1992) proposed a sample estimator based on a finite population of products. Suppose $X_1, X_2, \dots, X_n$ denote the sample measurements of product characteristics. It follows that yield and Q-yield are estimated by collected sample data and can be defined as follows:

$$\hat{Y} = \sum_{LSL \le X_i \le USL} \frac{1}{n}, \quad \hat{Y}_q = \sum_{LSL \le X_i \le USL} \left[\frac{1 - (X_i - T)^2/d^2}{n} \right].$$

The sampling distribution and sampling errors are investigated. The decision-maker would be interested in a lower bound on the Q-yield rather than just the sample point estimate. Further, as the rapid advancement of manufacturing technology and customers demand, when the fraction of defectives is very low, such as in parts per million (ppm), products almost all fall between LSL and USL, one cannot even observe a defective item on inspection for a reasonable sample size. Thus, such an approach is not applicable for the low defective processes (since the sample point estimate is almost certain to be zero). The Q-yield index Y_q can be rewritten as follows:

$$Y_q = Y - \int_{LSL}^{USL} \left[\frac{(x - T)^2}{d^2} \right] \mathrm{d}F(x) \ge Y - L_e.$$

Thus, the measure $Y - L_e$ provides a lower bound on the Q-yield Y_q. For processes with very low fraction of defectives, the approximation of Y_q using $Y - L_e$ would

C_{pk}	0.7	0.8	0.9	1	1.1	1.2	1.3	1.33
ppm	35729	16395	6934	2700	967	318	96	66
C_{pk}	1.4	1.5	1.6	1.67	1.7	1.8	1.9	2.0
ppm	27	6.795	1.587	0.544	0.34	0.067	0.012	0.002

Table 2. Some C_{pk} index values with the corresponding defective units (in ppm) for a normally distributed process.

be quite accurate. Subsequently, we discuss the estimators of process yield Y and process loss L_e.

3.1. *Estimation of process yield, Y*

The index C_{pk} is yield-based, which provides a lower bound on the process yield, i.e. $2\Phi(3C_{pk}) - 1 \le \text{yield} \le \Phi(3C_{pk})$ (Boyles 1991). Table 2 shows some indexes with two-sided specifications and the corresponding maximal non-conforming units (ppm) for a normally distributed process.

When $C_{pk} = C$, $b = d/\sigma$ can be expressed as $b = 3C + |\xi|$. Thus, the index C_{pk} can be expressed as a function of the characteristic parameter ξ:

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma} = \frac{d/\sigma - |\xi|}{3},$$

where $\xi = (\mu - M)/\sigma$.

Construction of the exact lower confidence bounds on C_{pk} is complicated since the distribution of $\hat{C}_{pk}$ involves the joint distribution of two non-central t-distributed random variables, or alternatively, the joint distribution of the folded-normal and the chi-square random variables, with an unknown process parameter even when the samples are given (Pearn *et al.* 1992). Numerous methods for obtaining approximate confidence bounds of C_{pk} have been proposed (e.g. Bissell 1990, Chou *et al.* 1990, Zhang *et al.* 1990, Porter and Oakland 1991, Kushler and Hurley 1992, Rodridguez 1992, Nagata and Nagahata 1994, Tang *et al.* 1997).

Using the integration technique similar to that presented in Vännman (1997), Pearn and Lin (2003) obtained an exactly explicit form of the cumulative distribution function of the natural estimator $\hat{C}_{pk}$ under the normal assumption, which is expressed in terms of a mixture of the chi-square distribution and the normal distribution, for $x > 0$, where $G(\cdot)$ is the cumulative distribution function of the chi-square distribution with degrees of freedom $n-1$, χ^2_{n-1} and $\phi(\cdot)$ is the probability density function of the standard normal distribution:

$$F_{\hat{C}_{pk}}(x) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n} - t)^2}{9nx^2}\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right] \mathrm{d}t.$$

(A brief derivation of the cumulative distribution function of $\hat{C}_{pk}$ is included in the appendix.) Hence, given the sample of size n, the confidence level γ, the estimated value $\hat{C}_{pk}$ and the parameter ξ, using numerical integration technique with iterations, the $100\gamma\%$ lower confidence bounds for C_{pk} and C_L, where $b_L = 3C_L + |\xi|$, can be obtained by solving the following equation:

$$\int_0^{b_L\sqrt{n}} G\left(\frac{(n-1)(b_L\sqrt{n} - t)^2}{9n\hat{C}_{pk}^2}\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right] \mathrm{d}t = 1 - \gamma.$$

A 100 γ% lower confidence bound on the process yield Y can then be expressed as $2\Phi(3C_L) - 1$.

However, since the process parameters μ and σ are unknown, then the distribution characteristic parameter, ξ is also unknown, which has to be estimated in real applications, naturally done by substituting μ and σ with the sample mean $\bar{X}$ and the sample standard deviation S. Such approach (and most existing methods) introduces additional sampling errors from estimating ξ in finding the lower confidence bounds, which certainly would make the decisions less reliable and provide less quality assurance to the customers. To eliminate the need for further estimating the distribution characteristic parameter ξ, we examine the behaviour of the lower confidence bound values C_L against the parameter ξ. The results indicate that the lower confidence bound is decreasing in ξ and reaches its minimum at $\xi = 1.00$ in all cases and stays at the same value for $\xi \geq 1.00$ with accuracy up to 10^{-4}. Figure 3a to d plots the curves of the lower confidence bound, C_L, versus the parameter $\xi = 0(0.05)3.00$, $n = 25$, 50, 75, 100, 150 and 200 with confidence level $\gamma = 0.95$, for $\hat{C}_{pk} = 1.00$, 1.33, 1.67 and 2.00, respectively. Hence, for practical purpose, we may solve the above equation with $\hat{\xi} = \xi = 1.00$ to calculate the required lower confidence bounds for given $\hat{C}_{pk}$, n and γ, without having to estimate further the parameter ξ. Thus, based on such an approach, the γ confidence level can be ensured and the decisions made are indeed more reliable.

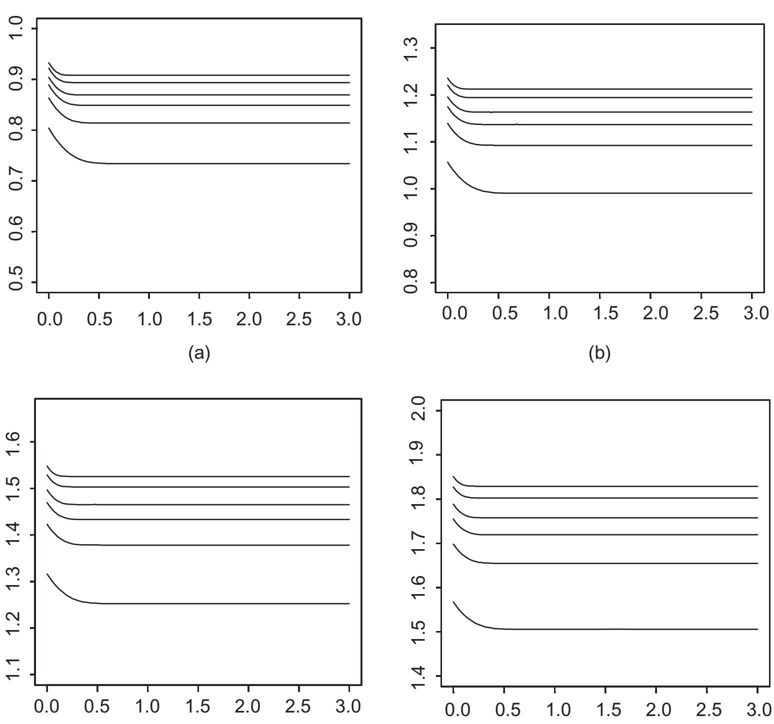

Figure 3. (a) Plots of C_L versus $|\xi|$ for $\hat{C}_{pk} = 1.00$, $n = 25$, 50, 75, 100, 150, 200 (bottom to top), (b) Plots of C_L versus $|\xi|$ for $\hat{C}_{pk} = 1.33$, $n = 25$, 50, 75, 100, 150, 200 (bottom to top), (c) Plots of C_L versus $|\xi|$ for $\hat{C}_{pk} = 1.67$, $n = 25$, 50, 75, 100, 150, 200 (bottom to top), (d) Plots of C_L versus $|\xi|$ for $\hat{C}_{pk} = 2.00$, $n = 25$, 50, 75, 100, 150, 200 (bottom to top).

3.2. *Estimation of process loss, L_e*

Johnson (1992) proposed the relative expected squared error loss L_e by approaching capability from the point of view of the loss function. However, here the opposite concept of worth was used. It was assumed that a characteristic achieves its maximum worth W_T, when $X = T$, with decreasing values of worth as X moves away from the target value T (eventually the worth becomes zero, then negative). The worth function can be described by $W_T - k(X-T)^2$, for $W_T \geq k(X-T)^2$, and it will become zero when $|X-T| = (W_T/k)^{1/2}$. Johnson viewed the pair $(T + (W_T/k)^{1/2}, T - (W_T/k)^{1/2})$ in the role of specification limits for C_{pm} and defined $\Delta = (W_T/k)^{1/2}$. The ratio of worth to maximum worth is called the relative worth and can be defined as:

$$W(X) = 1 - \frac{k(X-T)^2}{W_T} = 1 - \frac{(X-T)^2}{\Delta^2},$$

where $(X-T)/\Delta^2$ is the relative loss. The expected relative loss, $L_e = E[(X-T)]/\Delta^2$, is used to quantify capability and is effectively equivalent to C_{pm}, since:

$$L_e = \left(\frac{d}{3\Delta}\right)^2 / C_{pm}^2.$$

Suppose the product has zero worth outside the specifications by setting $\Delta = d$, the relationship between L_e and C_{pm} becomes $L_e = (3C_{pm})^{-2}$. A natural unbiased estimator of L_e is:

$$\hat{L}_e = \frac{1}{nd^2} \sum_{i=1}^{n} (X_i - T)^2.$$

4. Lower confidence bounds on Q-yield, Y_q

Now we deal with the lower confidence limit on the Q-yield. Given a sample of size n, confidence level γ, estimated value $\hat{C}_{pk}$ and the estimated relative loss $\hat{L}_e$, the lower confidence bounds of Y_q can be easily obtained by some mathematical manipulations. The $100\gamma\%$ lower confidence bound of Y_q can be expressed as:

$$L_{Y_q} = L_Y - U_{L_e} = 2\Phi(3C_L) - 1 - \left[\frac{n + \hat{\lambda}}{\chi_n'^2(1-\gamma_2; \hat{\lambda})}\right] \hat{L}_e,$$

where L_Y is a lower $100\gamma_1\%$ confidence bound on Y, U_{L_e} is an upper $100\gamma_2\%$ confidence bound on L_e and $\gamma = \gamma_1 \times \gamma_2$. All derivations are shown below. A lower $100\gamma\%$ confidence bound for Y_q and Y simultaneously can be derived as:

$$\begin{aligned} P(Y \geq L_Y, Y_q \geq L_{Y_q}) &= P(Y \geq L_Y) \times P(Y_q \geq L_{Y_q} | Y \geq L_Y) \\ &= P(Y \geq L_Y) \times P(L_e \leq Y - L_{Y_q} \mid Y \geq L_Y) \geq P(Y \geq L_Y) \\ &\quad \times P(L_e \leq L_Y - L_{Y_q}) \\ &= \gamma_1 \times \gamma_2. \end{aligned}$$

As noted above, the yield-based index C_{pk} gives a lower bound on the process yield. Hence, the probability $P(Y \geq L_Y)$ is equivalent to the probability $P(C_{pk} \geq C_L)$. Solve $P(Y \geq L_Y) = \gamma_1$ for L_Y, one obtains $L_Y = 2\Phi(3C_L) - 1$, where C_L is the $100\gamma_1\%$ lower confidence bound on C_{pk}. Next, we proceed with the expression

$P(L_{e} \leq L_Y - L_{Y_q}) = \gamma_2$. Under normal assumptions, $(n+\lambda)\hat{L}_e/L_e$ is distributed as $\chi_n^{'2}(\lambda)$, a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\lambda = n(\mu - T)^2/\sigma^2$. Let $U_{L_e} = U_{L_e}(X_1, X_2, \ldots, X_n)$ be a statistic calculated from the sample data satisfying $P(L_e \leq U_{L_e}) = \gamma_2$, where the confidence level γ_2 does not depend on L_e. Then, U_{L_e} is an 100 γ_2% upper confidence bound for L_e. Note that:

$$P(L_e \leq U_{L_e}) = \mathrm{P}((n+\lambda)\hat{L}_e/L_e \geq (n+\lambda)\hat{L}_e/U_{L_e}) \\ = \mathrm{P}(\chi_n^{'2}(\lambda) \geq (n+\lambda)\hat{L}_e/U_{L_e}) = \gamma_2.$$

Thus, $(n+\lambda)\hat{L}_e/U_{L_e} = \chi_n^{'2}(1-\gamma_2;\lambda)$, where $\chi_n^{'2}(1-\gamma_2;\lambda)$ is the (lower) $(1-\gamma_2)$th percentile of the $\chi_n^{'2}(\lambda)$ distribution. An $100\gamma_2$% upper confidence limit on L_e can be expressed, in terms of $\hat{L}_e$, as:

$$U_{L_e} = \left[\frac{n+\lambda}{\chi_n^{'2}(1-\gamma_2;\lambda)}\right]\hat{L}_e.$$

λ can be estimated by $\hat{\lambda} = n[(\bar{X} - T)/S_n]^2$, where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S_n = [\sum_{i=1}^{n}(X_i - \bar{X})^2/n]^{1/2}$. Substitute the results of L_Y and L_{Y_q} back to the equation, an 100 γ% lower confidence bound for Y_q and Y simultaneously can be expressed as:

$$P\left(Y \geq 2\Phi(3C_L) - 1, \quad Y_q \geq 2\Phi(3C_L) - 1 - \left[\frac{n+\hat{\lambda}}{\chi_n^{'2}(1-\gamma_2;\hat{\lambda})}\right]\hat{L}_e\right) \geq \gamma.$$

5. Application to amplified pressure sensor (APS)

Consider the following case taken from a manufacturing factory making a series of original equipment manufacturer (OEM) pressure sensors, which combines state-of-the-art pressure sensor technology with signal conditioning to produce a fully signal conditioned, amplified, temperature-compensated sensor in a dual in-line package (DIP) configuration. Combining the sensor and signal conditioning circuitry in a single package simplifies the use of advanced silicon micromachined pressure sensors. The pressure sensor can be directly mounted onto a standard printed circuit board and no additional components are required to obtain an amplified high-level, calibrated pressure measurement. The pressure sensors are based on highly stable, piezoresistive pressure sensor chips mounted on a ceramic substrate. Two different pin configurations of the APS part, one for classical through-hole printed circuit board applications the other for surface mount applications, are available (figure 4a, b). Note that the only difference between the two is the pins. The ceramic housing, cap and ports are identical between the two configurations.

An electronically programmable application-specific integrated circuit (ASIC) is contained in the same package to provide calibration and temperature compensation. The model is designed for operating pressure ranges from 0–5 to 0–100 psi. In addition, the sensor output is ratio metric with the supply voltage. Some features of the model are as follows: wide selection of full-scale ranges to 100 psi; low pressure (0–0.15 psi (full scale) FS) based on unique low-pressure die; amplified, calibrated, fully signal conditioned amplified output of 4.0 (volts direct current) VDC FS span (0.5–4.5 V signal); output ratio metric with supply voltage; temperature compensation for span and offset; gage, differential and absolute version; DIP package for convenient personal computer board mounting; and small, lightweight package.

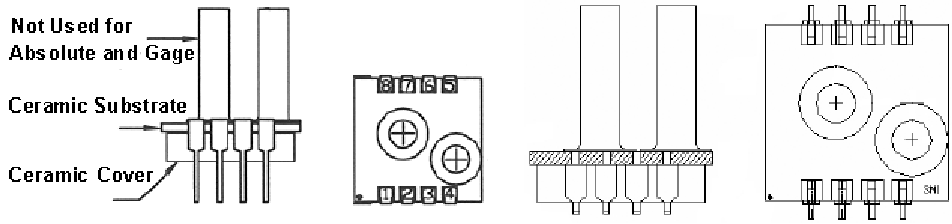

Figure 4. (a) Standard through hole pin configuration for the APS, (b) Surface-mount pin configuration for the APS.

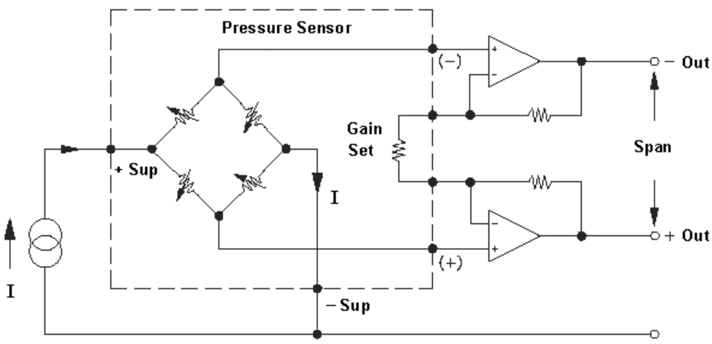

Figure 5. Application schematic with 1.5 mA drive at 25°C after a 10 s warm up.

Some typical applications are barometric measurement; medical instrumentation; pneumatic control; gas flow; respirators and ventilators and ventilating and air-conditioning.

5.1. *Amplified pressure product capabilities*

The series pressure product provides a significant advantage to the user due to a number of improvements associated with the technology used in fabricating this part. These advantages include integrated amplification, electronic-trim for more precise control of gain and offset, and fewer external support components. The following notes are meant as an aid to the user to document some of these improvements. The amplified configuration has some key advantages, but these also must be considered in designing the systems into which pressure sensor parts are used. For instance, a fairly standard 'trick' when using an unamplified part in such absolute applications as barometric measurements is to use a 5 PSIA part. This allows it to operate in the 15 PSIA range with effectively 10 psi overpressure to get three times more unamplified output from the part. The addition of amplification at the measurement site has several key advantages. One is the required support circuitry. The pressure sensor has been designed to eliminate the need for external components. It requires no external components. The pressure sensor model with the gain of the part testing is shown in figure 5.

One of the key features of the pressure sensor is that it is electronically trimmed. As such, the part can be tested and verified before the final trim parameters are programmed. With the conventional laser-trimmed components, the final performance is set by how well the test system can measure millivolt level signals and resistances ranging from < 50 Ohms to > 5 MOhms. All of this is done at the end of long test cables and this further makes measurements more uncertain. There are several alternative measures on the manufacturability of a part. Yield from a manufacturer's viewpoint is critical, but so to is the distribution of parts as manufactured. The tighter the distribution on key parameters, the higher the quality of the part and the lower is the probability that the end-customer will get a part that will not meet the published specification.

6. Q-yield calculation for pressure sensor product

To illustrate how the proposed Q-yield lower confidence bound could be established and applied to actual data collected from the factories, we consider the following example taken from a company located in the Science-Based Industrial Park, Taiwan, manufacturing and designing a pressure sensor product. For a particular model of amplified pressure sensor process, capability analysis with focus on two key characteristics, Span and Zero, are taken. Span limits are ± 100 mV about a 2.000 V target ($USL = 2.100$, $LSL = 1.900$, $T = 2.000$) and the Zero limits are set to ± 80 mV about a 2.500 V target ($USL = 2.580$, $LSL = 2.420$, $T = 2.500$). Tight control of Zero and Span during testing will make the part more capable. We test 100 parts in each key characteristic. The collected data are shown in table 3. Figure 6a and b shows the histogram with density of the 100 APS data measurements for the Zero and Span, respectively. Proceeding with the calculations with a 95% level of

Zero (V)				Span (V)			
2.5445	2.5310	2.5204	2.5406	2.0512	2.0532	2.0396	2.0035
2.5455	2.5305	2.5418	2.5390	2.0594	2.0507	2.0382	2.0512
2.5338	2.5721	2.5430	2.5570	2.0517	2.0050	2.0276	1.9956
2.5482	2.5573	2.5403	2.5539	2.0038	2.0300	2.0719	2.0038
2.5306	2.5329	2.5391	2.5493	2.0532	2.0318	1.9957	2.0629
2.5471	2.5495	2.5202	2.5452	2.0235	2.0308	2.0226	2.0409
2.5482	2.5355	2.5470	2.5528	2.0373	1.9684	2.0113	2.0092
2.5474	2.5611	2.5434	2.5335	2.0501	2.0037	2.0295	2.0524
2.5532	2.5419	2.5327	2.5416	2.0575	2.0557	2.0333	2.0584
2.5511	2.5455	2.5618	2.5506	2.0070	2.0374	2.0563	2.0094
2.5490	2.5476	2.5490	2.5382	1.9716	2.0152	2.0392	2.0113
2.5543	2.5375	2.5454	2.5225	2.0390	2.0504	2.0529	2.0463
2.5454	2.5466	2.5253	2.5405	2.0316	1.9912	2.0824	2.0307
2.5279	2.5333	2.5586	2.5432	2.0000	2.0243	2.0825	2.0180
2.5381	2.5364	2.5563	2.5521	2.0102	1.9842	2.0300	2.0433
2.5453	2.5396	2.5493	2.5402	2.0112	2.0482	2.0440	1.9793
2.5379	2.5486	2.5382	2.5432	2.0024	2.0277	2.0199	2.0255
2.5270	2.5484	2.5461	2.5409	2.0638	2.0252	2.0006	2.0227
2.5367	2.5289	2.5335	2.5429	2.0518	2.0668	2.0142	2.0239
2.5518	2.5346	2.5265	2.5409	2.0105	2.0254	1.9966	2.0359
2.5462	2.5432	2.5390	2.5358	2.0536	2.0377	2.0162	1.9897
2.5542	2.5583	2.5361	2.5454	2.0275	2.0231	2.0636	2.0289
2.5398	2.5331	2.5440	2.5424	1.9993	1.9831	2.0533	2.0238
2.5203	2.5464	2.5270	2.5607	1.9985	2.0519	2.0041	2.0499
2.5291	2.5445	2.5360	2.5502	2.0254	2.0709	2.0162	2.0156

Table 3. APS data of 100 measurements for the Zero and Span.

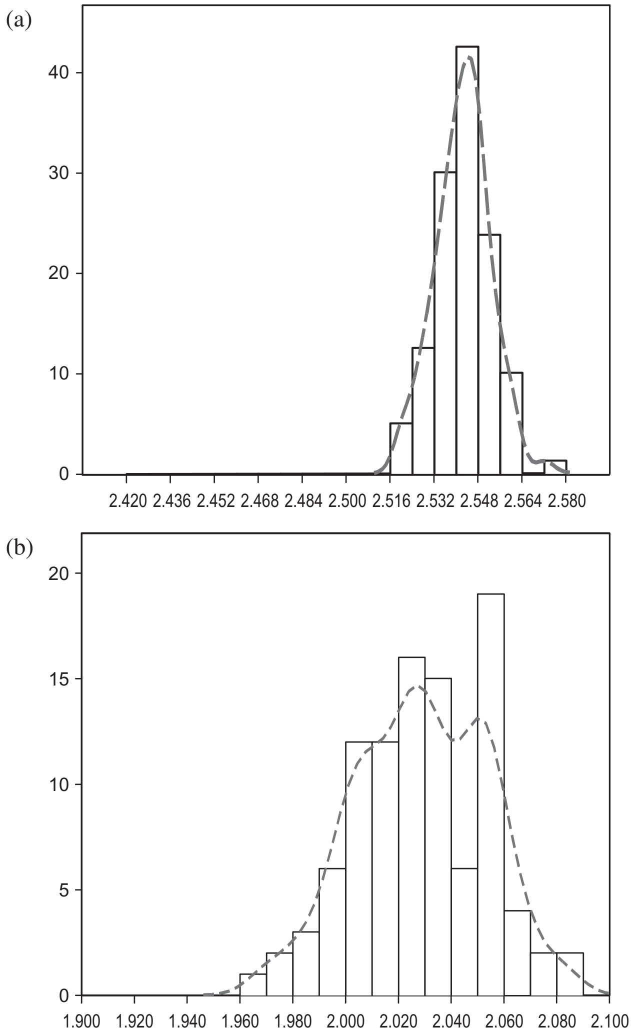

Figure 6. (a) Histogram of the APS data measurements for the Zero, (b) Histogram of the APS data measurements for the Span.

confidence, we obtain the calculated sample mean, sample derivation, estimated C_{pk} index values, C_{pk} lower confidence bounds, estimated L_e values, L_e upper confidence bounds, estimated yield, yield lower confidence bounds, estimated Q-yield and Q-yield lower confidence bounds (table 4).

Table 5 shows the manufacturing capabilities for the pressure sensor processes using the estimated yield, estimated Q-yield and their corresponding lower confidence bounds. The plot of Q-yield versus yield is shown in figure 7. These two dimensions of product quality are useful because one dimension represents customer satisfaction while the other represents factory fulfilment. The triangle with vertices (0, 0), (1, 0) and (1, 1) contains the set of all (Y, Y_q). The objective of quality improvement is to move towards the point (1, 1). The engineers can effectively monitor and get the most priority of all process characteristics simultaneously.

	$\bar{X}$	S	$\hat{C}_{pk}$	C_L	$\hat{L}_e$	U_{L_e}	$\hat{Y}$	L_Y	$\hat{Y}_q$	L_{Y_q}
Zero	2.5424	0.0099	1.2705	1.0821	0.2959	0.3983	1.0000	0.9999	0.7041	0.6016
Span	2.0286	0.0246	0.9660	0.8165	0.1418	0.1908	1.0000	0.9962	0.8582	0.8054

Table 4. Calculated statistics, estimated process capability measures and corresponding lower confidence bound of the APS products.

	Estimated yield	Yield LCB	Estimated Q-yield	Q-yield LCB
Zero	1.0000	0.9999	0.7041	0.6016
Span	1.0000	0.9962	0.8582	0.8054

Table 5. Comparison of estimated yield and Q-yield, associated with LCB for the APS products.

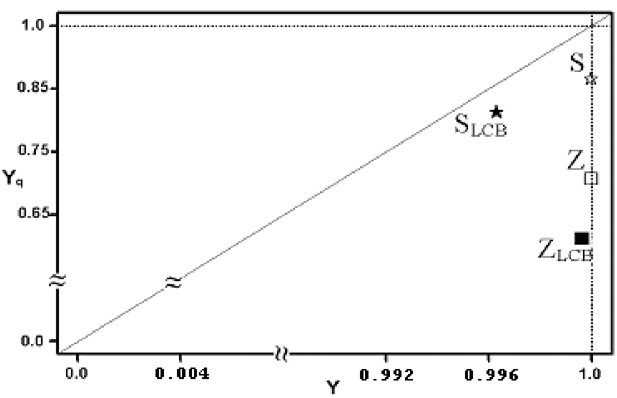

Figure 7. Plot of Q-yield versus yield.

7. Conclusions

Process capability indices, which establish the relationship between the actual process performance and the manufacturing specifications, have been the focus of recent research in quality assurance and capability analysis. The Q-yield, Y_q, has been proposed to calculate the process capability by taking customer loss into consideration. It penalizes yield for variation of the product characteristics from its target, combining the proportion of conformities and the average process loss. The present paper develops a reliable approach to obtain a lower confidence bound for Y_q, which can be applied to production processes with very low fraction of defectives where existing method cannot be applied. The lower confidence bound provides information about actual process performance for both the fraction of defective units and customer quality loss. The results obtained allow one to perform capability testing based on yield and customer satisfactions. A real-world application to the amplified pressure sensor manufacturing process is also presented for illustrative purposes.

Appendix: Derivation of the cumulative distribution function of $\hat{C}_{pk}$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from a normal distribution with mean μ and variance σ^2 measuring the characteristic under investigation.

The natural estimator $\hat{C}_{pk}$ is obtained by replacing the process mean μ and process standard deviation σ by their conventional estimators $\overline{X}$ and S, respectively. The following expression occurs:

$$\hat{C}_{pk} = \frac{d - \left|\bar{X} - M\right|}{3S}.$$

For the sake of deriving the cumulative distribution function of $\hat{C}_{pk}$, the following notations are introduced:

- $K=(n-1)S^2/\sigma^2$, which is distributed as χ^2_{n-1}.
- $Z' = \sqrt{n}(\bar{X} - M)/\sigma$, which is distributed as $N(\xi\sqrt{n}, 1)$ with $\xi=(\mu - M)/\sigma$.
- $H=|Z'|$, which is distributed as a folded-normal distribution with probability density function $f_H(h) = \phi(h + \xi\sqrt{n}) + \phi(h - \xi\sqrt{n})$ for $h \geq 0$, where $\phi(\cdot)$ is the probability density function of the standard normal distribution.

For $x > 0$, the cumulative distribution function of $\hat{C}_{pk}$ can be derived as follows:

$$\begin{aligned}
F_{\hat{C}_{pk}}(x) &= P\left(\hat{C}_{pk} \leq x\right) = P\left(\frac{\sqrt{n-1}(b\sqrt{n} - H)}{3\sqrt{nK}} \leq x\right) \\
&= 1 - P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - H)}{3x}\right) \\
&= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - H)}{3x} | H = h\right) f_H(h)\,\mathrm{d}h \\
&= 1 - \int_0^\infty P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - h)}{3x}\right) f_H(h)\,\mathrm{d}h,
\end{aligned}$$

where $b=d/\sigma$. Since K is distributed as χ^2_{n-1}:

$$P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - h)}{3x}\right) = 0 \text{ for } h > b\sqrt{n} \text{ and } x > 0.$$

Therefore,

$$\begin{aligned}
F_{\hat{C}_{pk}}(x) &= 1 - \int_0^{b\sqrt{n}} P\left(\sqrt{nK} < \frac{\sqrt{n-1}(b\sqrt{n} - h)}{3x}\right) f_H(h)\,\mathrm{d}h \\
&= 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n} - h)^2}{9nx^2}\right) f_H(h)\,\mathrm{d}h, \text{ for } x > 0,
\end{aligned}$$

where $G(\cdot)$ is the cumulative distribution function of χ^2_{n-1}. Substituting $f_H(t)$ leads to the result:

$$F_{\hat{C}_{pk}}(x) = 1 - \int_0^{b\sqrt{n}} G\left(\frac{(n-1)(b\sqrt{n} - t)^2}{9nx^2}\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right]\mathrm{d}t, \text{ for } x > 0.$$

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