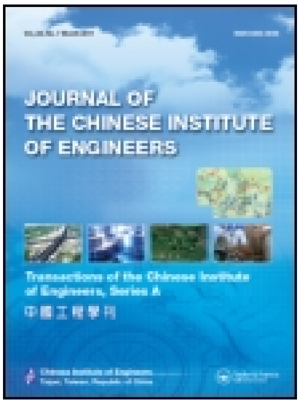

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Robust H 8 output feedback control for general nonlinear systems with structured uncertainty

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Short Paper

ROBUST H^∞ OUTPUT FEEDBACK CONTROL FOR GENERAL NONLINEAR SYSTEMS WITH STRUCTURED UNCERTAINTY

Jenq-Lang Wu* and Tsu-Tian Lee

ABSTRACT

In this paper, the robust H^∞ output feedback control problem for general nonlinear systems with L_2-norm-bounded structured uncertainties is considered. Sufficient conditions for the solvability of robust performance synthesis problems are represented in terms of two Hamilton-Jacobi inequalities with *n* independent variables. Based on these conditions, a state space characterization of a robust H^∞ output feedback controller solving the considered problem is proposed. An example is provided for illustration.

Key Words: H^∞ control, nonlinear systems, robust performance, structured uncertainty.

I. INTRODUCTION

In this paper, the robust H^∞ output feedback control problem for continuous-time general nonlinear systems with structured uncertainty will be considered. The block diagram of the considered problem is shown in Fig. 1, where ***P*** is the normal system, which is nonlinear and time-invariant, and Δ is the structured uncertainty. The design objective is to find an output feedback controller, ***K***, such that the closed-loop system is internally stable and its L_2-gain from ***w*** to ***z*** is less than, or equal to, some positive number, γ, for all possible structured uncertainties Δ.

In the case of no uncertainty, Δ, in the system, the problem becomes the well-known nonlinear H^∞ control problem, see, e.g., (Ball *et al*., 1993; Isidori and Astolfi, 1992; Isidori, 1994; Isidori and Kang, 1995; Isidori and Lin, 1998; Lu and Doyle, 1994; Van der Schaft, 1991; 1992; and Yung *et al*., 1996; 1998). It has been shown that the solution to the nonlinear H^∞ output feedback control problem can be obtained by solving two Hamilton-Jacobi equations (or inequalities), which are the nonlinear versions of the Riccati equations considered in the corresponding linear H^∞ control theory, see, e.g., (Doyle *et al*., 1989).

However, the H^∞ control problem for nonlinear systems with structured uncertainty is more difficult. The robust H^∞ control problem for linear systems with structured uncertainty has been considered in (Doyle, 1982; Lu *et al*., 1996; and Poola and Tikku, 1995). The studies of nonlinear H^∞ control problems with structured uncertainty are few. The only result is a state-space characterization of robustness analysis and synthesis for affine nonlinear systems provided by Lu and Doyle (1997). Specifically, sufficient conditions for the solvability of robustness synthesis problem are represented in terms of scaling Nonlinear Matrix Inequalities (NLMIs). However, in (Lu and Doyle, 1997), the considered system is assumed to be affine in control input and external input. Moreover, only the state feedback case is considered. In this paper, we first extend the results of the robust H^∞ state feedback control problem for affine nonlinear systems in (Lu and Doyle, 1997) to the case of general nonaffine nonlinear systems. Furthermore, not only state feedback but also output feedback cases are considered in this paper. Sufficient conditions for the solvability of robust H^∞ output feedback control problems for general nonlinear systems with structured uncertainty are represented in terms of two Hamilton-Jacobi inequalities with *n* independent variables. Finally, based

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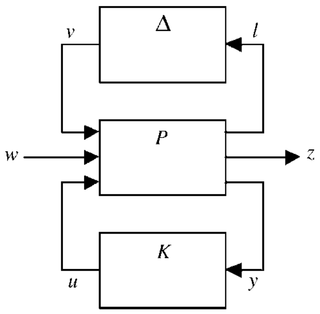

Fig. 1 Block diagram of the considered system

on these conditions, a state space characterization of an output feedback controller solving the considered problem is provided.

In what follows, $\|x\|^2$ denotes x^Tx; $\|x\|_M^2$ denotes x^TMx; $\|P\|_{L_2}$ denotes the L_2-norm of the system P; M>0 means that the matrix M is positive definite; M<0 means that the matrix M is negative definite.

II. PROBLEM FORMULATION

Consider a smooth uncertain nonlinear system shown in Fig. 1. The normal plant P is described by the following dynamic equations

$$P:\begin{cases}\dot{x}=F(x,w,v,u)\\ y=Y(x,w,v,u)\\ z=Z(x,w,v,u)\\ l_i=L_i(x,w,v,u),\ i=1,2,\cdots,N\end{cases} \quad (1)$$

where $x\in R^n$ represents the state, $u\in R^m$ is the control input, and $w\in R^r$ represents a set of exogenous inputs, $z\in R^s$ is the controlled variable, and $y\in R^p$ is the measured variable. Without losing of generality, assume that $F(0,0,0,0)=0$, $Z(0,0,0,0)=0$, $Y(0,0,0,0)=0$, and $L_i(0,0,0,0)=0$ for all i=1, 2, ⋯, N.

The uncertainty is described as

$$v_i=\Delta_i(l_i),\quad i=1,2,\cdots,N \quad (2)$$

where

$$\Delta_i\in B\Delta_i\equiv\{\Delta_i|\Delta_i \text{ is causal and asymptotically stable for } l_i=0\text{, and has } L_2\text{-gain}\le\rho_i\}$$

with ρ_i>0. Or equivalently,

$$v=\Delta(l)$$

where $l=[l_1^T\ \ l_2^T\ \ \cdots\ \ l_N^T]^T$, $v=[v_1^T\ \ v_2^T\ \ \cdots\ \ v_N^T]^T$, and

$$\Delta\in\boldsymbol{\Delta}\equiv\{\Delta=\text{diag}\{\Delta_1,\Delta_2,\ldots,\Delta_N\}|\Delta_i\in B\Delta_i\}$$

Suppose $l_i\in R^{n_i}$ and $v_i\in R^{m_i}$, i=1, 2, ⋯, N. Let $\overline{m}=\sum_{i=1}^{N}m_i$.

The design objective is to construct an output feedback controller which will asymptotically stabilize the resulting closed-loop system locally and render its L_2-gain (from w to z) less than or equal to γ for all $\Delta\in\boldsymbol{\Delta}$.

Suppose that the state of Δ_i is ς_i. Let $\varsigma\equiv[\varsigma_1^T\ \ \varsigma_2^T\ \ \cdots\ \ \varsigma_N^T]^T$. As in (Lu and Doyle, 1997), the following assumption is made.

Assumption (A1). For each $i\in\{1,2,\cdots,N\}$, Δ_i has a unique asymptotically stable equilibrium at ς_i=0 for l_i=0; in addition, there is a differentiable storage function $U_i(\varsigma_i)$ such that

$$\frac{dU_i(\varsigma_i(t))}{dt}\le\rho_i^2\|l_i(t)\|^2-\|v_i(t)\|^2+\varphi_i(\varsigma_i(t)), \quad (3)$$

with some negative definite function $\varphi_i(\cdot)$.

III. STATE FEEDBACK CASE

In this section, we will focus on the robust H^∞ state feedback control problem for system (1). Define a Hamiltonian function H_1: $R^n\times R^n\times R^r\times R^{\overline{m}}\times R^m\rightarrow R$ as

$$H_1(x,p_1,w,v,u)$$
$$=p_1^TF(x,w,v,u)+\|Z(x,w,v,u)\|^2-\gamma^2\|w\|^2$$
$$+\sum_{i=1}^{N}[\rho_i^2\|L_i(x,w,v,u)\|^2-\|v_i\|^2] \quad (4)$$

Let

$$\hat{Z}(x,w,v,u)=\begin{bmatrix}Z(x,w,v,u)\\ \rho_1L_1(x,w,v,u)\\ \cdot\\ \rho_NL_N(x,w,v,u)\end{bmatrix} \quad (5)$$

and set

$$D_{11}=\left.\frac{\partial\hat{Z}(x,w,v,u)}{\partial w}\right|_{(x,w,v,u)=(0,0,0,0)},$$

$$D_{12}=\left.\frac{\partial\hat{Z}(x,w,v,u)}{\partial v}\right|_{(x,w,v,u)=(0,0,0,0)},$$

$$D_{13}=\left.\frac{\partial\hat{Z}(x,w,v,u)}{\partial u}\right|_{(x,w,v,u)=(0,0,0,0)}$$

Suppose plant (1) satisfies the following assumption.

Assumption (A2). The matrix $D_{13}^TD_{13}$ is positive definite, and $\begin{bmatrix}D_{11}^TD_{11}-\gamma^2I & D_{11}^TD_{12}\\ D_{12}^TD_{11} & D_{12}^TD_{12}-I\end{bmatrix}$ is negative definite.

Assumption (A2) guarantees the existence and uniqueness of solutions $\boldsymbol{w}_*(\boldsymbol{x}, \boldsymbol{p}_1)$, $\boldsymbol{v}_*(\boldsymbol{x}, \boldsymbol{p}_1)$, and $\boldsymbol{u}_*(\boldsymbol{x}, \boldsymbol{p}_1)$, defined in the neighborhood of $(\boldsymbol{x}, \boldsymbol{p}_1)=(0, 0)$, satisfying

$$\frac{\partial H_1}{\partial \boldsymbol{w}}(\boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{w}_*(\boldsymbol{x}, \boldsymbol{p}_1), \boldsymbol{v}_*(\boldsymbol{x}, \boldsymbol{p}_1), \boldsymbol{u}_*(\boldsymbol{x}, \boldsymbol{p}_1))=0$$

$$\frac{\partial H_1}{\partial \boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{w}_*(\boldsymbol{x}, \boldsymbol{p}_1), \boldsymbol{v}_*(\boldsymbol{x}, \boldsymbol{p}_1), \boldsymbol{u}_*(\boldsymbol{x}, \boldsymbol{p}_1))=0$$

$$\frac{\partial H_1}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{w}_*(\boldsymbol{x}, \boldsymbol{p}_1), \boldsymbol{v}_*(\boldsymbol{x}, \boldsymbol{p}_1), \boldsymbol{u}_*(\boldsymbol{x}, \boldsymbol{p}_1))=0$$

with

$$\boldsymbol{w}_*(0, 0)=0,\ \boldsymbol{v}_*(0, 0)=0,\ \boldsymbol{u}_*(0, 0)=0$$

This can be drawn from the implicit function theorem.

Moreover, suppose the system (1) satisfies the "detectability" assumption given below.

Assumption (A3). Any bounded trajectory $\boldsymbol{x}(t)$ of the system $\dot{x}(t)=\boldsymbol{F}(\boldsymbol{x}(t), 0, \boldsymbol{v}(t), \boldsymbol{u}(t))$ satisfying $\boldsymbol{Z}(\boldsymbol{x}(t), 0, \boldsymbol{v}(t), \boldsymbol{u}(t))=0$ for all $t\geq 0$, is such that $\lim_{t\to\infty}\boldsymbol{x}(t)=0$.

Then, the following result holds.

Theorem 1. Consider system (1). Suppose Assumptions (A1), (A2), and (A3) hold. Suppose the following hypothesis also holds.

(H1) There exists a smooth, positive definite function $V(\boldsymbol{x})$, locally defined in the neighborhood of $\boldsymbol{x}=0$, such that the function

$$Y_1(\boldsymbol{x}) = H_1(\boldsymbol{x}, V_x^{\boldsymbol{T}}(\boldsymbol{x})), \boldsymbol{w}_*(\boldsymbol{x}, V_x^{\boldsymbol{T}}(\boldsymbol{x})), \boldsymbol{v}_*(\boldsymbol{x}, V_x^{\boldsymbol{T}}(\boldsymbol{x})), \boldsymbol{u}_*(x, V_x^{\boldsymbol{T}}(\boldsymbol{x})) \tag{6}$$

is negative semidefinite near $x=0$, where $V_x(\boldsymbol{x})$ denotes the row vector $\lfloor \partial V/\partial \boldsymbol{x}_1\ \ \partial V/\partial \boldsymbol{x}_2\ \dots\ \partial V/\partial \boldsymbol{x}_n \rfloor$.

Then the system (1) with the feedback law $\boldsymbol{u}=\boldsymbol{u}_*(\boldsymbol{x}, V_x^{\boldsymbol{T}}(\boldsymbol{x}))$ is locally internally stable and has L_2-gain (from $\boldsymbol{w}$ to $\boldsymbol{z}$) less than or equal to γ for all $\Delta\in\boldsymbol{\Delta}$.

Proof of Theorem 1: For simplicity of notation, we denote $\boldsymbol{w}_*(\boldsymbol{x})=\boldsymbol{w}_*(x, V_x^{\boldsymbol{T}}(\boldsymbol{x}))$, $\boldsymbol{v}_*(\boldsymbol{x})=\boldsymbol{v}_*(x, V_x^{\boldsymbol{T}}(\boldsymbol{x}))$ and $\boldsymbol{u}_*(\boldsymbol{x})=\boldsymbol{u}_*(x, V_x^{\boldsymbol{T}}(\boldsymbol{x}))$. Let $\hat{\boldsymbol{p}}\equiv[\boldsymbol{w}^{\boldsymbol{T}}\ \ \boldsymbol{v}^{\boldsymbol{T}}\ \ \boldsymbol{u}^{\boldsymbol{T}}]^{\boldsymbol{T}}$. Using the Taylor expansion theorem and noting (4) and (6), we have

$$H_1(\boldsymbol{x}, V_x^{\boldsymbol{T}}(\boldsymbol{x}), \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u})$$

$$= Y_1(\boldsymbol{x}) + \left\| \begin{bmatrix} \boldsymbol{w}-\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}-\boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix} \right\|^2_{\boldsymbol{R}(\boldsymbol{x})} + \boldsymbol{o}\left(\left\| \begin{bmatrix} \boldsymbol{w}-\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}-\boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix} \right\|^3 \right)$$

where

$$\boldsymbol{R}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{r}_{11}(\boldsymbol{x}) & \boldsymbol{r}_{12}(\boldsymbol{x}) & \boldsymbol{r}_{13}(\boldsymbol{x}) \\ \boldsymbol{r}_{21}(\boldsymbol{x}) & \boldsymbol{r}_{22}(\boldsymbol{x}) & \boldsymbol{r}_{23}(\boldsymbol{x}) \\ \boldsymbol{r}_{31}(\boldsymbol{x}) & \boldsymbol{r}_{32}(\boldsymbol{x}) & \boldsymbol{r}_{33}(\boldsymbol{x}) \end{bmatrix}$$

$$\equiv \frac{1}{2}\left.\frac{\partial^2 H_1(\boldsymbol{x}, V_x^{\boldsymbol{T}}(\boldsymbol{x}), \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u})}{\partial \hat{\boldsymbol{p}}^2}\right|_{\boldsymbol{w}=\boldsymbol{w}_*(\boldsymbol{x}),\ \boldsymbol{v}=\boldsymbol{v}_*(\boldsymbol{x}),\ \boldsymbol{u}=\boldsymbol{u}_*(\boldsymbol{x})}$$

It is easy to show that

$$\boldsymbol{R}(0) = \begin{bmatrix} \boldsymbol{D}_{11}^{\boldsymbol{T}}\boldsymbol{D}_{11}-\gamma^2\boldsymbol{I} & \boldsymbol{D}_{11}^{\boldsymbol{T}}\boldsymbol{D}_{12} & \boldsymbol{D}_{11}^{\boldsymbol{T}}\boldsymbol{D}_{13} \\ \boldsymbol{D}_{12}^{\boldsymbol{T}}\boldsymbol{D}_{11} & \boldsymbol{D}_{12}^{\boldsymbol{T}}\boldsymbol{D}_{12}-\boldsymbol{I} & \boldsymbol{D}_{12}^{\boldsymbol{T}}\boldsymbol{D}_{13} \\ \boldsymbol{D}_{13}^{\boldsymbol{T}}\boldsymbol{D}_{11} & \boldsymbol{D}_{13}^{\boldsymbol{T}}\boldsymbol{D}_{12} & \boldsymbol{D}_{13}^{\boldsymbol{T}}\boldsymbol{D}_{13} \end{bmatrix}$$

which is nonsingular by Assumption (A2).

Consider the candidate storage function $U(\boldsymbol{x}, \boldsymbol{\varsigma}) =V(\boldsymbol{x})+\sum_{i=1}^{N} U_i(\varsigma_i)$, which is positive definite. Setting $\boldsymbol{u}=\boldsymbol{u}_*(\boldsymbol{x})$ in (1) yields a closed-loop system satisfying

$$\frac{dU}{dt} + \|\boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{x}))\|^2 - \gamma^2\|\boldsymbol{w}\|^2$$

$$\leq Y_1(\boldsymbol{x}) + \begin{bmatrix} \boldsymbol{w}-\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \end{bmatrix}^{\boldsymbol{T}} \begin{bmatrix} \boldsymbol{r}_{11}(\boldsymbol{x}) & \boldsymbol{r}_{12}(\boldsymbol{x}) \\ \boldsymbol{r}_{21}(\boldsymbol{x}) & \boldsymbol{r}_{22}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \boldsymbol{w}-\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \end{bmatrix}$$

$$+\boldsymbol{o}\left(\left\| \begin{bmatrix} \boldsymbol{w}-\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \end{bmatrix} \right\|^3 \right) + \sum_{i=1}^{N} \varphi_i(\varsigma_i) \tag{7}$$

Then, from hypothesis (H1) and Assumptions (A1) and (A2), we immediately have the following dissipation inequality

$$\frac{dU}{dt} + \|Z(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{x}))\|^2 - \gamma^2\|\boldsymbol{w}\|^2 \leq 0$$

in the neighborhood of the origin. Thus, the closed-loop system has L_2–gain$\leq\gamma$. It remains to prove that the closed-loop system is locally asymptotically stable. To this end, letting $\boldsymbol{w}=0$ in (7), it yields

$$\frac{dU}{dt} \leq -\|\boldsymbol{Z}(\boldsymbol{x}, 0, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{x}))\|^2 + Y_1(\boldsymbol{x})$$

$$+\begin{bmatrix} -\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \end{bmatrix}^{\boldsymbol{T}} \begin{bmatrix} \boldsymbol{r}_{11}(\boldsymbol{x}) & \boldsymbol{r}_{12}(\boldsymbol{x}) \\ \boldsymbol{r}_{21}(\boldsymbol{x}) & \boldsymbol{r}_{22}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} -\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \end{bmatrix}$$

$$+\boldsymbol{o}\left(\left\| \begin{bmatrix} -\boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v}-\boldsymbol{v}_*(\boldsymbol{x}) \end{bmatrix} \right\|^3 \right) + \sum_{i=1}^{N} \varphi_i(\varsigma_i) \tag{8}$$

which is negative semidefinite near the origin by hypothesis (H1) and Assumptions (A1) and (A2). This proves that the equilibrium $\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\varsigma} \end{bmatrix}=0$ of the closed-loop system is stable. To prove the asymptotic stability of the closed-loop system, note that equation (8) implies

$$\frac{dU}{dt} \le -\left\| \boldsymbol{Z}(\boldsymbol{x}, 0, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{x})) \right\|^2 + \sum_{i=1}^{N} \varphi_i(\varsigma_i) \le 0 \qquad (9)$$

in the neighborhood of the origin. Therefore, the asymptotic stability can be concluded by LaSalle's invariance principle.

IV. OUTPUT FEEDBACK CASE

The major contribution of this paper is to propose an output feedback controller

$$\dot{\boldsymbol{\xi}} = \boldsymbol{F}(\boldsymbol{\xi}, \boldsymbol{u}_*(\boldsymbol{\xi}), \boldsymbol{v}_*(\boldsymbol{\xi}), \boldsymbol{w}_*(\boldsymbol{\xi})) + \boldsymbol{G}(\boldsymbol{\xi})(\boldsymbol{y} - \boldsymbol{Y}(\boldsymbol{\xi}, \boldsymbol{u}_*(\boldsymbol{\xi}), \boldsymbol{v}_*(\boldsymbol{\xi}), \boldsymbol{w}_*(\boldsymbol{\xi})))$$

$$\boldsymbol{u} = \boldsymbol{u}_*(\boldsymbol{\xi}) \qquad (10)$$

to solve the considered robust H^∞ control problem, where $\boldsymbol{\xi} \in R^n$ is defined in the neighborhood of the origin, and the output injection gain $\boldsymbol{G}(\boldsymbol{\xi})$ is a matrix to be determined.

For convenience, the corresponding closed-loop system is expressed as

$$\dot{\boldsymbol{x}}^{\boldsymbol{o}} = \boldsymbol{F}^{\boldsymbol{o}}(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v})$$

$$\boldsymbol{y} = \boldsymbol{Y}^{\boldsymbol{o}}(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v})$$

$$\boldsymbol{z} = \boldsymbol{Z}^{\boldsymbol{o}}(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v})$$

$$\boldsymbol{l}_i = \boldsymbol{L}_i^o(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v})$$

$$\boldsymbol{v}_i = \Delta_i(\boldsymbol{l}_i), \quad i = 1, 2, \cdots, N$$

where $\boldsymbol{x}^o = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\xi} \end{bmatrix}$,

$$\boldsymbol{F}^o(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v}) = \begin{bmatrix} \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{\xi})) \\ \tilde{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{v}) \end{bmatrix}$$

$$\boldsymbol{Y}^o(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{\xi}))$$

$$\boldsymbol{Z}^o(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{\xi}))$$

$$\boldsymbol{L}_i^o(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{L}_i(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{\xi}))$$

and

$$\tilde{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{F}(\boldsymbol{\xi}, \boldsymbol{w}_*(\boldsymbol{\xi}), \boldsymbol{v}_*(\boldsymbol{\xi}), \boldsymbol{u}_*(\boldsymbol{\xi})) + \boldsymbol{G}(\boldsymbol{\xi})(\boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{\xi})) - \boldsymbol{Y}(\boldsymbol{\xi}, \boldsymbol{w}_*(\boldsymbol{\xi}), \boldsymbol{v}_*(\boldsymbol{\xi}), \boldsymbol{u}_*(\boldsymbol{\xi})))$$

In what follows, we shall show how to asymptotically stabilize the closed-loop system locally and render its L_2–gain$\le\gamma$ (from $\boldsymbol{w}$ to $\boldsymbol{z}$) for all $\Delta \in \boldsymbol{\Delta}$.

Define a Hamiltonian function H_2: $R^{2n} \times R^{2n} \times R^r \times R^{\overline{m}} \to R$ as

$$H_2(\boldsymbol{x}^o, \boldsymbol{p}_2, \boldsymbol{w}, \boldsymbol{v}) = \boldsymbol{p}_2^T \boldsymbol{F}^o(\boldsymbol{x}^o, \boldsymbol{w}, \boldsymbol{v}) + \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}_*(\boldsymbol{\xi}) - \boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix}^T \cdot \begin{bmatrix} (1-\varepsilon_1)\boldsymbol{r}_{11}(\boldsymbol{x}) & (1-\varepsilon_1)\boldsymbol{r}_{12}(\boldsymbol{x}) & \boldsymbol{r}_{13}(\boldsymbol{x}) \\ (1-\varepsilon_1)\boldsymbol{r}_{21}(\boldsymbol{x}) & (1-\varepsilon_1)\boldsymbol{r}_{22}(\boldsymbol{x}) & \boldsymbol{r}_{23}(\boldsymbol{x}) \\ \boldsymbol{r}_{31}(\boldsymbol{x}) & \boldsymbol{r}_{32}(\boldsymbol{x}) & (1+\varepsilon_3)\boldsymbol{r}_{33}(\boldsymbol{x}) \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}_*(\boldsymbol{\xi}) - \boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix} \qquad (11)$$

where $0<\varepsilon_1<1$ and $\varepsilon_3>0$. Then, by implict function theorem, there exist unique smooth functions $\hat{\boldsymbol{w}}(\boldsymbol{x}^o, \boldsymbol{p}_2)$ and $\hat{\boldsymbol{v}}(\boldsymbol{x}^o, \boldsymbol{p}_2)$, defined in the neighborhood of $(\boldsymbol{x}^o, \boldsymbol{p}_2)=(0, 0)$, satisfying

$$\left.\frac{\partial H_2(\boldsymbol{x}^o, \boldsymbol{p}_2, \boldsymbol{w}, \boldsymbol{v})}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w} = \hat{\boldsymbol{w}}(\boldsymbol{x}^o, \boldsymbol{p}_2),\ \boldsymbol{v} = \hat{\boldsymbol{v}}(\boldsymbol{x}^o, \boldsymbol{p}_2)} = 0$$

$$\left.\frac{\partial \boldsymbol{H}_2(\boldsymbol{x}^o, \boldsymbol{p}_2, \boldsymbol{w}, \boldsymbol{v})}{\partial \boldsymbol{v}}\right|_{\boldsymbol{w} = \hat{\boldsymbol{w}}(\boldsymbol{x}^o, \boldsymbol{p}_2),\ \boldsymbol{v} = \hat{\boldsymbol{v}}(\boldsymbol{x}^o, \boldsymbol{p}_2)} = 0$$

with $\hat{\boldsymbol{w}}(0, 0) = 0$ and $\hat{\boldsymbol{v}}(0, 0) = 0$.

As a result, we have the following theorem.

Theorem 2. Consider system (1). Suppose Assumptions (A1), (A2) and (A3), and hypothesis (H1) in Theorem 1 hold. Suppose the following hypothesis also holds.

(H2) There exists a smooth real-valued function $\boldsymbol{M}(\boldsymbol{x}^o)$, which is locally defined in the neighborhood of $\boldsymbol{x}^o=0$, and which vanishes at $\boldsymbol{x}=\boldsymbol{\xi}$ and is positive elsewhere, such that the function

$$Y_2(\boldsymbol{x}^o) = H_2(\boldsymbol{x}^o, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x}^o), \hat{\boldsymbol{w}}(\boldsymbol{x}^o, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x}^o)), \hat{\boldsymbol{v}}(\boldsymbol{x}^o, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x}^o)))$$

vanishes at $\boldsymbol{x}=\boldsymbol{\xi}$ and is negative elsewhere.

Then the system (1) with the output feedback controller (10) is locally internally stable and has L_2-gain (from $\boldsymbol{w}$ to $\boldsymbol{z}$) less than or equal to γ for all $\Delta \in \boldsymbol{\Delta}$.

Proof of Theorem 2: Using the Taylor expansion theorem yields

$$H_2(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o}), \boldsymbol{w}, \boldsymbol{v})$$

$$= Y_2(\boldsymbol{x^o}) + \left\| \begin{bmatrix} \boldsymbol{w} - \hat{\boldsymbol{w}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \\ \boldsymbol{v} - \hat{\boldsymbol{v}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \end{bmatrix} \right\|^2_{\boldsymbol{R}_2(\boldsymbol{x^o})}$$

$$+ o\left(\left\| \begin{bmatrix} \boldsymbol{w} - \hat{\boldsymbol{w}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \\ \boldsymbol{v} - \hat{\boldsymbol{v}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \end{bmatrix} \right\|^3 \right) \tag{12}$$

where

$$\boldsymbol{R}_2(\boldsymbol{x^o})$$

$$= \frac{1}{2} \begin{bmatrix} \dfrac{\partial^2 \boldsymbol{H}_2}{\partial \boldsymbol{w}^2} & \dfrac{\partial^2 \boldsymbol{H}_2}{\partial \boldsymbol{v} \partial \boldsymbol{w}} \\ \dfrac{\partial^2 \boldsymbol{H}_2}{\partial \boldsymbol{w} \partial \boldsymbol{v}} & \dfrac{\partial^2 \boldsymbol{H}_2}{\partial \boldsymbol{v}^2} \end{bmatrix} \Bigg|_{w = \hat{w}(x^o, M_{x^o}^T(x^o)),\ v = \hat{v}(x^o, M_{x^o}^T(x^o))}$$

which is negative definite near the origin.

Consider the candidate storage function

$$U^o(\boldsymbol{x^o}, \boldsymbol{\varsigma}) = M(\boldsymbol{x^o}) + V(\boldsymbol{x}) + \sum_{i=1}^{N} U_i(\boldsymbol{\varsigma_i})$$

which is positive for all $\begin{bmatrix} \boldsymbol{x^o} \\ \boldsymbol{\varsigma} \end{bmatrix} \neq 0$. Along the trajectories of the closed-loop system, we have

$$\frac{dU^o}{dt} + \| \boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}_*(\boldsymbol{\xi})) \|^2 - \gamma^2 \| \boldsymbol{w} \|^2$$

$$\leq Y_1(\boldsymbol{x}) + Y_2(\boldsymbol{x^o}) + \sum_{i=1}^{N} \varphi_i(\boldsymbol{\varsigma_i}) + \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}_*(\boldsymbol{\xi}) - \boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix}^{\boldsymbol{T}}$$

$$\cdot \begin{bmatrix} \varepsilon_1 \boldsymbol{r}_{11}(\boldsymbol{x}) & \varepsilon_1 \boldsymbol{r}_{12}(\boldsymbol{x}) & 0 \\ \varepsilon_1 \boldsymbol{r}_{21}(\boldsymbol{x}) & \varepsilon_1 \boldsymbol{r}_{22}(\boldsymbol{x}) & 0 \\ 0 & 0 & -\varepsilon_3 \boldsymbol{r}_{33}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}_*(\boldsymbol{\xi}) - \boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix}$$

$$+ \boldsymbol{o}\left(\left\| \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ \boldsymbol{u}_*(\boldsymbol{\xi}) - \boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix} \right\|^3 \right)$$

$$+ \left\| \begin{bmatrix} \boldsymbol{w} - \hat{\boldsymbol{w}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \\ \boldsymbol{v} - \hat{\boldsymbol{v}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \end{bmatrix} \right\|^{\boldsymbol{2}}_{\boldsymbol{R}_2(\boldsymbol{x})}$$

$$+ \boldsymbol{o}\left(\left\| \begin{bmatrix} \boldsymbol{w} - \hat{\boldsymbol{w}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \\ \boldsymbol{v} - \hat{\boldsymbol{v}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})) \end{bmatrix} \right\|^3 \right) \leq 0 \tag{13}$$

in the neighborhood of the origin. Then, we can prove this Theorem via a procedure similar to the proof of Theorem 1. So, the detailed proof is omitted here for saving space.

The function $Y_2(x^o)$ thus obtained has $2n$ independent variables and actually involves the undetermined matrix $\boldsymbol{G}(\boldsymbol{\xi})$. In what follows, we shall show how to reduce the number of independent variables in $Y_2(x^o)$, and how to determine the output injection gain $\boldsymbol{G}(\boldsymbol{\xi})$.

To this end, define a Hamiltonian function H_3: $R^n \times R^n \times R^p \times R^r \times R^{\overline{m}} \to R$ as

$$H_3(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{w}, \boldsymbol{v})$$

$$= \boldsymbol{p}_3^T \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, 0) - \boldsymbol{p}_4^T \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, 0) + \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ -\boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix}^{\boldsymbol{T}}$$

$$\cdot \begin{bmatrix} (1-\varepsilon_1)\boldsymbol{r}_{11}(\boldsymbol{x}) & (1-\varepsilon_1)\boldsymbol{r}_{12}(\boldsymbol{x}) & \boldsymbol{r}_{13}(\boldsymbol{x}) \\ (1-\varepsilon_1)\boldsymbol{r}_{21}(\boldsymbol{x}) & (1-\varepsilon_1)\boldsymbol{r}_{22}(\boldsymbol{x}) & \boldsymbol{r}_{23}(\boldsymbol{x}) \\ \boldsymbol{r}_{31}(\boldsymbol{x}) & \boldsymbol{r}_{32}(\boldsymbol{x}) & (1+\varepsilon_3)\boldsymbol{r}_{33}(\boldsymbol{x}) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \boldsymbol{w} - \boldsymbol{w}_*(\boldsymbol{x}) \\ \boldsymbol{v} - \boldsymbol{v}_*(\boldsymbol{x}) \\ -\boldsymbol{u}_*(\boldsymbol{x}) \end{bmatrix} \tag{14}$$

and suppose the plant (1) satisfies the following additional assumption.

Assumption (A4). The measurement output $\boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u})$ is such that the matrix $\boldsymbol{D}_{21}$=[$\boldsymbol{Y_w}$(0, 0, 0, 0) $\boldsymbol{Y_v}$(0, 0, 0, 0)] has full row rank.

Then, by implict function, theorem there exist unique smooth functions $\tilde{\boldsymbol{w}}(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4)$ and $\tilde{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4)$, defined in the neighborhood of $(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4)$=(0,0,0), satisfying

$$\frac{\partial H_3(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{w}, \boldsymbol{v})}{\partial \boldsymbol{w}} \Bigg|_{w = \tilde{w}(x, p_3, p_4),\ v = \tilde{v}(x, p_3, p_4)} = 0$$

$$\tilde{\boldsymbol{w}}(0, 0, 0) = 0$$

$$\frac{\partial H_3(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{w}, \boldsymbol{v})}{\partial \boldsymbol{v}} \Bigg|_{w = \tilde{w}(x, p_3, p_4),\ v = \tilde{v}(x, p_3, p_4)} = 0$$

$$\tilde{\boldsymbol{v}}(0, 0, 0) = 0$$

Moreover, it is easy to verify that

$$\frac{\partial^2 \boldsymbol{H}_3(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4, \tilde{\boldsymbol{w}}(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4), \tilde{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4))}{\partial \boldsymbol{p}_4^2} \Bigg|_{(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4) = (0, 0, 0)}$$

$$= \frac{1}{2(1-\varepsilon_1)} \boldsymbol{D}_{21} \begin{bmatrix} \boldsymbol{r}_{11}(0) & \boldsymbol{r}_{12}(0) \\ \boldsymbol{r}_{21}(0) & \boldsymbol{r}_{22}(0) \end{bmatrix}^{-1} \boldsymbol{D}_{21}^{\boldsymbol{T}} < 0$$

which is nonsingular. Thus, there exists a smooth function $\boldsymbol{p}_{4*}(\boldsymbol{x}, \boldsymbol{p}_3)$, defined in the neighborhood of (0,0) such that

$$\frac{\partial \boldsymbol{H}_3(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4, \tilde{\boldsymbol{w}}(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4), \tilde{\boldsymbol{v}}(\boldsymbol{x}, \boldsymbol{p}_3, \boldsymbol{p}_4))}{\partial \boldsymbol{p}_4} \Bigg|_{\boldsymbol{p}_4 = \boldsymbol{p}_{4*}(\boldsymbol{x}, \boldsymbol{p}_3)}$$

$$=0 \qquad \boldsymbol{p}_{4*}(0, 0)=0$$

Then, we have the following result.

Theorem 3. Consider system (1). Suppose Assumptions (A1), (A2), and (A3), and hypothesis (H1) in Theorem 1 hold. Suppose the following hypothesis also holds.

(H3) There exists a smooth, positive definite function $Q(\boldsymbol{x})$, locally defined in the neighborhood of the origin in R^n, such that the function

$$Y_3(\boldsymbol{x}) = H_3(\boldsymbol{x}, Q_x^T(\boldsymbol{x}), \boldsymbol{p}_{4*}(\boldsymbol{x}, Q_x^T(\boldsymbol{x})),\ \tilde{\boldsymbol{w}}(\boldsymbol{x}, Q_x^T(\boldsymbol{x}), \boldsymbol{p}_{4*}(\boldsymbol{x}, Q_x^T(\boldsymbol{x}))),\ \tilde{\boldsymbol{v}}(\boldsymbol{x}, Q_x^T(\boldsymbol{x}), \boldsymbol{p}_{4*}(\boldsymbol{x}, Q_x^T(\boldsymbol{x})))) \tag{15}$$

is negative definite near $\boldsymbol{x}$=0, and its Hessian matrix is nonsingular at $\boldsymbol{x}$=0. If the equation

$$Q_x(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x}) = p_{4*}(\boldsymbol{x}, Q_x^T(\boldsymbol{x})) \tag{16}$$

has a smooth solution $\boldsymbol{G}(\boldsymbol{x})$, then the system (1) with the output feedback controller (10) is locally internally stable and has L_2-gain (from $\boldsymbol{w}$ to $\boldsymbol{z}$) less than or equal to γ for all $\Delta\in\boldsymbol{\Delta}$.

Proof of Theorem 3: It suffices to prove that $M(\boldsymbol{x^o})\equiv Q(\boldsymbol{x}-\boldsymbol{\xi})$ with $Q_x(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x})=\boldsymbol{p}_{4*}(\boldsymbol{x}, Q_x^T(\boldsymbol{x}))$ which satisfies hypothesis (H2). Clearly, $M(\boldsymbol{x^o})\equiv Q(\boldsymbol{x}-\boldsymbol{\xi})$ vanishes at $\boldsymbol{x}=\boldsymbol{\xi}$ and is positive elsewhere. Set

$$\overline{Y}_2(\boldsymbol{x^o}) = H_2(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o}), \hat{\boldsymbol{w}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})),\ \hat{\boldsymbol{v}}(\boldsymbol{x^o}, M_{\boldsymbol{x^o}}^{\boldsymbol{T}}(\boldsymbol{x^o})))$$

where $M(\boldsymbol{x^o})=Q(\boldsymbol{x}-\boldsymbol{\xi})$ and $Q_x(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x})=\boldsymbol{p}_{4*}(\boldsymbol{x}, Q_x^T(\boldsymbol{x}))$.

It remains to prove that $\overline{Y}_2(\boldsymbol{x^o})$ vanishes at $\boldsymbol{x}=\boldsymbol{\xi}$ and is negative elsewhere. This can be proven via a similar procedure as that presented in (Yung *et al.*, 1998). Therefore, it is omitted for saving the space.

V. EXAMPLE

Consider a system (denoted by *system* $\boldsymbol{P}$) which has the following realization

$$\dot{x}_1 = 4x_1x_2 - 2x_1^3 - x_1^3x_2^2 + x_2w - 2x_1x_2v_1 + x_1u^2$$

$$\dot{x}_2 = -14x_2 - 5x_2^3 + \frac{3}{4}x_1^2x_2 - v_2 + 2(1 + x_1^2)u$$

$$y = \frac{20}{3}x_1 + 5x_1x_2 + 2x_1^3x_2 + w$$

$$z = x_1^2 + u$$

$$l_1 = x_2^2$$

$$l_2 = x_1^2$$

$$v_1 = \Delta_1(l_1)$$

$$v_2 = \Delta_2(l_2)$$

where the uncertain terms Δ_1 and Δ_2 satisfy $\|\Delta_1\|_{L_2}\le 3$ and $\|\Delta_2\|_{L_2}\le 1$. Suppose Assumption (A1) holds. Here we want to find a state feedback controller and an output feedback controller, respectively, such that the closed loop system is internally stable and has L_2–*gain*≤1 (form $\boldsymbol{w}$ to $\boldsymbol{z}$) for all possible Δ_1 and Δ_2.

State Feedback Case:

For system $\boldsymbol{P}$, it is easy to verify that Assumption (A2) holds. From Theorem 1, it can be shown that the positive definite function

$$V(\boldsymbol{x}) = \frac{1}{2}x_1^2 + x_2^2$$

satisfies Hypothesis (H1). The corresponding $Y_1(\boldsymbol{x})$ is

$$Y_1(\boldsymbol{x}) = -31x_2^2 - x_2^4 - \frac{9x_1^2x_2^2}{4} - \frac{x_1^4}{1 + x_1^2}$$

which is negative definite for all $\boldsymbol{x}\neq 0$. The worst-case disturbance w is

$$w = w_*(\boldsymbol{x}) = \frac{1}{2}x_1x_2$$

Moreover, if we choose the state feedback controller as

$$u = u_*(\boldsymbol{x}) = -\frac{x_1^2}{1 + x_1^2} - 2x_2$$

then the closed-loop system will be internally stable and its L_2–*gain* (from $\boldsymbol{w}$ to $\boldsymbol{z}$) will be less than or equal to 1.

Output Feedback Case:

For output feedback, it can be shown that Assumption (A4) holds. Moreover, the positive definite function $Q(\boldsymbol{x})=x_1^2 + \frac{1}{2}x_2^2$ satisfies Hypothesis (H3) with $\varepsilon_1=\varepsilon_3=\frac{1}{2}$. The corresponding $Y_3(X)$ is

$$Y_3(\boldsymbol{x}) = (\frac{3}{1 + x_1^2} - 4)x_1^4 - \frac{3}{4}x_2^2 - 5x_2^4 - \frac{5}{2}x_1^2x_2^2 - x_1^2(\frac{20}{3} + 5x_2 + 2x_1^2x_2)^2$$

which is negative definite near $\boldsymbol{x}$=0. Moreover,

$$p_{4*}(\boldsymbol{x}) = \frac{40}{3}x_1 + 12x_1x_2 + 4x_1^3x_2$$

Let the output injection gain $\boldsymbol{G}(\boldsymbol{x})=\begin{bmatrix} G_1(\boldsymbol{x}) \\ G_2(\boldsymbol{x}) \end{bmatrix}$. Since

$$Q_x(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x})=p_{4*}(\boldsymbol{x})$$

i.e.

$$[2x_1\ \ x_2]\cdot\begin{bmatrix} G_1(\boldsymbol{x}) \\ G_2(\boldsymbol{x}) \end{bmatrix} = \frac{40}{3}x_1 + 12x_1x_2 + 4x_1^3x_2$$

has a smooth solution

$$\begin{bmatrix} G_1(\boldsymbol{x}) \\ G_2(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \frac{20}{3} + 6x_2 \\ 4x_1^3 \end{bmatrix}$$

So the output feedback H^∞ control problem is solvable. Moreover, with the output feedback controller

$$\dot{\xi}_1 = 4\xi_1\xi_2 - 2\xi_1^3 + \xi_1^3\xi_2^2 + \frac{1}{2}\xi_1\xi_2^2 + \xi_1(\frac{\xi_1^2}{1+\xi_1^2} + 2\xi_2)^2$$

$$+ (\frac{20}{3} + 6\xi_2)\cdot(y - \frac{20}{3}\xi_1 - \frac{11}{2}\xi_1\xi_2 - 2\xi_1^3\xi_2)$$

$$\dot{\xi}_2 = -13\xi_2 - 5\xi_2^3 + \frac{3}{4}\xi_1^2\xi_2 - 2(\xi_1^2 + 2(1+\xi_1^2)\xi_2)$$

$$+ 4\xi_1^3(y - \frac{20}{3}\xi_1 - \frac{11}{2}\xi_1\xi_2 - 2\xi_1^3\xi_2)$$

$$u = -\frac{\xi_1^2}{1+\xi_1^2} - 2\xi_2$$

the closed-loop system will be internally stable and its L_2-*gain* (from $\boldsymbol{w}$ to z) will be less than or equal to 1.

VI. CONCLUSIONS

In this paper, a state-space characterization of robust H^∞ output feedback controllers for general nonlinear systems with L_2-gain-bounded structured uncertainties has been proposed. Sufficient conditions for the solvability of robust performance synthesis problems have been represented in terms of two Hamilton-Jacobi inequalities with n independent variables. Based on these conditions, an output feedback H^∞ controller solving the considered problem has been provided. The example shows that the provided method is useful.

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