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Some Qualitative Properties of the Riemann Problem in Gas Dynamical Combustion

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We study the Chapman–Jouguet (CJ) model and the selfsimilar Zeldovich–von Neumann–Döring (SZND) model in chemically reacting gas flows. We discover some basic relationships among ignition temperature T_i, total chemical binding energy Q, and the adibatic exponent γ of polytropic gas. From these relations, we can determine when temperatures along the SZND burning solutions are higher than the ignition temperature T_i. We also study the all possible selfsimilar solutions for the SZND-model. From these results, we can determine when selfsimilar solutions for the CJ-model are the limits of selfsimilar solutions of the SZND model when the reaction rate tends to infinity. © 1997 Academic Press

1. INTRODUCTION

Two well-known mathematical models have been used frequently to study combustion phenomena in chemically reactive gas flows: the Chapman–Jouguet (CJ) model and the Zeldovich–von Neumann–Döring (ZND) model (see [2, 10]). In Lagrangian coordinates, the CJ-model is expressed as

$$
\text{(CJ)}\quad
\begin{cases}
u_t + p_x = 0, \\
\tau_t - u_x = 0, \\
E_t + (pu)_x = 0, \\
q(x,t) = 0 & \text{if} \quad \sup_{0 \leqslant s \leqslant t} T(x,s) > T_i; \\
\qquad\quad = q(x,0), & \text{otherwise,}
\end{cases}
$$

where u, p, τ, T, T_i, q, and E are respectively velocity, pressure, specific volume, temperature, ignition temperature, chemical binding energy, and specific total energy. More precisely, $E = (1/2)\,u^2 + e + q$, where e is the

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internal energy. For polytropic gases, the internal energy $e=e(T)=p\tau/(\gamma-1)$, where γ is the adiabatic exponent with $\gamma\in(1,5/3)$ for media occurring. Temperature T satisfies Boyle and Gay-Lussac's law, $p\tau=RT$. R is a constant that depends on the effective weight of particular gases. For simplicity, we assume R and γ remain unchanged during the reaction, and that $R=1$. The CJ-model is based on two physical assumptions:

(i) the reaction rate is infinitely large (i.e., the reaction zones are infinitely thin),

(ii) the effects of viscosity and heat conduction are negligible.

In the ZND-model, on the other hand, a finity but large reaction rate is assumed although the effects of viscosity and heat condution are still ignored. The ZND-model is expressed as

$$\text{(ZND)}\qquad \begin{cases} u_t+p_x=0, \\ \tau_t-u_x=0, \\ E_t+(pu)_x=0, \\ q_t=-k\varphi(T)q, \end{cases}$$

where

$$\varphi(T)=\begin{cases} 1 & \text{if} \quad T>T_i, \\ 0 & \text{if} \quad T\leqslant T_i, \end{cases} \tag{1.1}$$

and k is positive constant related to the reaction rate. It is natural to ask whether or not the CJ-model is a limit of the ZND-model as $k\to\infty$. The question is still unsolved due to the mathematical difficulty of obtaining the existence of global (in time) weak solutions for the ZND-model and then studying their asymptotic behavior as $k\to\infty$.

However, Ying and Terng [11] studied the following simplified scalar combustion model

$$\text{(A1)}\qquad \begin{cases} (u+Qz)_t+f(u)_x=0, \\ z_t=-k\varphi(u)z. \end{cases}$$

They were able to prove the existence and uniqueness of a solution for the Riemann problem. Furthermore, they proved the existence of limits on the solutions as $k\to\infty$ and found that the limit function is a solution of the Riemann problem for the corresponding scalar CJ-model. Later, Jäger, Yang and Zhang [4] studied the following selfsimilar scalar ZND-model

$$\text{(A2)}\qquad \begin{cases} (u+qz)_t+f(u)_x=0, \\ z_t=-\dfrac{k}{t}\varphi(u)z. \end{cases}$$

They proved that all selfsimilar solutions for the scalar CJ-model are the limits of the solutions of the Riemann problem stated in (A2). Based on these results, Tan and Zhang [8] then studied the following selfsimilar ZND (SZND) model

$$(\text{SZND}) \quad \begin{cases} u_t + p_x = 0, \\ \tau_t - u_x = 0, \\ E_t + (pu)_x = 0, \\ q_t = -\frac{k}{t}\varphi(t)q. \end{cases}$$

Previously, Courant and Friedrichs [2] proved that any combustional shock wave (deflagration and detonation) must satisfy Jouguet's rule. However, only quite recently has a complete solution that satisfies Jouguet's rule for the Riemann problem as it relates to the CJ-model been obtained by Zhang and Zheng [13]. The number of solutions may be (at most) nine for some initial data. Later, Tan and Zhang [8] proved that these selfsimilar solutions for the CJ-model are limits of the SZND solutions as $k \to \infty$ assuming the following:

(TZ-1) the selfsimilar solutions for the SZND-model are of a special type; (we call them simple types in this paper),

(TZ-2) temperatures along the SZND burning solutions are higher than the ignition temperature T_i.

Due to the discontinuity of φ at T_i, initial-value problems involving selfsimilar solutions for the SZND model may yield non-unique results at $T = T_i$. In this paper, we discuss this issue in detail and identify the solutions obtained in [8] as simple solutions.

We discovered the answers for assumption (TZ-2), lie in the relationships among ignition temperature T_i, total chemical binding energy Q, and the adiabatic exponent γ of polytropic gas. More precisely, it depends on the relation between T_i and $Q_*(\gamma) = ((1-9\mu^4)/2\mu^2)Q$ with $\mu^2 = (\gamma-1)/(\gamma+1)$. These are intrinsic properties of the CJ-model. We then divide all unburnt states into three classes: A, B and C. (TZ-2) is always true for the Jouguet diagrams of class A unburnt states (see Section 2 for further details). (TZ-2) is only partially true for the Jouguet diagrams associated with class B and C unburnt states. From these observations, we can determine when selfsimilar solutions for the CJ-model are the limits of simple solutions of the SZND-model.

Since many studies of combustion theory exist, we mention only those few that are closely related to our work.

Based on the work of Ying and Terng [11], Liu and Zhang [6] obtained a set of entropy conditions for the scalar CJ-model. This sect of entropy conditions consists of two parts—pointwise and global, and they were able to prove the existence and uniqueness of solutions. In [7], Majda studied the combustional profile of scalar combustion model with finite reaction rate and diffusion. Using singular perturbation methods, Wagner [9] and Gasser and Szmolyan [3] studied combustional problems involving low viscosity, heat induction and diffusion. In [1], Chen proved the existence of global generalized solutions to the compressible Navier–Stokes equation for a reacting mixture with discontinuous Arrhenius functions.

The paper is organized as follows. In Section 2, we study (CJ) and obtain some new properties of it, including some relationships between T_i, Q and γ. We then divide all unburnt states into three classes in order to study temperatures along the burning solution. In Section 3, we study (SZND) and establish the existence of global selfsimilar solutions. The solutions at $T = T_i$ are classified as simple and non-simple. The simple solutions are used to approximate (CJ) as $k \to \infty$. In Section 4, after improving the strength of the results obtained by Tan and Zhang [8], we derive a complete answer when the CJ-model is a limit of the SZND-model as $k \to \infty$.

2. SOME BASIC PROPERTIES OF THE CJ-MODEL

In this section we shall review some known properties of the CJ-model and provide some new results which are interesting in themselves and also useful in studying the SZND-model.

For a given burnt state $(u_0, p_0, \tau_0, 0)$, all states $(u, p, \tau, 0)$ that can be linked by shock (S) or rarefaction waves (R) to $(u_0, p_0, \tau_0, 0)$ are given by

$$R: \quad p\tau^\gamma = p_0 \tau_0^\gamma, \qquad 0 < p \leqslant p_0, \tag{2.1}$$

$$u = u_0 - \frac{1-\mu^4}{\mu^4} \tau_0^{1/2} p_0^{1/2\gamma} (p^{(\gamma-1)/2\gamma} - p_0^{(\gamma-1)/2\gamma}), \tag{2.2}$$

$$S: \quad (p + \mu^2 p_0)(\tau - \mu^2 \tau_0) = (1 - \mu^4) p_0 \tau_0, \qquad p > p_0, \tag{2.3}$$

$$u = u_0 - (p - p_0)\{(1 - \mu^2) \tau_0 / (p + \mu^2 p_0)\}^{1/2}, \tag{2.4}$$

where

$$\mu^2 = (\gamma - 1)/(\gamma + 1).$$

Here, Lax entropy conditions are assumed, see [5]. However, for a given unburnt state (u_0, p_0, τ_0, Q), in addition to the non-combustional shock waves (S) and rarefaction waves (R) (u, p, τ, Q) given (2.1) $\sim$ (2.4), we also have combustional shock waves: detonation waves (DT) and deflagration waves (DF) $(u, p, \tau, 0)$ that can be linked to (u_0, p_0, τ_0, Q) and lie on a Hugoniot curve:

$$(p+\mu^2 p_0)(\tau-\mu^2\tau_0)=(1-\mu^4)\,p_0\tau_0+2\mu^2 Q. \tag{2.5}$$

$\mathrm{DT}\equiv\mathrm{DT}((\tau_0, p_0))$ is the upper portion of the curve, i.e., $p \geqslant p_A$, where

$$p_A=p_0+\frac{\alpha^2}{(1-\mu^2)\,\tau_0} \tag{2.6}$$

and $\alpha=(2\mu^2 Q)^{1/2}$. $\mathrm{DF}\equiv\mathrm{DF}((\tau_0, p_0))$ is the lower portion of the curve, i.e., $p \leqslant p_0$. Furthermore, there is a unique Rayleigh line

$$-\eta_c^2=\frac{p-p_0}{\tau-\tau_0}, \qquad \eta_c>0, \tag{2.7}$$

which starts from (τ_0, p_0) and is tangent to DT at point (τ_c, p_c). (τ_c, p_c) is called the Chapman–Jouguet detonation point (CJDT) of state (τ_0, p_0).

Moreover, the CJDT point (τ_c, p_c) divides DT into two parts: strong detonation (SDT) for $p>p_c$ and weak detonation (WDT) for $p_A \leqslant p<p_c$. Similarly, there is a Chapman–Jouguet deflagration point (CJDF) (τ^c, p^c) on DF that divides DF into two parts: weak deflagration (WDF) for $p^c<p\leqslant p_0$ and strong deflagration (SDF) for $0<p<p^c$. For details, see [2, 8, 13] and Fig. 1.

For any $\eta \in [\eta_c, \infty)$, the associated Rayleigh line

$$-\eta^2=\frac{p-p_0}{\tau-\tau_0} \tag{2.8}$$

intersects SDT at $(\tau(\eta), p(\eta))$ and S at $(\tau^{\#}(\eta), p^{\#}(\eta))$ uniquely. $(\tau^{\#}(\eta), p^{\#}(\eta))$ is called the von Neumann point. They are very important in our study of the CJ-model and the explicit expressions of these points are given below.

The Riemann problem stated below has material in a burnt state on the left and material in an unburnt state on the right,

$$\text{burnt state}(-)\colon (u_-, p_-, \tau_-, 0) \qquad \text{for} \quad x<0,$$

and

$$\text{unburnt state}(+)\colon (u_+, p_+, \tau_+, Q) \qquad \text{for} \quad x>0,$$

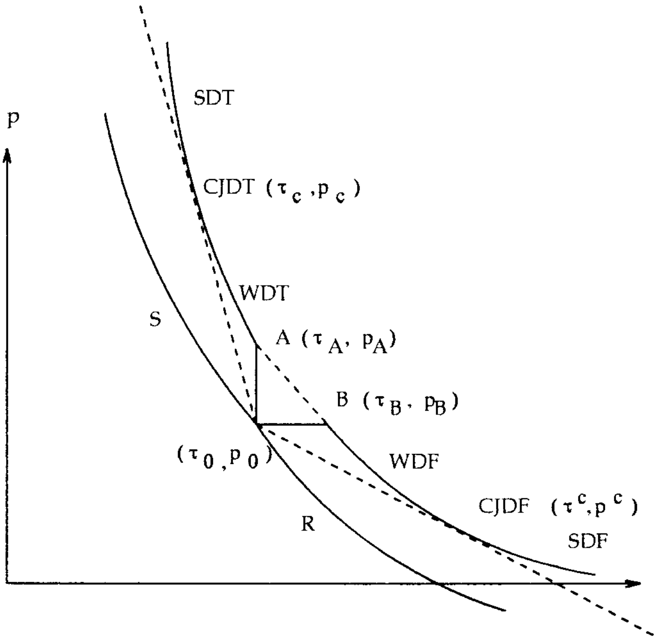


FIG. 1. Chapman–Jouguet's diagram.

with

$$T_- \equiv p_- \tau_- \geqslant T_i > T_+ \equiv p_+ \tau_+ .$$

The state $(-)$ of Hugoniot curve $S(-) \cup R(-)$ is given by (2.1) ~ (2.4) with u_0, p_0 and τ_0 being replaced by u_-, p_-, and τ_-, respectively.

Courant and Friedrichs [2] pointed out that the WDT and SDF are not stable for the unburnt state $(+)$. If we assume as in [8, 13] that the temperature at the front WDF bank is exactly at the ignition temperature T_i, then Jouguet's rule implies three different kinds of wave series can be linked to state $(+)$:

(i) $S(+)$ or $R(+)$ (with $q = Q$) (containing no combustion waves);

(ii) $(i) +$ WDF(i) or $(i) +$ CJDF$(i) + R$(CJDF(i)) (containing no DT waves); and

(iii) SDT$(+)$ or CJDT$(+) + R$(CJDT$(+)$).

Here $i \equiv i(+) \equiv (u_i, p_i, \tau_i, Q) \equiv (u_i(+), p_i(+), \tau_i(+), Q)$ is the state at $S(+)$ with ignition temperature T_i. (The symbol "+" between two states in (ii) and (iii) means "followed by".) Substituing states $(+)$ and (i) into (2.1) and (2.3), we obtain the following expressions.

$$R(+):\quad p\tau^{\gamma}=p_{+}\tau_{+}^{\gamma},\qquad p<p_{+}. \tag{2.9}$$

$$S(+):\quad (p+\mu^2 p_{+})(\tau-\mu^2\tau_{+})=(1-\mu^4)\,p_{+}\tau_{+},\quad p>p_{+}. \tag{2.10}$$

$$\mathrm{SDT}(+):\quad (p+\mu^2 p_{+})(\tau-\mu^2\tau_{+})=(1-\mu^4)\,p_{+}\tau_{+}+2\mu^2 Q,\qquad p>p_{c}. \tag{2.11}$$

$$\mathrm{WDF}(i):\quad (p+\mu^2 p_{i})(\tau-\mu^2\tau_{i})=(1-\mu^4)\,T_{i}+2\mu^2 Q,\qquad p_i^c<p\leqslant p_i. \tag{2.12}$$

$$R(\mathrm{CJDT}(+)):\quad p\tau^{\gamma}=p_{c}\tau_{c}^{\gamma},\qquad p\leqslant p_{c}. \tag{2.13}$$

$$R(\mathrm{CJDF}(i)):\quad p\tau^{\gamma}=p_{i}^{c}(\tau_{i}^{c})^{\gamma},\qquad p\leqslant p_{i}^{c}. \tag{2.14}$$

Hence the set of all possible states can be linked to state $(+)$ is

$$J(+)\equiv R(+)\cup S(+)\cup \mathrm{JDT}(+)\cup \mathrm{JDF}(+),$$

where

$$\mathrm{JDT}(+)\equiv \mathrm{SDT}(+)\cup R(\mathrm{CJDT}(+))$$

and

$$\mathrm{JDF}(+)\equiv \mathrm{WDF}(i)\cup R(\mathrm{CJDF}(i)).$$

$J(+)$ is called the Jouguet diagram just because it relates to Jouguet's rule [2, 8, 13]. A typical Jourguet diagram is given in Fig. 2.

From the above observations, there are at most three solutions for our Riemann problem, see [8, 13]. Our main task in this paper is to study which one of them can be a limit of solutions for the SZND-model as $k\to\infty$.

From now on, for simplicity, we shall omit dependence on $(+)$ wherever the omission does not cause any confusion. We first establish some explicit expressions for CJDT$(+)$ and CJDF$(+)$.

PROPOSITION 2.1. *The coordinates* (τ_c, p_c) *and* (τ^c, p^c) *of* CJDT$(+)$ *and* CJDF$(+)$ *are given by*

$$p_c=p_{+}+\frac{\alpha(\alpha+\beta)}{(1-\mu^2)\,\tau_{+}},\qquad \tau_c=\tau_{+}+\frac{\alpha(\alpha-\beta)}{(1+\mu^2)\,p_{+}}, \tag{2.15}$$

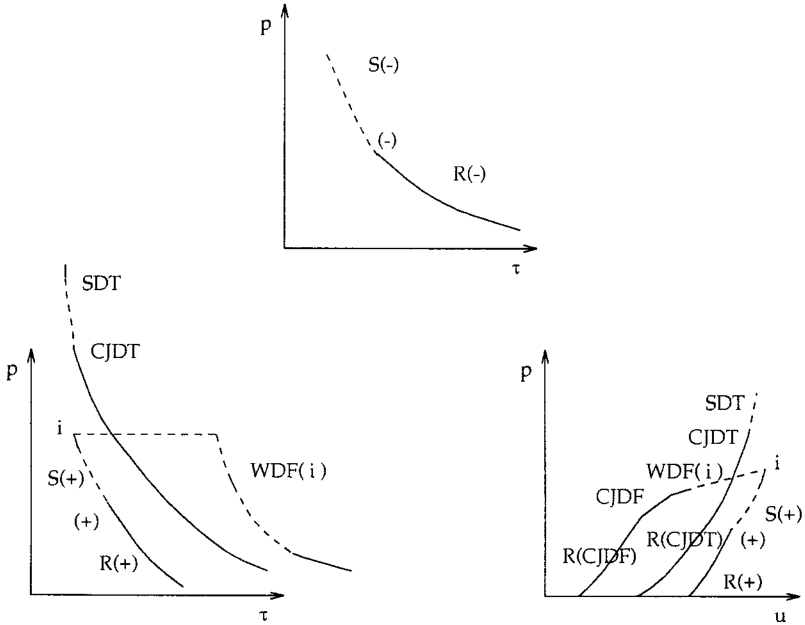


FIG. 2. Jouguet's diagrams.

and

$$p^c = p_+ + \frac{\alpha(\alpha - \beta)}{(1-\mu^2)\,\tau_+}, \qquad \tau^c = \tau_+ + \frac{\alpha(\alpha+\beta)}{(1+\mu^2)\,p_+}, \tag{2.16}$$

where

$$\alpha = (2\mu^2 Q)^{1/2} \qquad \textit{and} \qquad \beta = \beta(+) = \{2\mu^2 Q + (1-\mu^4)\,p_+\tau_+\}^{1/2}. \tag{2.17}$$

Furthermore,

$$T_c \equiv T_c(+) \equiv p_c \tau_c = p_+\tau_+ + \frac{2\alpha(\alpha+\mu^2\beta)}{1-\mu^4} - \alpha^2. \tag{2.18}$$

Proof. On SDT(+), taking $p = p(\tau)$, it is easy to verify that

$$\frac{dp}{d\tau} = -\frac{p+\mu^2 p_+}{\tau - \mu^2 \tau_+}. \tag{2.19}$$

Since CJDT(+) is the point of tangency between the Rayleigh line and the Hugoniot curve (2.11). (τ_c, p_c) satisfies

$$\frac{p_c - p_+}{\tau_c - \tau_+} = -\frac{p_c + \mu^2 p_+}{\tau_c - \mu^2 \tau_+}. \tag{2.20}$$

From (2.11) and (2.20), after a direct computation we can obtain

$$\tau_c = (1+\mu^2)\, p_c \tau_+ / 2p_c - (1-\mu^2)\, p_+$$

and

$$(1-\mu^2)\,\tau_+ p_c^2 + \{2(\mu^2-1)\, p_+ \tau_+ - 2\alpha^2\}\; p_c + (1-\mu^2)\, p_+ (p_+ \tau_+ + \alpha^2) = 0.$$

Then (2.15) follows from the last two equations.

We can obtain (2.16) in similar fashion, thus the details are omitted, and the proof is complete.

Next, it is useful to distinguish SDT(+) from WDT(+) by comparing η with $\gamma p/\tau$ as follows.

LEMMA 2.2. *If* $(\tau, p) \in \mathrm{DT}(+)$ *and* η *satisfies*

$$-\eta^2 = \frac{p - p_+}{\tau - \tau_+}, \tag{2.21}$$

then, on SDT(+), *we have* $\eta^2 < \gamma p/\tau$, *at* CJDT(+), *we have* $\eta^2 = \gamma p/\tau$, *and on* WDT(+), *we have* $\eta^2 > \gamma p/\tau$. *Similarly, if* $(\tau, p) \in \mathrm{DF}(+)$ *and* η *also satisfy* (2.21), *then on* WDF(i), *we have* $\eta^2 < \gamma p/\tau$, *at* CJDF(i), *we have* $\eta^2 = \gamma p/\tau$, *and on* SDF(i), *we have* $\eta^2 > \gamma p/\tau$.

Proof. On DT(+), taking $p = p(\tau)$ and $\eta = \eta(\tau)$ on (2.21), we may define

$$g(\tau) = \gamma p(\tau) - \eta^2(\tau)\,\tau$$

on $(\mu^2 \tau_+, \tau_+)$. Then it is easy to verify that

$$\frac{dg}{d\tau} = \left\{ \frac{(\tau - \tau_+) + (\tau - \mu^2 \tau_+)}{(1-\mu^2)(\tau_+ - \tau)} \right\} \frac{p + \mu^2 p_+}{\tau - \mu^2 \tau_+} - \tau_+ \frac{(p - p_+)}{(\tau - \tau_+)^2}$$

$$< \frac{p + \mu^2 p_+}{(1-\mu^2)(\tau - \mu^2 \tau_+)} - \tau_+ \frac{(p - p_+)}{(\tau - \tau_+)^2} = 0,$$

which is strictly negative in $(\mu^2\tau_+, \tau_+)$. Now, (2.20) and (2.21) implies

$$\eta_c^2 = \frac{\gamma p_c}{\tau_c},$$

i.e., $g(\tau_c) = 0$. Hence, the result follows for DT(+). We can obtain the results for DF(+) in similar fashion, and the proof is complete.

Explicit von Neumann points can also be computed as follows.

LEMMA 2.3. *For each* $\eta \in [\eta_c(+), \infty)$ *the von Neumann point* $(\tau^{\#}(\eta), p^{\#}(\eta))$ *is given by*

$$p^{\#}(\eta) = p(\eta) + \frac{\alpha^2}{\tau_+ - \tau(\eta)}, \tag{2.22}$$

and

$$\tau^{\#}(\eta) = \tau(\eta) + \frac{\alpha^2}{p_+ - p(\eta)}, \tag{2.23}$$

where $(\tau(\eta), p(\eta))$ *is the associated* SDT(+) *point. Furthermore, the temperature at the associated von Neumann point* $(\tau_c^{\#}, p_c^{\#})$ *with* CJDT *point* (τ_c, p_c) *is*

$$T_c^{\#}(+) \equiv p_c^{\#}\tau_c^{\#} = p_+\tau_+ + \frac{4\alpha(\alpha + \mu^2\beta)}{1 - \mu^4} - 4\alpha^2. \tag{2.24}$$

Proof. Since $(\tau^{\#}(\eta), p^{\#}(\eta)) \in S(+)$ and

$$-\eta^2 = \frac{p^{\#} - p_+}{\tau^{\#} - \tau_+}, \tag{2.25}$$

we can easily obtain

$$p^{\#} = p_+ - \eta^2(\tau^{\#} - \tau_+)$$

and

$$\eta^2(\tau^{\#})^2 - (1 + \mu^2)(\eta^2\tau_+ + p_+)\,\tau^{\#} + (1 + \mu^2)\,p_+\tau_+ + \mu^2\eta^2\tau_+^2 = 0.$$

From last two equations, we have

$$p^{\#}(\eta) = (1 - \mu^2)\,\eta^2\tau_+ - \mu^2 p_+, \tag{2.26}$$

and

$$\tau^{\#}(\eta) = (1+\mu^2)\frac{p_+}{\eta^2} + \mu^2 \tau_+. \tag{2.27}$$

Substituting (2.21) into (2.26) and (2.27), (2.22) and (2.23) then follow.

Finally, let $\eta = \eta_c(+)$. We then have

$$T_c^{\#}(+) = \left(p_c + \frac{\alpha^2}{\tau_+ - \tau_c}\right)\left(\tau_c + \frac{\alpha^2}{p_+ - p_c}\right). \tag{2.28}$$

(2.24) follows from (2.15) and (2.28). The proof is complete.

For convenience, we denote the set of all von Neumann points of state $(+)$ as

$$\mathrm{VN}(+) \equiv \{(\tau^{\#}(\eta), p^{\#}(\eta) \mid \eta \in [\eta_c(+), \infty)\}.$$

From the explicit expressions above, we can easily recover the following well-known result: SDT is composed of a shock wave followed by a weak deflagration wave.

PROPOSITION 2.4. *For $\eta \in [\eta_c, \infty)$, let $(\tau(\eta), p(\eta)) \in \mathrm{SDT}(+)$ and the associated von Neumann point $(\tau^{\#}(\eta), p^{\#}(\eta)) \in \mathrm{VN}(+)$. Then $(\tau(\eta), p(\eta)) \in \mathrm{WDF}((\tau^{\#}(\eta), p^{\#}(\eta))$.*

Proof. $(\tau^{\#}(\eta), p^{\#}(\eta))$ and $(\tau(\eta), p(\eta))$ satisfy (2.10) and (2.11) respectively, i.e., we have

$$(p^{\#} + \mu^2 p_+)(\tau^{\#} - \mu^2 \tau_+) = (1-\mu^4)\, p_+ \tau_+,$$

and

$$(p + \mu^2 p_+)(\tau - \mu^2 \tau_+) = (1-\mu^4)\, p_+ \tau_+ + 2\mu^2 Q.$$

On the other hand, both of them also lie on the same Rayleigh line with slope $-\eta^2$, i.e.,

$$\frac{p^{\#} - p_+}{\tau^{\#} - \tau_+} = \frac{p - p^{\#}}{\tau - \tau^{\#}} = -\eta^2.$$

By direct computation, we can obtain

$$(p + \mu^2 p^{\#})(\tau - \mu^2 \tau^{\#}) = (1-\mu^4)\, p^{\#} \tau^{\#} + 2\mu^2 Q.$$

Since $p < p^{\#}$ and $\tau > \tau^{\#}$, we have $(\tau, p) \in \mathrm{DF}((\tau^{\#}, p^{\#}))$. Furthermore, $\eta^2 < \gamma p/\tau$ and Lemma 2.2 imply $(\tau, p) \in \mathrm{WDF}((\tau^{\#}, p^{\#}))$. The proof is complete.

Usually, the Hugoniot curve SDF$(+)$ is parametrized by $\tau \in [\mu^2\tau_+, \tau_c)$. However, it can also be parametrized by $\eta \in [\eta_c(+), \infty)$ through (2.21). In particular, as in (2.22) and (2.23), the von Neumann set VN$(+)$ are already parametrized by $\eta \in [\eta_c(+), \infty)$. We can also use (2.21) to parametrize the von Neumann set VN$(+)$ by $\tau \in [\mu^2\tau_+, \tau_c)$ whenever we like. From now on, any quantity defined on SDT$(+)$ or VN$(+)$ can be seen either as a function of τ or η. For example, the temperature $T = p\tau$ can be seen either as $T(\tau) = p(\tau)\tau$ or $T(\eta)$.

It is known that temperatures along $S(+)$ or SDT$(+)$ strictly decrease as τ increases [12]. Indeed, we have:

LEMMA 2.5. *On $S(+)$ and* SDT$(+)$, *we have*

$$\frac{dT}{d\tau}(\tau) < 0. \tag{2.29}$$

Furthermore, T and $T^{\#}$ strictly increase in (η_c, ∞), with

$$T_c^{\#}(+) > T_+. \tag{2.30}$$

Proof. By (2.19), we have

$$\frac{dT}{d\tau} = -\mu^2 \left(\frac{p\tau_+ + p_+\tau}{\tau - \mu^2\tau_+} \right).$$

Hence, (2.29) follows. From (2.29) and $d\tau/d\eta < 0$, T and $T^{\#}$ strictly increase in (η_c, ∞). Finally, (2.30) follows by (2.28). The proof is complete.

DEFINITION 2.6. Given $Q > 0$ and $\gamma \in (1, 2)$, define

$$Q_* = Q_*(\gamma) = \frac{1 - 9\mu^4}{2\mu^2} Q. \tag{2.31}$$

The positive quantity Q_* plays a very important role in studying the temperatures T and $T^{\#}$ along SDT$(+)$ and VN$(+)$, respectively. Indeed, we have following theorem.

THEOREM 2.7. *Let Q and $T_+ > 0$ be given. Then,*

$$\text{(i)} \quad T_+ < Q_* \quad \textit{if and only if} \quad T_c^\#(+) < T_c(+), \tag{2.32}$$

$$\text{(ii)} \quad T_+ = Q_* \quad \textit{if and only if} \quad T_c^\#(+) = T_c(+), \tag{2.33}$$

$$\text{(iii)} \quad T_+ > Q_* \quad \textit{if and only if} \quad T_c^\#(+) > T_c(+). \tag{2.34}$$

Furthermore, in (i) *and* (ii),

$$T(\eta) > T^\#(\eta) \qquad \textit{on } (\eta_c(+), \infty).$$

In (iii), *there exists* $\hat{\eta}(+) \in (\eta_c(+), \infty)$ *such that*

$$T(\eta) > T^\#(\eta) \qquad \textit{in } (\hat{\eta}(+), \infty),$$

$$T(\hat{\eta}(+)) = T^\#(\hat{\eta}(+)),$$

and

$$T(\eta) < T^\#(\eta) \qquad \textit{in } (\eta_c(+), \hat{\eta}(+)).$$

Proof. By (2.18) and (2.24), $T_c^\#(+) < T_c(+)$ if and only if

$$\frac{4\alpha(\alpha + \mu^2\beta)}{1-\mu^4} - 4\alpha^2 < \frac{2\alpha(\alpha+\mu^2\beta)}{1-\mu^4} - \alpha^2.$$

By (2.17) and a straightforward computation, the last inequality is shown to be equivalent to $T_+ < Q_*$. Similarly, (ii) and (iii) hold. It remains to show that T and $T^\#$ intersect at most once in $(\eta_c(+), \infty)$. Using (2.22) and (2.23), we obtain

$$p\tau - p^\#\tau^\# = \frac{\alpha^2 G(p)}{Z(p)(p-p_+)(\tau_+ - \tau)},$$

where

$$Z(p) \equiv p + \mu^2 p_+ > 0,$$

and

$$G(p) \equiv (p - p_+)\{(1-2\mu^2)\, p - p_+\}\ \tau_+ + \alpha^2\{(1-\mu^2)\, p_+ - p\}$$

are defined on (p_A, ∞), and p_A is given in (2.6). Since G is quadratic in $(-\infty, \infty)$ and

$$G(p_A) = \frac{\alpha^2\mu^2}{1-\mu^2}\,(\mu^2 p_+ - p_A) < 0,$$

$p_c > p_A$ implies G has at most one zero in (p_c, ∞). Hence, the result follows, and the proof is complete.

When $p_+ = 0$ or $\tau_+ = 0$, (2.17) then implies

$$\alpha = \beta = (2\mu^2 Q)^{1/2}.$$

In this case, by (2.18) and (2.28), we have

$$T_c(0) \equiv T_c(p_+ \tau_+ = 0) = 2\gamma\mu^2 Q, \tag{2.35}$$

and

$$T_c^{\#}(0) \equiv T_c^{\#}(p_+ \tau_+ = 0) = \frac{8-\mu^2}{1-\mu^2} Q. \tag{2.36}$$

On the other hand, when $T_+ = Q_*$, then Theorem 2.7(ii) implies $T_c^{\#}(+) = T_c(+)$. In this case, it can be computed that $T_c^{\#}(+) = (1-\mu^4)/(2\mu^2) Q$, allowing us to denote

$$T_c(Q_*) = T_c^{\#}(p_+ \tau_+ = Q_*) = \frac{1-\mu^4}{2\mu^2} Q. \tag{2.37}$$

By using (2.32), (2.33) and (2.34) and a straightforward computation, we can obtain

PROPOSITION 2.8.

$$T_c^{\#}(0) < T_c(0) < Q_* < T_c(Q_*). \tag{2.38}$$

As is known from the previous work of Tan and Zhang [8], it is important to determine whether or not the temperatures T and $T^{\#}$ along SDT$(+)$ and VN$(+)$ are always higher than the ignition temperature T_i. We will show the answer depends on the value relationships among T_i, $T_c(0)$, and $T_c^{\#}(0)$.

To begin with we divide all unburnt states (τ_+, p_+) into three classes according only to the relationships among T_i, $T_c(+)$ and $T_c^{\#}(+)$ as follows.

DEFINITION 2.9. If $T_+ = p_+ \tau_+ < Q_*(\gamma)$, then the unburnt state (τ_+, p_+) belongs to

class A	if	$T_i \leqslant T_c^{\#}(+)$,
class B	if	$T_i \in (T_c^{\#}(+), T_c(+)]$,
class C	if	$T_c(+) < T_i$.

Similarly, if $T_+ > Q_*(\gamma)$, the unburnt state (τ_+, p_+) belongs to

$$\begin{array}{lll} \text{class A} & \text{if} & T_i \leqslant T_c(+), \\ \text{class B} & \text{if} & T_i \in (T_c(+), T_c^{\#}(+)], \\ \text{class C} & \text{if} & T_i(+) > T_c^{\#}(+). \end{array}$$

See Fig. 3, with T and $T^{\#}$ along SDT(+) and VN(+), respectively.

For the fixed ignition temperature T_i, denote the set of all unburnt states as

$$U \equiv U(i) \equiv \{(\tau_+, p_+): 0 \leqslant p_+ \tau_+ < T_i.$$

Now, we can have a complete classification of all unburnt states as follows.

THEOREM 2.10. (I) *If $T_i \leqslant Q_*(\gamma)$, then we have three casses*:

(i) *if $T_i \leqslant T_c^{\#}(0)$, then each state in U is of class A.*

(ii) *if $T_c^{\#}(0) < T_i \leqslant T_c(0)$, the equi-temperature curve $\Gamma_{AB} \equiv \Gamma_{AB}(i) \equiv \{(\tau_+, p_+) \in U : T_c^{\#}(+) = T_i\}$ divides U into two simply-connected sets, U_A and U_B. Each state in U_X is of class X for $X = A$ and B.*

(iii) *if $T_c(0) < T_i$, then there are two disjoint equi-temperature curves $\Gamma_{AB} \equiv \Gamma_{AB}(i) \equiv \{(\tau_+, p_+) \in U : T_c^{\#}(+) = T_i\}$ and $\Gamma_{BC} \equiv \Gamma_{BC}(i) \equiv \{(\tau_+, p_+) \in U : T_c(+) = T_i\}$. Γ_{AB} and Γ_{BC} divide U into three disjoint simply-connected sets U_A, U_B and U_C. Each state in U_X is of class X for $X = A$, B and C.*

Similarly, for (II) $T_i > Q_*(\gamma)$, *we have two cases*:

(i) *if $T_i < T_c(Q_*)$, then there exist two disjoint equi-temperature curves Γ_{AB} and Γ_{BC} defined as in* (I)(ii) *that divide U into three disjoint simply-connected sets U_A, U_B, U_C. Each state in U_X is of class X for $X = A$, B and C. Furthermore, the curve $\{(\tau_+, p_+) \in U : T_+ = Q_*\}$ lies in U_A.*

(ii) *if $T_i > T_c(Q_*)$, then there exist two disjoint equi-temperature curves $\tilde{\Gamma}_{AB} = \{(\tau_+, p_+) \in U : T_c(+) = T_i\}$ and $\tilde{\Gamma}_{BC} = \{(\tau_+, p_+) \in U : T_c^{\#}(+) = T_i\}$ also divide U into three disjoint simply-connected sets $U_A U_B$ and U_C. Each state in U_X is of class X for $X = A$, B and C.*

See Fig. 4.

Proof. (I) If $T_i \leqslant Q_*$ then any unburnt state (τ_+, p_+) satisfies $T_+ \leqslant Q_*$. Thus, by Theorem 2.7, we have

$$T_c^{\#}(+) \leqslant T_c(+) \tag{2.39}$$

(I) $T_+ < Q_*(\gamma)$

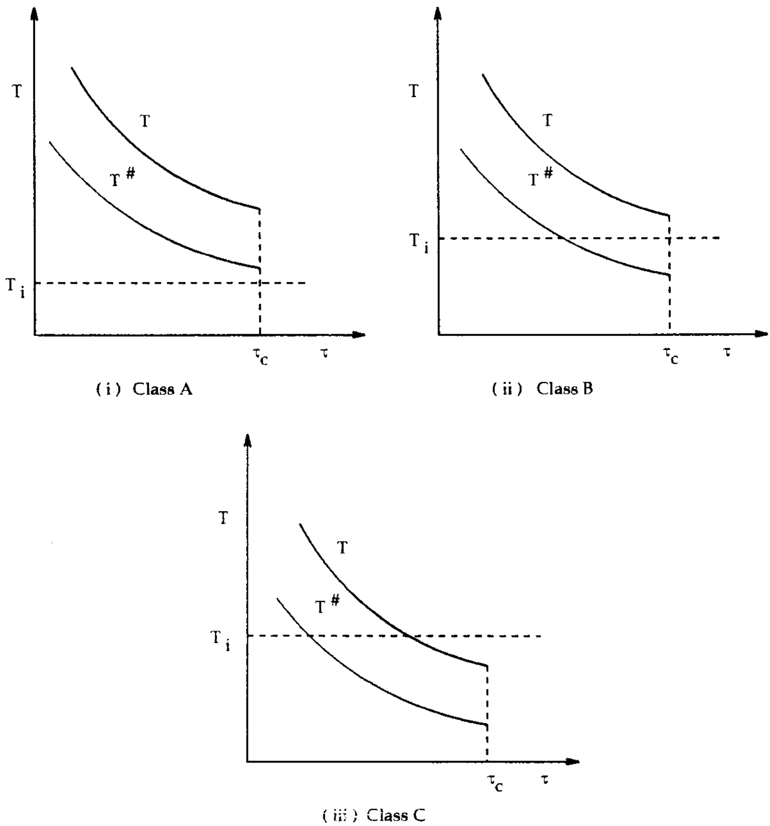


FIG. 3. Temperatures along SDT(+) and VN(+).

for each $(\tau_+, p_+) \in U$. On the other hand, (2.24) implies

$$T_c^{\#}(T_i) \equiv T_c^{\#}(p_+ \tau_+ = T_i) > T_i. \tag{2.40}$$

(i) If $T_c^{\#}(0) \geqslant T_i$, then $T_c^{\#}(+) > T_c^{\#}(0)$ implies $T_c^{\#}(+) > T_i$. Hence each state of U is of class A.

(ii) If $T_c^{\#}(0) < T_i \leqslant T_c(0)$, then the equi-temperature curve $\Gamma_{AB} = \{(\tau_+, p_+) \in U : T_c^{\#}(+) = T_i\}$ is non-empty. Furthermore, (2.40) implies $\Gamma_{AB} \subset U$. It is easy to verify that Γ_{AB} is an unbounded, continuous Jordan curve in U. Therefore, Γ_{AB} divides U into two disjoint open sets,

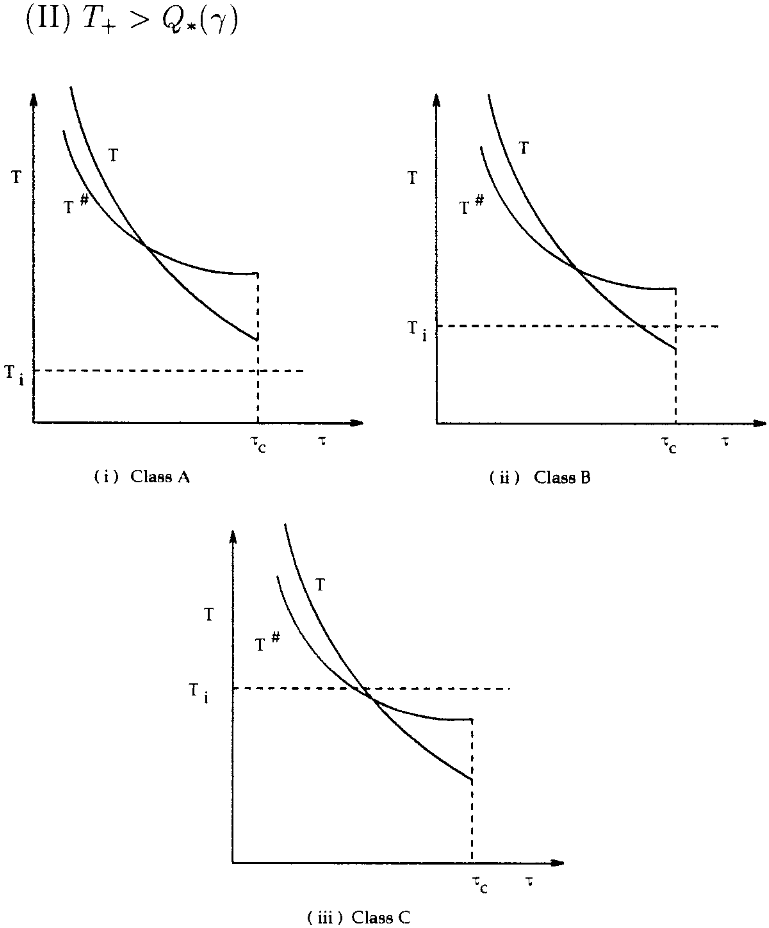


FIG. 3—*Continued*

U_A and U_B, such that $U_A = \{(\tau_+, p_+) \in U : T_c^\#(+) < T_i\}$ and $U_B = \{(\tau_+, p_+) \in U: T_c^\#(+) > T_i\}$. (2.39) now implies $T_c(+) > T_i$ too. Hence, each state in U_X is of class X for $X = A$ and B.

(iii) If $T_c(0) < T_i$, then Γ_{AB} and Γ_{BC} are non-empty in U. Now

$$U_A = \{(\tau_+, p_+) \in U : T_c^\#(i) > T_i\},$$

$$U_B = \{(\tau_+, p_+) \in U : T_c^\#(+) < T_i < T_c(+)\}, \text{ and}$$

$$U_C = \{(\tau_+, p_+) \in U : T_c(+) < T_i\}.$$

It is clear that each state in U_X is of class X for $X = A$, B and C.

(I) $T_i < Q_*(\gamma)$

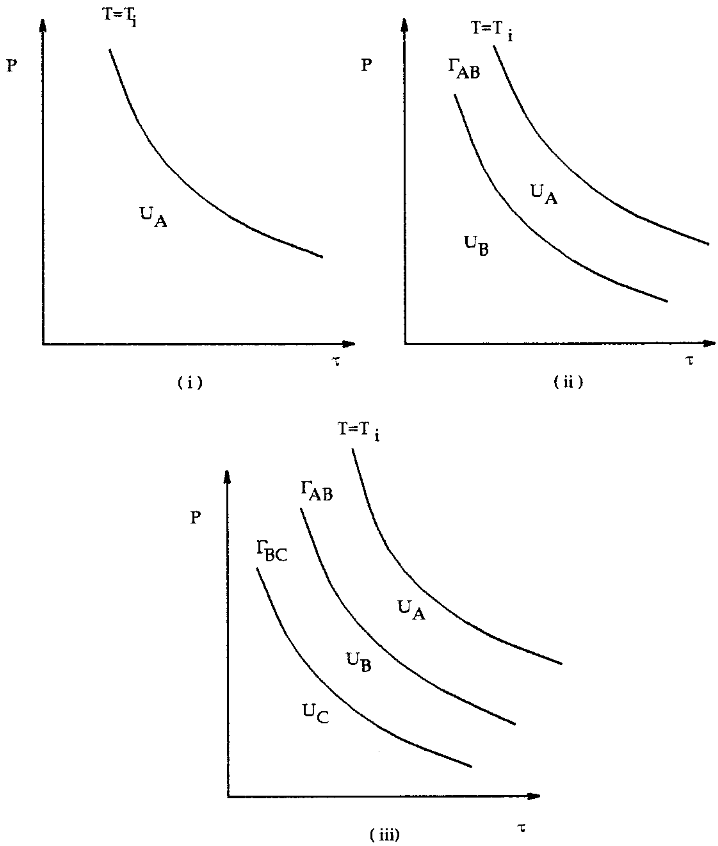


FIG. 4. Classification of unburnt states.

(II) If $Q_*(\gamma) < T_i$, then the curve $\Gamma_* = \{(\tau_+, p_+) \in U : T_+ = Q_*(\gamma)\} \subset U$.

(i) If $T_i \leqslant T_c(Q_*)$, then each state in

$$U^* = \{(\tau_+, p_+) \in U : T_+ > Q_*(\gamma)\}$$

is of class A by virtue of Theorems 2.7(iii) and (2.18). Hence, it remains to determine the states in

$$U_* = \{(\tau_+, p_+) \in U : T_+ < Q_*(\gamma)\}.$$

(II) $T_i > Q_*(\gamma)$

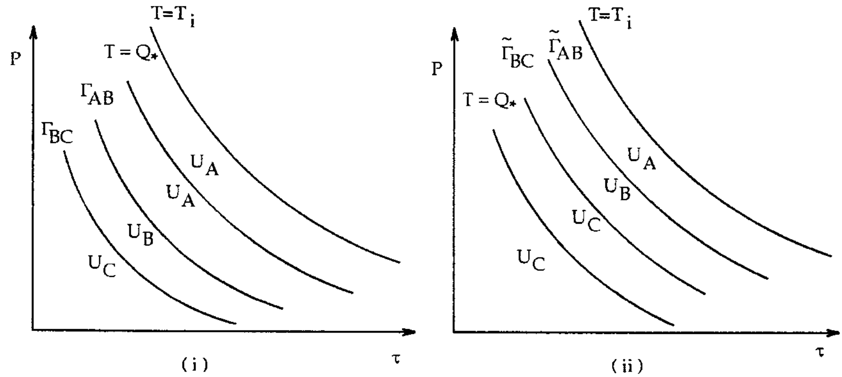


FIG. 4—*Continued*

Again by Theorem 2.7(i), each state in U_* satisfies

$$T_c^{\#}(+) < T_c(+).$$

Proposition 2.8 and $Q_* < T_i$ imply Γ_{AB} and Γ_{BC} are not empty in U_*. A similar argument as in (I) (iii) also holds in U_*. Hence, the result follows.

(ii) If $T_i > T_c(Q_*)$, then by Theorem 2.7(iii) and (2.38), each state in

$$U^* = \{(\tau_+, p_+) \in U : T_+ > Q_*\}$$

satisfies

$$T_c^{\#}(+) > T_c(+).$$

On $\Gamma_+ = \{(\tau_+, p_+) \in U : T_+ = Q_*\}$, we have $T_c^{\#}(+) = T_c(+) = T_c(Q_*) < T_i$ implying that there are two curves,

$$\tilde{\Gamma}_{AB} = \{(\tau_+, p_+): T_c(+) = T_i\}$$

and

$$\tilde{\Gamma}_{BC} = \{(\tau_+, p_+): T_c^{\#}(+) = T_i\}$$

in U^*, dividing U^* into three regions:

$$U_A = \{(\tau_+, p_+) \in U^* : T_c(+) > T_i\},$$
$$U_B = \{(\tau_+, p_+) \in U^* : T_c(+) < T_i < T_c^{\#}(+)\}, \text{ and}$$
$$\tilde{U}_C = \{(\tau_+, p_+) \in U^* : T_c^{\#}(+) < T_i\}.$$

In each set, the state is of the indicated class. Finally, in

$$U_* = \{(\tau_+, p_+) \in U : T_+ < Q_*\},$$

each state has $T_c^{\#}(+) < T_c(+)$ and $T_c(+) < T_c(Q_*) < T_i$. Hence each state in U_* is of class C. The results hold by letting

$$U_C = U_* \cup \tilde{U}_C \cup \Gamma_*.$$

The proof is complete.

For deflagration waves, we also have results similar to those from Theorem 2.7.

DEFINITION 2.11. Given $Q > 0$ and $\gamma \in (1, 2)$, denote

$$Q^* \equiv Q^*(\gamma) = \frac{1-\mu^4}{2\mu^2} Q.$$

The positive quantity Q^* plays a role similar to that of Q_* for detonation waves. Indeed, we have the following theorem.

THEOREM 2.12. *Let Q and T_i be given. Then,*

$$\text{(i)} \quad T(\mathrm{CJDF}(i(+))) > T_i \quad \textit{if and only if} \quad T_i < Q^*. \tag{2.41}$$

$$\text{(ii)} \quad T(\mathrm{CJDF}(i(+))) = T_i \quad \textit{if and only if} \quad T_i = Q^*. \tag{2.42}$$

$$\text{(iii)} \quad T(\mathrm{CJDF}(i(+))) < T_i \quad \textit{if and only if} \quad T_i > Q^*. \tag{2.43}$$

See Fig. 5.

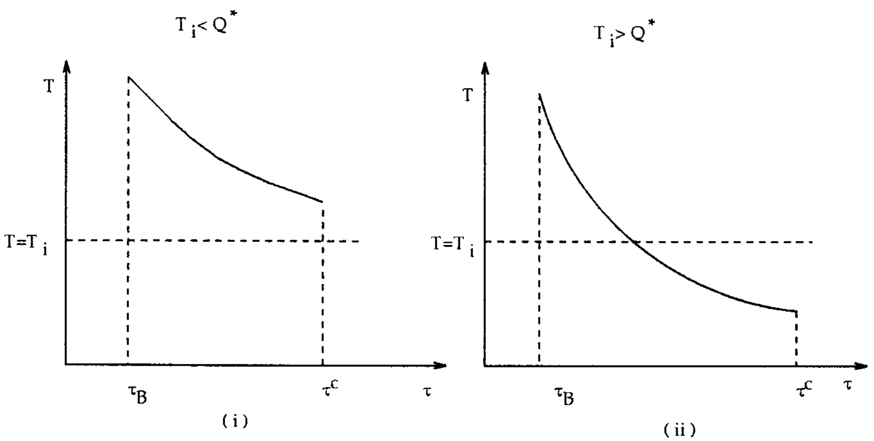


FIG. 5. Temperature along WDF.

Proof. By Proposition 2.1, we have

$$T(\mathrm{CJDF}(i(+)) - T_i = \alpha(2\alpha - 2\mu^2\beta)/(1-\mu^4) - \alpha^2. \tag{2.44}$$

Thus the sign of $T(\mathrm{CJDF}(i(+))) - T_i$ is the same as the sign of $Q^* - T_i$. The proof is complete.

3. SELFSIMILAR SOLUTIONS OF SZND MODEL

In this section, we study the selfsimilar solutions for the SZND-model. Indeed, let $\xi = x/t$, if solution (u, p, τ, q) for (SZND) depends only on ξ, then it satisfies the following equations:

$$\xi u' - p' = 0, \tag{3.1}$$

$$\xi\tau' + u' = 0, \tag{3.2}$$

$$\xi E' - (up)' = 0, \tag{3.3}$$

$$\xi q' = k\varphi(T) q, \tag{3.4}$$

where

$$\varphi(T) = \begin{cases} 0, & T \leqslant T_i, \\ 1, & T > T_i. \end{cases} \tag{3.5}$$

The Riemann data becomes

$$(u, p, \tau, q)(-\infty) = (u_-, p_-, \tau_-, 0), \tag{3.6}$$

and

$$(u, p, \tau, q)(+\infty) = (u_+, p_+, \tau_+, Q). \tag{3.7}$$

When $\xi \neq 0$ and the solution is smooth, then (3.1) $\sim$ (3.4) can also be expressed as

$$\tau' = -\frac{\gamma - 1}{\gamma p - \xi^2 \tau} q', \tag{3.8}$$

$$p' = -\xi^2 \tau', \tag{3.9}$$

$$q' = \frac{k}{\xi} \varphi(T) q, \tag{3.10}$$

where $\gamma p - \xi^2 \tau \neq 0$. If $\gamma p - \xi^2 \tau = 0$, then we require $q' = 0$. For any $\xi_0 \in \mathbf{R}^1$, we may supply (3.8) ~ (3.10) with the initial conditions

$$\tau(\xi_0) = \tau_0 > 0, \qquad p(\xi_0) = p_0 > 0, \qquad q(\xi_0) = q_0 \geqslant 0. \tag{3.11}$$

When solution $(u, p, \tau, q)(\xi)$ for the SZND-model has a discontinuity at $\xi = \eta$, it then satisfies the Rankine–Hugoniot condition,

$$\text{(RH):} \quad \eta[u] = [p], \tag{3.12}$$

$$\eta[\tau] = -[u], \tag{3.13}$$

$$[e+q] + \frac{p_r + p_l}{2}[\tau] = 0, \tag{3.14}$$

where $[w] = w_r - w_l$, $w_r = w(\eta + 0)$, and $w_l = w(\eta - 0)$.

By direct computation, we can determine (u_l, p_l, τ_l) in terms of (u_r, p_r, τ_r), η and $[q]$ as follows.

$$p_l = \frac{1}{2}\{(1-\mu^2)(\eta^2\tau_r + p_r)\} + \frac{1}{2}\{((1-\mu^2)\,\eta^2\tau_r - (1+\mu^2)\,p_r)^2 + 8\mu^2\eta^2[q]\}^{1/2}, \tag{3.15}$$

$$\tau_l = \tau_r + (p_r - p_l)/\eta^2, \tag{3.16}$$

$$u_l = u_r + \frac{1}{\eta}(p_l - p_r). \tag{3.17}$$

When $[q] = 0$, (3.15) ~ (3.17) can be simplified to

$$p_l = (1-\mu^2)\,\eta^2\tau_r - \mu^2 p_r, \tag{3.18}$$

$$\tau_l = (1+\mu^2)\,p_r/\eta^2 + \mu^2\tau_r, \tag{3.19}$$

$$u_l = u_r + (1-\mu^2)\,\eta\tau_r - (1+\mu^2)\,p_r/\eta. \tag{3.20}$$

Since state $(-)$ is burnt, we may consider the solution (u, p, τ, q) is also burnt in $(-\infty, 0)$, i.e. $q(\xi) = 0$ for any $\xi \in (-\infty, 0]$. Then (3.4) is automatically satisfied, and (3.1) ~ (3.3) is the classic, non-combustion equation. Therefore, for any data at $\xi = 0$,

$$\tau(0) = \tau_0 > 0, \qquad p(0) = p_0 > 0, \qquad u(0) = u_0, \qquad q(0) = 0. \tag{3.21}$$

Then (3.1) ~ (3.3), (3.6) and (3.21) yield unique solutions, (see Chapter 3 in [12]). Therefore, the (SZND) Riemann problem is reduced to finding suitable initial data $(u_0, p_0, \tau_0, 0)$ at $\xi = 0$ such that solutions for (3.1) ~ (3.4), (3.7) and (3.21) exist in $(0, \infty)$. Hence, it is necessary to study the initial-value problem (3.8) $\sim$ (3.11) when $\xi_0 \geqslant 0$. We first study the existence of global smooth (continuous) solutions in (ξ_0, ∞), and then study solutions with discontinuities at some $\eta > \xi_0$. The solutions for (3.1) $\sim$ (3.4) depend on $k(>0)$. For simplicity we shall omit the dependence of k wherever such omission does not cause confusion.

We first have the following simple observation.

LEMMA 3.1. *Let $q(\xi)$ be the solution of* (3.10) *in (ξ_0, ξ_1) with $q(\xi_0) = q_0 \geqslant 0$. Then*:

(i) *If $T > T_i$ in $(\xi_0, \xi_1]$ and $\eta \in (\xi_0, \xi_1]$, then*

$$q(\xi) = q(\eta)\left(\frac{\xi}{\eta}\right)^k \qquad \text{in } (\xi_0, \xi_1]. \tag{3.22}$$

(ii) *If $T \leqslant T_i$ in (ξ_0, ξ_1), then*

$$q(\xi) = q_0 \qquad \text{in } (\xi_0, \xi_1). \tag{3.23}$$

However, if $T(\xi_0) = T_i$, then we may have a non-unique solution since φ is discontinuous at $T = T_i$. This is clarified below.

For more efficient study of (3.8)–(3.11), it is convenient to divide $(\mathbf{R}^+)^3$ into different regions, then study the equations for each region separately.

DEFINITION 3.2. On $(\mathbf{R}^+)^3$, denote as

$$\begin{aligned} G &= \{(\tau, p, \xi) : \gamma p - \xi^2\tau > 0\}, & I^0 &= \{(\tau, p, \xi) : p\tau = T_i\}, \\ H &= \{(\tau, p, \xi) : \gamma p - \xi^2\tau < 0\}, & I^+ &= \{(\tau, p, \xi) : p\tau > T_i\}, \\ P &= \{(\tau, p, \xi) : \gamma p - \xi^2\tau = 0\}, & I^- &= \{(\tau, p, \xi) : p\tau < T_i\}, \end{aligned}$$
$$G^+ = G \cap I^+, \text{ etc...}$$

Therefore, $(\mathbf{R}^+)^3$ consists of 4 open regions, G^+, G^-, H^+, H^-, 4 surfaces, P^+, P^-, G^0, H^0 and one curve, P^0.

For simplicity, denote the solution of (3.8) $\sim$ (3.10) as

$$X(\xi) = (\tau(\xi), p(\xi), \xi), \tag{3.24}$$

with initial condition

$$X(\xi_0) = X_0 \equiv (\tau_0, p_0, \xi_0). \tag{3.25}$$

We have the following demonstration of solution existence and uniqueness for all regions except G^0 and P^0.

PROPOSITION 3.3. *Assume $k > 2$. We then have*:

(I) (i) *If $X_0 \in H^-$, then $X(\xi) = X_0$ in (ξ_0, ∞).*

(ii) *If $X_0 \in H^0$, then $X(\xi) = X_0$ in (ξ_0, ∞).*

(iii) *If $X_0 \in H^+$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) \in H^+$ in $(\xi_0, \xi_1]$ and $X(\xi_1) \in H^0$.*

(II) (i) *If $X_0 \in P^-$, then there exists a $\xi_1 > \xi_0$ such that*

$$p(\xi) = C_1 \xi^{(2\gamma)/(\gamma+1)} \qquad \text{and} \qquad \tau(\xi) = C_2 \xi^{(-2)/(\gamma+1)} \tag{3.26}$$

for $\xi \in (\xi_0, \xi_1]$ and $X(\xi_1) \in P^0$, for some positive constants C_1 and C_2.

(ii) *If $X_0 \in P^+$ and $q_0 > 0$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) \in H^+$ in (ξ_0, ξ_1).*

(III) (i) *If $X_0 \in G^-$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) = X_0$ in $(\xi_0, \xi_1]$ and $X(\xi_1) \in P^-$.*

(ii) *If $X_0 \in G^+$, then there exists a $\xi_1 > \xi_0$ such that $X(\xi) \in G^+$ in (ξ_0, ξ_1) and $X(\xi_1) \in G^0$.*

Proof. (I) (i) Since $X_0 \in H^-$, $T(\xi_0) < T_i$ and $\gamma p_0 - \xi_0^2 \tau_0 < 0$, (3.10) and (3.8) imply $q' = 0$ and $\tau' = 0$ for $\xi > \xi_0$. Hence, $X(\xi) = X_0$ for $\xi > \xi_0$.

(ii) The proof is similar to that for (i).

(iii) When $\gamma p - \xi^2 \tau \neq 0$, we then have

$$\frac{d}{d\xi} T(\xi) = -\frac{(\gamma-1)k}{\xi} \varphi(T) \frac{p - \xi^2 \tau}{\gamma p - \xi^2 \tau} \cdot q. \tag{3.27}$$

If $X(\xi) \in H^+$, then (3.27) implies $T'(\xi) < 0$. Furthermore, we also have

$$\frac{d}{d\xi} (\gamma p - \xi^2 \tau) = -2\xi\tau - (\gamma+1)\, \xi^2 \tau' < 0$$

in H^+. Therefore, $\xi_1 > \xi_0$ exists such that $T(\xi_1) = T_i$ and $X(\xi) \in H^+$ in (ξ_0, ξ_1).

(II) (i) If $X(\xi) \in P^-$, we then have

$$\gamma p - \xi^2 \tau = 0. \tag{3.28}$$

For polytropic gas we also have

$$p\tau^\gamma = C > 0. \tag{3.29}$$

From (3.28) and (3.29), (3.26) follows. Since

$$T(\xi) = C_1 C_2 \xi^{2((\gamma-1)/(\gamma+1))}, \tag{3.30}$$

$T(\xi_1) = T_i$ for some $\xi_1 > \xi_0$. Hence, $X(\xi_1) \in P^0$ and $X(\xi) \in P^-$ in (ξ_0, ξ_1).

(ii) Since $X_0 \in P^+$, $\xi_0 > 0$. If there exists $\xi_1 > \xi_0$ such that $X(\xi) \in P^+$ in (ξ_0, ξ_1), i.e., (3.28) holds, then (3.30) implies $T(\xi) > T_i$. $q_0 > 0$ now implies $q' > 0$ in (ξ_0, ξ_1), a contradiction, therefore the result holds.

(III) (i) Since $X_0 \in G^-$, $\varphi(T) = 0$, thus $X(\xi) = X_0$ in $(\xi_0, \xi_1]$, where $\gamma p_0 - \xi_1^2 \tau_0 = 0$.

(ii) If $X(\xi) \in G^+$, then $\tau' < 0$. From (4.11) on p. 344 in [8], we have

$$\tfrac{1}{2}(\gamma+1)\,\xi^2\tau^2(\xi) - \gamma\left\{p_0 + 2\int_{\xi_0}^{\xi} \tau(s)\,ds\right\}\tau(\xi)$$
$$+\gamma p_0 \tau_0 + (\gamma-1)\int_{\xi_0}^{\xi} s\tau^2(s)\,ds - q_0(\gamma-1)(\xi/\xi_0)^k = 0. \tag{3.31}$$

We may assume $q_0 > 0$ in (3.31). Otherwise, we replace (τ_0, p_0, q_0) and ξ_0 with $(\tau(\hat{\xi}), p(\hat{\xi}), q(\hat{\xi}))$ and $\hat{\xi}$ in (3.31) for some $\hat{\xi} > \xi_0$, and closed to it with $q(\hat{\xi}) > 0$. Since $k > 2$, it can be verified by using (3.31) that $X(\xi)$ can not stay at G^+ forever. Thus, there exists $\xi_1 > \xi_0$ such that $X(\xi) \in G^+$ in (ξ_0, ξ_1), and either $X(\xi_1) \in G^0$, or $X(\xi_1) \in P^+$. The latter case can be ruled out according to Theorem 3.1 [8]. The proof is complete.

Now it remains to study the problem of $X_0 \in P^0$ or G^0.

PROPOSITION 3.4. *If $X_0 \in P^0$, then either*

(i) *$X(\xi) = X_0$ for $\xi > \xi_0$, and so, $X(\xi)$ stays at H^0 forever, or*

(ii) *$X(\xi) \in G^+$ for $\xi \in (\xi_0, \xi_1)$ some $\xi_1 > \xi_0$. In this case, we have*

$$\tau(\xi) = \tau_0 - C_0(\xi-\xi_0)^{1/2} + o((\xi-\xi)^{1/2}), \tag{3.32}$$

$$p(\xi) = p_0 + C_0\xi_0^2(\xi-\xi_0)^{1/2} + o((\xi-\xi_0)^{1/2}), \tag{3.33}$$

$$T(\xi) = T_i + C_0\,|p_0 - \xi_0^2\tau_0|\,(\xi-\xi_0)^{1/2} + o((\xi-\xi_0)^{1/2}), \tag{3.34}$$

for $\xi \sim \xi_0^+$, where

$$C_0 = \{2(\gamma-1)\,kq_0/(\gamma+1)\,\xi_0^3\}^{1/2}.$$

Proof. It can be verified that (3.32) and (3.33) hold for $\xi \sim \xi_0^+$. $X(\xi) \in G^+$ is equivalent to $T(\xi) > T_i$ which is guaranteed by (3.34). The proof is complete.

Similarly, we have a result for $X_0 \in G^0$.

PROPOSITION 3.5. *If* $X_0 \in G^0$, *then either*

(i) $X(\xi) = X_0$ *in* (ξ_0, ξ_1) *with* $X(\xi_1) \in P^0$, *or*

(ii) $X(\xi) \in G^+$ *in* (ξ_0, ξ_1) *for some* $\xi_1 > \xi_0$, *and* (3.32) $\sim$ (3.34) *hold.*

Proof. If (i) does not hold, then by an argument similar to that in Proposition 3.2(III)(ii) we can prove (ii), the details are omitted. The proof is complete.

The case of $\xi_0 = 0$ is the most interesting to us, since we have to solve the Riemann problem in $(0, \infty)$. When $\xi_0 = 0$, we always assume (3.21), and then $X_0 \in G^+ \cup G^0$. If $X(\xi) \in G^+$, for $\xi > 0$, then (3.10) implies

$$q(\xi) = \tilde{q}_0 \xi^k \tag{3.35}$$

for $\xi \sim 0^+$, where $\tilde{q}_0 \geqslant 0$ is a parameter. In view of (3.35), we always have the freedom to chose $\tilde{q}_0 \in [0, \infty)$ and it may be possible to find an appropriate $\tilde{q}_0$ to fit the boundary conditions at some point $\eta(\leqslant \infty)$ when it is needed. This is stated more precisely below. The following proposition is very important in studying temperature of $X(\xi)$ at G^+, which was essentially proven in [8].

PROPOSITION 3.6. *If* $\xi_0 > 0$, $X(\xi_0) \in G^+$ *and*

$$T'(\xi_0) = 0, \tag{3.36}$$

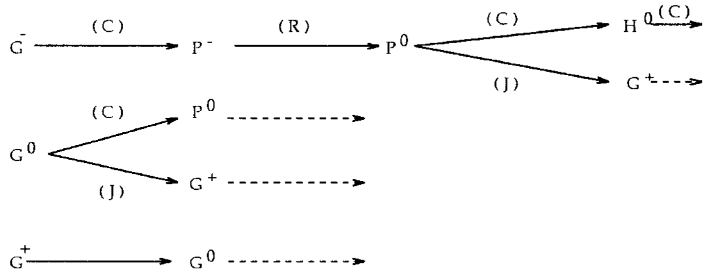
then there exists $\xi_1 > \xi_0$ *such that* $X(\xi) \in G^+$

$$T'(\xi) < 0 \qquad \text{in } (\xi_0, \xi_1) \qquad \text{and} \qquad T(\xi_1) = T_i, \tag{3.37}$$

i.e., $X(\xi_1) \in G^0$.

Proof. By Theorem 3.2 in [8], there is no $\tilde{\xi} > \xi_0$ such that $X(\xi) \in G^+$ and $T'(\tilde{\xi}) = 0$. Therefore, $T'(\xi) < 0$ as far as $X(\xi) \in G^+$. Now, according to Proposition 3.2 (III)(ii), there exists $\xi_1 > \xi_0$ such that $T(\xi_1) = T_i$. The proof is complete.

Remark 3.7. If $\xi_0 = 0$ and $X_0 \in G^+$, then $T'(0) = 0$, which does not contradict Proposition 3.5. Furthermore, according to Proposition 3.5, the temperature $T(\xi)$ either strictly decreases or has exactly one maximum in $(0, \xi_1)$ with $X(\xi_1) \in G^0$, where ξ_1 is the first ξ such that $T(\xi) = T_i$. Summarizing the above results, we have the following diagram for constructing the solutions starting at $\xi_0 \geqslant 0$, see Fig. 6.



Notation: $\rightarrow$ followed by, (C) remains constant, (J) jumps to G^+, (R) rarefaction wave (3.29). $--\rightarrow$ repeat an appropriate route.

FIG. 6. Routes in different regions.

Combining the results of Propositions 3.3, 3.4 and 3.5, we have the following global existence of solutions for (3.8) $\sim$ (3.10).

THEOREM 3.8. *Given* $\tau(0) > 0$, $p(0) > 0$ *and* $q(0) = 0$, *then there is a smooth* (*continuous*) *solution* $X(\xi)$ *for* (3.8) $\sim$ (3.10) *in* $(0, \infty)$. *Furthermore, there is* $\bar{\xi} \leqslant \infty$ *such that* $q(\xi) \leqslant Q$ *in* $(0, \bar{\xi})$ *and* $q(\bar{\xi}) = Q$ *if* $\bar{\xi} < \infty$.

Due to the non-uniqueness of solutions when $X_0 \in P^0 \cup G^0$, we shall pay much more attention to the following simplest solutions which were considered in [8].

DEFINITION 3.9. A solution $X(\xi)$ for (3.8) $\sim$ (3.10) is called simple

(i) if $X(0) \in G^0 \cup G^+$, then $X(\xi)$ can not jump to G^+ in $(0, \bar{\xi})$,

(ii) if $X(0) \in G^-$, then $X(\xi)$ jumps exactly once in $(0, \bar{\xi})$.

Otherwise, $X(\xi)$ is called a non-simple solution.

Therefore, we have four types of simple solutions when $q(\xi) \leqslant Q$; see Fig. 7. From (3.10), because $q(\xi)$ never decreases, $X(\xi)$ is a physical solution of (SZND), and it is necessary that $q(\xi) \leqslant Q$. It is possible to get a non-simple solution while $q \leqslant Q$.

For type (i) solutions, in [8] Tan and Zhang considered $q(\eta_0) = Q$ for some $\eta_0 > 0$. (3.8) $\sim$ (3.10) are then equivalent to

$$\tau' = -\frac{k(\gamma - 1)}{\gamma p - \xi^2 \tau} \left(\frac{\xi}{\eta_0}\right)^{k-1} \frac{1}{\eta_0}, \tag{3.38}$$

$$p' = -\xi^2 \tau', \tag{3.39}$$

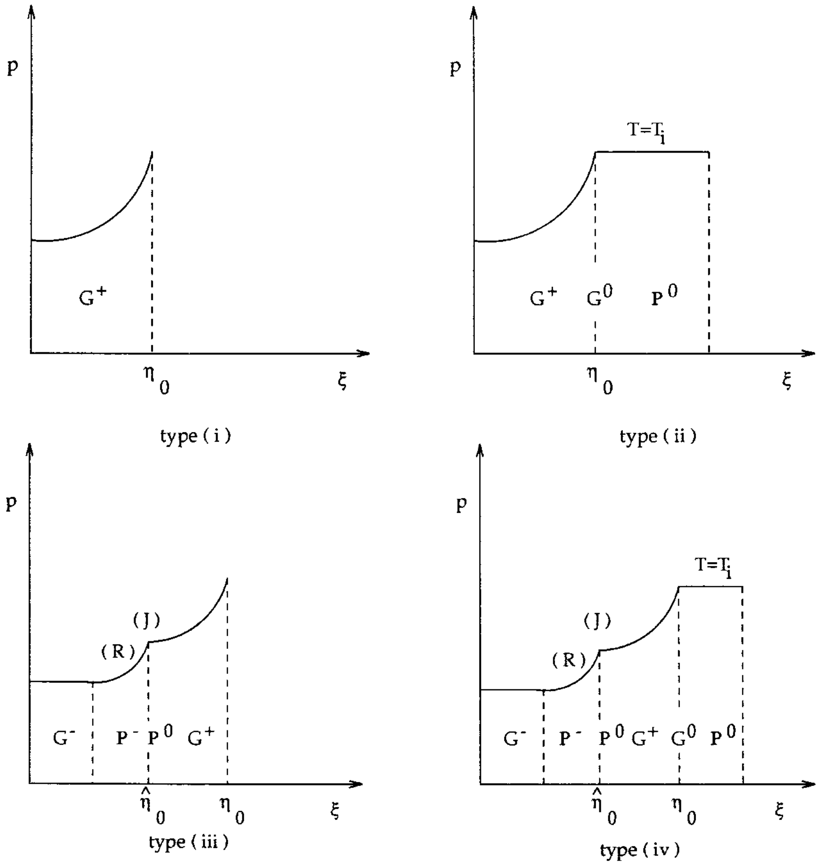


FIG. 7. Typical diagrams for simple solutions.

where

$$q(\xi) = Q\left(\frac{\xi}{\eta_0}\right)^k. \tag{3.40}$$

Based on (3.38) $\sim$ (3.40), Tan and Zhang were able to prove their results, the details are presented in the next section.

4. LIMITS OF SZND

In this section, we shall study the limits of selfsimilar simple solutions of (SZND) and the locate the states in $J(+)$ which can be limits of (SZND) as $k \to \infty$. The methods of proof are similar to those used in [8], here we only sketch a necessary modification and omit other details.

Let

$$\tau_k(0) = \tau_{0k} > 0, \qquad p_k(0) = p_{0k} > 0, \qquad q_k(0) = 0, \tag{4.1}$$

and $X_k(\xi) = (\tau_k(\xi), p_k(\xi), \xi)$ be the solutions of (3.8) ~ (3.10). Let $\eta_k < \infty$ such that

$$q_k(\eta_k) = Q. \tag{4.2}$$

If $X_k(\xi) \in G^+$ in $(0, \eta_k)$, i.e.,

$$T_k(\xi) > T_i \qquad \text{in } (0, \eta_k), \tag{4.3}$$

Then

$$q_k(\xi) = Q\left(\frac{\xi}{\eta_k}\right)^k, \tag{4.4}$$

and $X_k(\xi)$ satisfies

$$\tau_k'(\xi) = -\frac{kQ(\gamma-1)}{\gamma p_k - \xi^2 \tau_k}\left(\frac{\xi}{\eta_k}\right)^{k-1}\frac{1}{\eta_k}, \tag{4.5}$$

$$p_k'(\xi) = -\frac{kQ(\gamma-1)}{\gamma p_k - \xi^2 \tau_k}\,\eta_k\left(\frac{\xi}{\eta_k}\right)^{k+1}, \tag{4.6}$$

and

$$u_k' = \frac{-kQ(\gamma-1)}{\gamma p_k - \xi^2 \tau_k}\left(\frac{\xi}{\eta_k}\right)^k. \tag{4.7}$$

Condition (4.3) implies the solution $X(\xi)$ under consideration is necessarily of type (i) or (ii). For type (iii) or (iv), (4.3) may be replaced by

$$T_k(\xi) > T_i \qquad \text{in } (\hat{\eta}_k, \eta_k),$$

with $T_n(\hat{\eta}_k) = T_i$. All results obtained below for type (i) or (ii) can also be applied to type (iii) or (iv) on $(\hat{\eta}_k, \eta_k)$. Therefore, we only consider type (i) or (ii) solutions.

The problem of studying the limits of (4.4) $\sim$ (4.7) as $k \to \infty$ is necessarily a singularity problem. For example, for type (i) or (ii) solutions, if

$$\eta_k \to \eta_0 (<\infty) \qquad \text{as} \quad k \to \infty,$$

then (4.4) implies

$$q_k(\xi) \to q^*(\xi) = 0 \qquad \text{in} \quad (0, \eta_0) \quad \text{as} \quad k \to \infty, \tag{4.8}$$

and

$$q_k(\eta_k) = Q = q^*(\eta_0) \qquad \text{for all} \quad k.$$

Hence, the limit q^* is 0 in $(0, \eta_0)$ and Q at η_0, a jump at η_0. This phenomena is also brought to the limit (τ^*, p^*) of (τ_k, p_k) as $k \to \infty$, whenever (τ^*, p^*) exists. This is discussed in more detail below. We first state a monotonicity result for τ_k and T_k in k when $X_k \in G^+$.

PROPOSITION 4.1. *Assume* (4.1), (4.2), (4.3) *and*

$$(\tau_{0k}, p_{0k}, \eta_k) \to (\tau_0, p_0, \eta_0) \qquad \textit{as} \quad k \to \infty. \tag{4.9}$$

Then for sufficient large k, the solution X_k satisfies

$$\text{(i)} \quad \frac{\partial \tau_k}{\partial k} \geqslant 0 \qquad \textit{in } (0, \eta_0),$$

$$\text{(ii)} \quad \frac{\partial T_k}{\partial k} \geqslant 0 \qquad \textit{in} \quad (\tilde{\eta}, \eta_0) \qquad \textit{for some} \quad 0 < \tilde{\eta} < \eta_0.$$

Proof. (i) This can be proven by the same argument used in [8] since (4.9) holds, the detail is omitted.

(ii) By (3.9), we have

$$p_k(\xi) = -\xi^2 \tau_k(\xi) + 2 \int_0^{\xi} s \tau_k(s)\, ds.$$

Now,

$$\frac{\partial p_k}{\partial k}(\xi) = -\xi^2 \frac{\partial \tau_k}{\partial k}(\xi) + 2 \int_0^{\xi} s \frac{\partial \tau_k}{\partial k}(s)\, ds$$

implies

$$\frac{\partial T_k}{\partial k}(\xi)=(p_k(\xi)-\xi^2\tau_k(\xi))\frac{\partial \tau_k}{\partial k}(\xi)+2\tau_k\int_0^{\xi} s\,\frac{\partial \tau_k}{\partial k}(s)\,ds.$$

Then (ii) follows by (i) and $X_k \in G^+$. The proof is complete.

The above theorem enables us to define the limits of solutions X_k.

DEFINITION 4.2. Let (4.1), (4.2), (4.3) and (4.9) hold, and X_k by type (i), define

$$p^* = \lim_{k\to\infty} p_k(\xi), \qquad \tau^*(\xi)=\lim_{k\to\infty}\tau_k(\xi), \qquad \text{in } (0,\eta_0),$$

$$p^*(\eta_0)=\lim_{k\to\infty} p_k(\eta_k) \qquad \text{and} \qquad \tau^*(\eta_0)=\lim_{k\to\infty}\tau_k(\eta_k).$$

For type (ii) solution, we also assume $T(\eta_0)=T_i$.

PROPOSITION 4.3. *Assume* (4.1), (4.2), (4.3) *and* (4.9) *hold.*

(i) *If* $\eta_0^2 \leqslant \gamma p_0/\tau_0$, *then* $(\tau^*, p^*)=(\tau_0, p_0)$ *in* $(0,\eta_0)$ *and the convergence of* (τ_k, p_k) *to* (τ^*, p^*) *is uniformly in any compact subinterval of* $[0,\eta_0)$.

(ii) *If* $\eta_0^2 > \gamma p_0/\tau_0$, *and* $\tilde{\eta}_0=(\gamma p_0/\tau_0)^{1/2}$, *then* $(\tau^*, p^*)=(\tau_0, p_0)$ *in* $[0,\tilde{\eta}_0]$ *and a rarefaction wave given in* (3.26) *in* $[\tilde{\eta}_0, \eta_0)$.

Proof. The results can be obtained by an argument similar to that used in [8], so details are omitted.

The main part of the following theorem was essentially proven in [8].

THEOREM 4.4.(I)*If* $(\tau_0, p_0)\in \mathrm{SDT}(+)$ *with* $-\eta_0^2=(p_0-p_+)/(\tau_0-\tau_+)$, *then*

$$T^{\#}(\eta_0)\geqslant T_i, \quad T(\eta_0)\geqslant T_i, \quad \tau^*(\eta_0)=\tau^{\#}(\eta_0) \quad \text{and} \quad p^*(\eta_0)=p^{\#}(\eta_0) \tag{4.10}$$

hold if and only if

$$\lim_{k\to\infty}(\tau_{0k}, p_{0k}, \eta_k)=(\tau_0, p_0, \eta_0) \tag{4.11}$$

such that the associated solution X_k *is of type* (i).

(II) *If* $(\tau_0, p_0) \in R(\mathrm{CJDT}(+))$ *with*

$$-\eta_c^2 = \frac{p_c - p_+}{\tau_c - \tau_+} \qquad \textit{and} \quad \eta_c^2 > \gamma p_0/\tau_0.$$

Case 1.

$$T_0 \geqslant T_i, \tag{4.12}$$

$$T^{\#}(\eta_c) \geqslant T_i, \qquad \tau^*(\eta_c) = \tau^{\#}(\eta_c) \qquad \text{and} \qquad p^*(\eta_c) = p^{\#}(\eta_c) \tag{4.13}$$

hold if and if (4.11) holds such that X_k is of type (i).

Case 2.

$$T_0 < T_i \tag{4.14}$$

and (4.14) hold if and only if (4.11) holds such that X_k is of type (iii).

(III) If $(\tau_0, p_0) \in \mathrm{WDF}(i(+))$ with $-\eta_0^2 = (p_0 - p_i)/(\tau_0 - \tau_i)$, the

$$-\eta^*(\eta_0) = \tau_i \qquad \text{and} \qquad p^*(\eta_0) = p_i \tag{4.15}$$

hold if and only if (4.11) holds such that X_k is of type (ii).

(IV) If $(\tau_0, p_0) \in R(\mathrm{CJDF}(i(+))$ with

$$-(\eta^c)^2 = \frac{p^c - p_i}{\tau^c - \tau_i} \qquad \text{and} \qquad (\eta^c)^2 > \gamma p_0/\tau_0.$$

Case 1.

$$T_0 \geqslant T_i, \qquad \tau^*(\eta^c) = \tau_i \qquad \text{and} \qquad p^*(\eta^c) = p_i \tag{4.16}$$

hold if and only if (4.11) holds such that X_k is of type (ii).

Case 2. (4.14) and (4.16) hold if and only if (4.11) holds such that X_k is of type (iv).

Proof. The proofs of (I), (II), and (III) are almost the same as those given in [8]. However, η_k is not necessarily equal to η_0, and $q(\eta_k)$ may be less than Q in general, which is assumed in [8]. In proving (IV), we need the following result:

Suppose (τ^c, p^c) is the $\mathrm{CJDF}(i(+))$ and $(\eta^c)^2 = -(p^c - p_+)/(\tau^c - \tau_+)$. Then (4.16) holds.

Indeed, by Lemma 2.1, we have $(\eta^c)^2 = (\beta - \alpha)/(1 - \mu^2)\,\tau_+$. Then, from [8], we have $p^*(\eta^c) = p^c + \eta^c \alpha$ and $\tau^*(\eta^c) = \tau^c - \alpha/\eta^c$. A direct computation implies (4.16). The proof is complete.

Now, combining the classification theorems in Section 2 and Theorem 4.4, we obtain the following complete result for $J(+)$.

THEOREM 4.5. (I) *Concerning* JDT(+):

(i) *If unburnt state* (+) *is of class* A, *then any state* $(\tau_0, p_0) \in$ JDT(+) *is a limit of the simple solution* X_k *of* (SZND) *as* $k \to \infty$. *Furthermore*, X_k *is of type* (i) *when* $T_0 \geqslant T_i$, *and of type* (iii) *when* $T_0 < T_i$.

(ii) *If unburnt state* (+) *is of class* B *or* C, *then state* $(\tau_0, p_0) \in$ SDT(+) *is a limit of the simple solution* X_k *of* (SZND) *as* $k \to \infty$ *if and only if*

$$\eta_0 \geqslant \eta_*, \tag{4.17}$$

where

$$T^{\#}(\eta_*) = T_i. \tag{4.18}$$

In this case, X_k *is of type* (i). *Furthermore, any state in* R(CJDT(+)) *cannot be a limit of the simple solution of* (SZND).

(II) *Concerning* JDF(+):

(i) *If* $Q^*(\gamma) \geqslant T_i$ *then for any unburnt state* (+), *any state* $(\tau_0, p_0) \in$ JDF(+) *can be a limit of the simple solution* X_k *of* (SZND) *as* $k \to \infty$. *Furthermore*, X_k *is of type* (ii) *when* $T_0 \geqslant T_i$, *and of type* (iv) *when* $T_0 < T_i$.

(ii) *If* $Q^*(\gamma) < T_i$ *and* (+) *is an unburnt state, then* $(\tau_0, p_0) \in$ WDF(i(+)) *is a limit of the simple solution* X_k *of* (SZND) *as* $k \to \infty$ *if and only if*

$$\eta_0 \geqslant \eta^*, \tag{4.19}$$

where

$$T(\eta^*) = T_i. \tag{4.20}$$

In this case, X_k is of type (ii). Furthermore, any state in R(CJDF(i(+))) cannot be a limit of the simple solution of (SZND).

Remark 4.6. It is of interest to study the limit of non-simple solutions of (SZND) as $k \to \infty$.

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