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Note on inventory model with a mixture of back orders and lost sales

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Abstract

Competitiveness is an important means of determining whether a company will prosper. Business organizations compete with one another in a variety of ways. Among these competitive methods are time and cost factors. The purpose of this paper is to examine the inventory models presented by Padmanabhan and Vrat [International Journal of Systems Sciences 21 (1990) 1721] with a mixture of back orders and lost sales. We develop the criterion for the optimal solution for the total cost function. If the criterion is not satisfied, this model will degenerate into one cycle inventory model with a finite inventory period. This implies an extension of shortage period as long as possible to produce lower cost. However, we know that time is another important factor in company competitiveness. Customers will not indefinitely wait for back orders. A tradeoff will be made between the two most important factors; time and cost. The minimum total cost is evaluated under the diversity cycle time and illustrations are applied to explain the calculation process. This work provides a reference for decision-makers.
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1. Introduction

The success of business tactics has close relationship with controlling inventory. Too much inventory reduces capital and increases storage expenses and risk. Not enough inventory produces a supply inefficient for meeting demand. This will produce lost sales opportunity and ill will. Determining the correct inventory level at the right time and reducing cost and satisfying customer demand are the most important issues for managers. In many situations, the customers of certain suppliers have high faith and loyalty. When there is a shortage, these customers are willing to wait for back orders. However, if the customer becomes impatient and turns to other suppliers lost sales will result. Many scholars have devoted study to inventory models with a mixture of back orders and lost sales.

Montgomery et al. (1973) solved continuous review and periodic review inventory models that

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considered a mixture of back orders and lost sales first. Kim and Park (1985) considered a continuous review system with constant lead-time where a fraction of the unfilled demand was back ordered and the back order cost was assumed proportional to the length of time the back order existed. Padmanabhan and Vrat (1990) developed an inventory model with a mixture of back orders and lost sales such that the backlogged demand rate was dependent upon the negative inventory level during the stock out period. Padmanabhan and Vrat (1995) presented inventory models for deteriorating items with stock-dependent selling rates and derived the profit functions with and without backlogging and complete backlogging cases. Arreola-Risa and DeCroix (1998) studied a stochastic-demand inventory system where the product's supply is randomly disrupted for periods of random duration. They considered the stochastic-demand inventory system will become a mixture of back orders and lost sales during demand shortage. DeCroix and Arreola-Risa (1998) explored the potential benefits of offering economic incentives to back order as a strategy for inventory management when the system involves an unreliable supply. Ouyang and Chuang (1999) investigated an inventory model with a mixture of back orders and lost sales in which the back order rate was a random variable and the quantity was discounted on the inventory model effect. Chung et al. (2000) considered the Padmanabhan and Vrat problem (1995). They developed necessary and sufficient conditions for the optimal profit per unit time function solutions. Abad (2000) considered the problem of determining the lot size for a perishable good under finite production, exponential decay, partial back ordering and lost sales.

In supply shortage cases, the lead-time issue becomes more important. Some scholars performed research on this matter. Moon and Gallego (1994) assumed unfavorable lead-time demand distribution and solved both the continuous review model and the periodic review model with a mixture of back orders and lost sales using the minmax distribution free approach. Ouyang et al. (1996) assumed that shortages were allowed and constructed a variable lead-time from a mixed inventory model with back orders and lost sales. Moon

and Choi (1998) and Lan et al. (1999) pointed at the problem of Ouyang et al. (1996). They found individually optimal order quantities and optimal lead-time for a mixed inventory model and simplified the solution procedure. Wu and Tsai (2001) considered that the lead-time demands of different customers are not identical. They developed a mixed inventory model with back orders and lost sales for variable lead-time demand with a mixed normal distribution. Pan and Hsiao (2001) presented inventory models with back order discount and variable lead-time to ensure that customers were willing to wait for the back orders.

From the business competitiveness standpoint, the inventory model with a mixture of back orders and lost sales means that time and cost are two important competitive factors. This paper examines Padmanabhan and Vrat's (1990) inventory models with a mixture of back orders and lost sales. In Section 2, the notation and assumptions are defined. In Section 3, the results from Padmanabhan and Vrat are reviewed. In Section 4, we will analyze and improve Padmanabhan and Vrat's method. We will prove that the minimum solution must occur inside the interior section such that solving the first partial derivative system will attain the minimum solution. Utilizing this approach, a theorem to determine the criterion for proving the existence and uniqueness of the minimum solution is developed. In Section 5, an illustration to show the accurate calculation process is applied. If the criterion is not satisfied, we explain the realistic meanings for tradeoff time and cost.

2. Notation and assumptions

We used the same notation and assumptions as Padmanabhan and Vrat (1990).

α constant demand rate
T cycle time
R fixed opportunity cost for a lost sale
C unit cost
A ordering cost per order
i inventory carrying cost fraction per unit per unit time
C_2 shortage cost per unit per unit time

t_1 period up to which the inventory is posi-
tive
Q order quantity
S maximum inventory level
$C_T(T, t_1)$ the total cost per unit time

The following assumptions were made in developing the model.

(a) The demand rate, unit cost, ordering cost and inventory carrying cost fraction are known and constant.
(b) The backlogged demand rate follows $D(t) = \alpha + \delta I(t)$ during the stock out period where $I(t)$ is the negative inventory level and δ is a positive constant.
(c) The rate of replenishment is infinite and the lead-time is zero.

3. The work of Padmanabhan and Vrat

The results from Padmanabhan and Vrat (1990), who constructed the total cost per unit time for an inventory model with a mixture of back orders and lost sales such that the backlogged demand rate is dependent on the negative inventory level during the stock out period, are reviewed.

$$C_T(T, t_1)$$
$$= \frac{1}{T} \left[A + \frac{Ci\alpha}{2} t_1^2 + \frac{\alpha C_2}{\delta^2} (e^{\delta(t_1 - T)} + \delta(T - t_1) - 1) \right.$$
$$\left. + \left\{ \alpha(T - t_1) + \frac{\alpha}{\delta} (e^{\delta(t_1 - T)} - 1) \right\} R \right]. \quad (1)$$

They proved that $C_T(T, t_1)$ is convex and the necessary conditions for optimality are:

$$\frac{\partial C_T}{\partial T} = 0 \quad \text{and} \quad \frac{\partial C_T}{\partial t_1} = 0.$$

These result in the following:

$$\alpha T(1 - e^{\delta(t_1 - T)}) \left(R + \frac{C_2}{\delta} \right)$$
$$- \left[A + \frac{Ci\alpha}{2} t_1^2 + \frac{\alpha C_2}{\delta^2} (e^{\delta(t_1 - T)} + \delta(T - t_1) - 1) \right.$$
$$\left. + \left\{ \alpha(T - t_1) + \frac{\alpha}{\delta} (e^{\delta(t_1 - T)} - 1) \right\} R \right] = 0, \quad (2)$$

$$\left(R + \frac{C_2}{\delta} \right) (e^{\delta(t_1 - T)} - 1) + Cit_1 = 0. \quad (3)$$

They claimed that the above non-linear Eqs. (2) and (3) can be solved using numerical methods for the optimum values for T and t_1.

4. Our analysis and improvement

We rewrite the total cost per unit time $C_T(T, t_1)$ as follows:

$$C_T(T, t_1) = \frac{1}{T} \left[A + \frac{iC\alpha}{2} t_1^2 + \frac{\alpha}{\delta} \left(\frac{C_2}{\delta} + R \right) (e^{\delta(t_1 - T)} \right.$$
$$\left. + \delta(T - t_1) - 1) \right]. \quad (4)$$

We solve the minimum solution for $C_T(T, t_1)$ for $0 \leqslant t_1 \leqslant T$ and $(T, t_1) \neq (0, 0)$.

Lemma 1. For each fixed T, $C_T(T, t_1)$ with $0 \leqslant t_1 \leqslant T$ has a minimum value at the interior $0 < t_1 < T$.

Proof. When T is fixed, we know that

$$\lim_{t_1 \to T} \frac{d}{dt_1} C_T(T, t_1) > 0, \quad \lim_{t_1 \to 0} \frac{d}{dt_1} C_T(T, t_1) < 0 \quad \text{and}$$
$$\frac{d}{dt_1} C_T(T, t_1) = \frac{\delta}{T} \left(R + \frac{C_2}{\delta} \right) (e^{\delta(t_1 - T)} - 1) + \frac{\delta}{T} Cit_1.$$

Moreover, since

$$\frac{d^2}{dt_1^2} C_T(T, t_1) = \frac{\delta}{T} (R\delta + C_2)(e^{\delta(t_1 - T)} - 1) + \frac{\delta}{T} Ci > 0,$$

we obtain that $\frac{d}{dt_1} C_T(T, t_1)$ is increasing from negative values to positive values. This completes the proof for Lemma 1. □

From Lemma 1, for the minimum problem, only the interior points $0 < t_1 < T$ are considered. We try to solve $\frac{\partial}{\partial T} C_T(T, t_1) = 0$ and improve Eq. (2) as

$$\alpha T(1 - e^{\delta(t_1 - T)}) \left(R + \frac{C_2}{\delta} \right)$$
$$= A + \frac{iC\alpha}{2} t_1^2 + \frac{\alpha}{\delta} \left(\frac{C_2}{\delta} + R \right)$$
$$\times (e^{\delta(t_1 - T)} + \delta(T - t_1) - 1). \quad (5)$$

If Eq. (3) has solutions, then

$$Cit_1 = \left(R + \frac{C_2}{\delta}\right)(1 - e^{-\delta(T-t_1)}) < \left(R + \frac{C_2}{\delta}\right),$$

hence, if $t_1 > \frac{R\delta + C_2}{iC\delta}$, then $\frac{\partial}{\partial t_1} C_T(T, t_1) > 0$, so we can derive the following lemma.

Lemma 2

(a) If a t_1 is given with $0 < t_1 < \frac{R\delta + C_2}{iC\delta}$, then there is a unique T as

$$T = t_1 + \frac{1}{\delta} \ln\left(\frac{R\delta + C_2}{R\delta + C_2 - iC\delta t_1}\right) \qquad (6)$$

such that $T > t_1$ and T satisfies Eq. (3).

(b) If $t_1 \geqslant \frac{R\delta + C_2}{iC\delta}$, then Eq. (3) does not have solutions.

Lemma 2 shows that the minimum solution will occur for $0 < t_1 < \frac{R\delta + C_2}{iC\delta}$. From Eq. (3), we substitute $\frac{iC\delta t_1}{R\delta + C_2} = 1 - e^{\delta(t_1 - T)}$ and Eq. (6) into Eq. (5), and then have

$$\frac{iC\alpha}{2} t_1^2 + \frac{iC\alpha}{\delta} t_1 + \frac{\alpha}{\delta^2}(iC\delta t_1 - (R\delta + C_2))$$
$$\times \ln\left(\frac{R\delta + C_2}{R\delta + C_2 - iC\delta t_1}\right) = A. \qquad (7)$$

Motivated by Eq. (7), we define a new function, $f(t_1)$, as follows:

$$f(t_1) = \frac{iC\alpha}{2} t_1^2 + \frac{iC\alpha}{\delta} t_1 + \frac{\alpha}{\delta^2}(iC\delta t_1 - (R\delta + C_2))$$
$$\times \ln\left(\frac{R\delta + C_2}{R\delta + C_2 - iC\delta t_1}\right). \qquad (8)$$

Lemma 3

(a) When

$$\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) > A,$$

it shows that $f(t_1) = A$ has solution t_1^* and $C_T(T^*, t_1^*) = iC\alpha t_1^*$.

(b) If

$$\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) \leqslant A,$$

then $f(t_1) = A$ has no solutions in $0 < t_1 < \frac{R\delta + C_2}{iC\delta}$, and the minimum value is $\lim_{T \to \infty} C_T(T, t_1) = \alpha R + \frac{\alpha}{\delta} C_2$.

Proof. Since

$$\lim_{t_1 \to \frac{R\delta + C_2}{iC\delta}} f(t_1) = \frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right),$$
$$f(0) = 0$$

and for $0 < t_1 < \frac{R\delta + C_2}{iC\delta}$,

$$\frac{d}{dt_1} f(t_1) = iC\alpha t_1 + iC\frac{\alpha}{\delta} \ln\left(\frac{R\delta + C_2}{R\delta + C_2 - iC\delta t_1}\right) > 0,$$

then we have that if

$$\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) > A,$$

then $f(t_1^*) = A$ has solution t_1^*. When we compute the minimum value plunging (2) and (3) into (1), it implies $C_T(T^*, t_1^*) = iC\alpha t_1^*$.

If $\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) \leqslant A$, then $f(t_1) = A$ has no solutions for $0 < t_1 < \frac{R\delta + C_2}{iC\delta}$. Recalling Lemma 1, if the minimum point, say (T^*, t_1^*), satisfies $T^* < \infty$ then there is the point t_1^* satisfying Eq. (3). However, we have pointed out that when $\frac{\alpha}{2iC}(R + \frac{C_2}{\delta})^2 + \frac{\alpha}{\delta}(R + \frac{C_2}{\delta}) \leqslant A$, Eq. (3) does not have solutions. Hence, the minimum value will occur for $T^* = \infty$ and the minimum value is $\lim_{T \to \infty} C_T(T, t_1) = \alpha R + \frac{\alpha}{\delta} C_2$. This completes the proof for Lemma 3. $\square$

Now, we state our main theorem.

Theorem 1

(a) If

$$\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) > A,$$

then $C_T(T^*, t_1^*) = iC\alpha t_1^*$ is the minimum value where t_1^* satisfies Eq. (7) and (T^*, t_1^*) satisfies Eq. (6).

(b) If

$$\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) \leqslant A,$$

then $C_T(T, t_1)$ has minimum value $\alpha R + \frac{\alpha}{\delta} C_2$, when $T \to \infty$.

Proof. From Lemma 1, we know that the minimum solution will occur at the interior point. Using Lemma 3, we obtain the minimum solution for two different cases, respectively.

When the set-up cost, the unit cost and the inventory carrying cost are relatively high such that $\frac{\alpha}{2iC}\left(R + \frac{C_2}{\delta}\right)^2 + \frac{\alpha}{\delta}\left(R + \frac{C_2}{\delta}\right) \leqslant A$, the best policy is to prolong the shortage period as long as possible to attain the minimum cost. This is a new discovery that deserves more consideration. $\square$

5. Numerical examples

In this section, we begin from Theorem 1 to illustrate different examples under the respective criterion. While examining the Padmanabhan and Vrat inventory model, we construct a precise judgment criterion and demonstrate using the practical inventory systems shown below.

Our first illustration is to satisfy Theorem 1(a) under its existence and uniqueness for the minimum solution. We quote the same numerical example of Padmanabhan and Vrat (1990) with the following characteristics:

$i = 0.3, \quad C = $ Rs. 10, $\quad A = $ Rs. 50,

$C_2 = $ Rs. 1, $\quad R = $ Rs. 2 and $\quad \alpha = 200$ units.

We consider the problem for $\delta = 0.5$, 1, 2 and 3, since Theorem 1 (a) applies and list the results in Table 1.

For the optimal maximum inventory level, $S_{opt} = \alpha t_1^*$, our numerical results are different from Padmanabhan and Vrat (1990). However, from our Eqs. (7) and (8), we can check our results. For example, let $g(t) = f(t) - A$, if $g(t_1^*) > 0$ then we show that $g'(t_1^* - 10^{-6}) < 0$ and if $g(t_1^*) < 0$ then $g'(t_1^* + 10^{-6}) > 0$. Hence, we can demonstrate that our t_1^* is very accurate.

In our second illustration, for $\delta = 3$, we change the values of the set up cost, $A = $ Rs. 200, unit cost, $C = $ Rs. 40 and inventory carrying cost fraction, $i = 0.5$ such that Theorem 1(b) applies. We know that the first order partial derivative system does not have solutions and the optimal value will be attained when $T \to \infty$. However, in the practical sense, we cannot use the optimal cycle time $T^* = \infty$. We need to choose a finite cycle time as large as possible to reduce the averaged total cost. For a fixed T, using Lemma 1, there is a point, say $t_1(T)$, satisfying Eq. (3). The results are listed in the following Table 2. The description in Table 2 is similar to the trade-off time and cost for the CPM (critical path method) in project management (Lee and Larry, 1999).

As Table 2 shows, the best solution for the total cost per unit time appears when the cycle time is $T \to \infty$. The model will degenerate into an inventory model with one cycle. When a company makes this decision, a longer stock shortage time would lower the total cost. However, in cases when a

Table 1
Effect of δ on t_1^*, S_{opt} and $C_T(T^*, t_1^*)$

δ	t_1^*	$g(t_1^*)$	$g'(t_1^* \pm 10^{-6})$	S_{opt}	$C_T(T^*, t_1^*)$
0.5	0.252941	2×10^{-4}	$g(t_1^* - 10^{-6}) = -2 \times 10^{-4}$	50.588	151.765
1	0.281076	-8×10^{-5}	$g(t_1^* + 10^{-6}) = 3 \times 10^{-4}$	56.215	168.646
2	0.313688	-1×10^{-4}	$g(t_1^* + 10^{-6}) = 2 \times 10^{-4}$	62.738	188.213
3	0.332510	-9×10^{-5}	$g(t_1^* + 10^{-6}) = 2 \times 10^{-4}$	66.502	199.506

Table 2
Effect of choosing T on $t_1(T)$ and $C_T(T, t_1(T))$

T	1	10	100	1000	∞
$t_1(T)$	0.108621	0.116667	0.116667	0.116667	0.116667
$C_T(T, t_1(T))$	494.746	468.389	466.839	466.684	466.667

back order occurs, the company will delay delivery to the customer. The stock shortage time cannot be extended infinitely. Customers might wait willingly or turn to other sellers. If the business has time competitive superiority, it will deliver quickly to its customers and win market share. Conversely, if the business has cost competitive superiority, it promotes the customers willingness to purchase and expands its market. Businesses should develop a variety of customer-oriented strategies to fit the customer's demands and create competitive time and cost trade-off. In that way, the company can create a maximum profit for its business.

6. Conclusion

Firms that maintain a backlog of orders as a normal business practice can be successful. To maintain capacity balance, the backlog must grow during high demand periods with delivery promised to customers at some future date. Under this situation, they are able to lose some orders. Managers consider goodwill and expected loss. One can foresee that a back order and lost sales mixed model will become more important.

This paper explored inventory models with a mixture of back orders and lost sales. This is the same as that discussed in Padmanabhan and Vrat (1990). Theorem 1 shows the criterion for the existence and uniqueness of the optimal solution. If the set-up cost, the unit cost and the inventory carrying cost are relatively high, so that the criterion is not satisfied, the inventory model will be reduced to one cycle to maintain the shortage period as long as possible. We evaluated the minimum total cost under the cycle time limit. The result is easy to implement, intuitive and provides managerial insights and a better understanding of the effect of the relations among different costs. In addition, we suggest an accurate formula for deriving the optimal solution parameters.

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