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Distributional and Inferential Properties of the Process Loss Indices

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ABSTRACT *Johnson (1992) developed the process loss index L_e, which is defined as the ratio of the expected quadratic loss to the square of half specification width. Tsui (1997) expressed the index L_e as $L_e = L_{pe} + L_{ot}$, which provides an uncontaminated separation between information concerning the potential relative expected loss (L_{pe}) and the relative off-target squared (L_{ot}), as the ratio of the process variance and the square of the half specification width, and the square of the ratio of the deviation of mean from the target and the half specification width, respectively. In this paper, we consider these three loss function indices, and investigate the statistical properties of their natural estimators. For the three indices, we obtain their UMVUEs and MLEs, and compare the reliability of the two estimators based on the relative mean squared errors. In addition, we construct 90%, 95%, and 99% upper confidence limits, and the maximum values of $\hat{L}_e$ for which the process is capable, 90%, 95%, and 99% of the time. The results obtained in this paper are useful to the practitioners in choosing good estimators and making reliable decisions on judging process capability.*

KEY WORDS: MLE, potential relative expected loss, relative expected loss, relative mean squared error, relative off-target squared, UMVUE

Introduction

Process capability indices (PCIs), the purpose of which is to provide numerical measures of whether or not the ability of a manufacturing process meets a predetermined level of production tolerance, have received considerable research attention and increased usage in process assessments and purchasing decisions in the automotive industry during last decade. Those indices are effective tools for process capability analysis and quality assurance, and the formula for those indices are easy to understand and straightforward to apply. Kane (1986) developed the two most commonly used process capability indices, C_p and C_{pk}. However, they are not related to the cost of failing to meet customer desires. Boyles (1991) noted that C_p and C_{pk} are yield-based indices, which are independent of the target value T, and may fail to consider process centring. In order to take into account the departure of the process mean from the target, the index C_{pm} is proposed by Chan *et al.* (1988). Actually, the denominator of the index

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C_{pm} is the expected quadratic loss, which is closely related to process departure. For on-target processes, the value of C_{pm} index reaches its maximum, implying that the process capability runs under the desired condition. On the other hand, smaller values of C_{pm} mean higher expected loss and the poorer process capability. Therefore, the index C_{pm} is considered to be more sensitive than C_p and C_{pk} in reflecting the process loss. Pearn *et al.* (1992) investigated the index C_{pmk}, which takes into account the process yield as well as the process loss. Those four well-known index indices have been defined explicitly as:

$$C_p = \frac{USL - LSL}{6\sigma} \tag{1}$$

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} \tag{2}$$

$$C_{pm} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}} \tag{3}$$

$$C_{pmk} = \min\left\{\frac{USL-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-LSL}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\} \tag{4}$$

The index C_p considers the overall process variability relative to the manufacturing tolerance, reflecting product quality consistency. The index C_{pk} takes the process mean into consideration but can fail to distinguish between on-target processes from off-target processes, which is a yield-based index providing lower bounds on process yield. The index C_{pm} takes the proximity of process mean from the target value into account, which is more sensitive to process departure than C_{pk}. Since the design of C_{pm} is based on the average process loss relative to the manufacturing tolerance, the index C_{pm} provides an upper bound on the average process loss, which has been alternatively called the Taguchi index. The index C_{pmk} is constructed from combining the modifications to C_p that produced C_{pk} and C_{pm}, which inherits the merits of both indices.

Yield Index

One of the commonly understood basic criteria for interpretations of the process capability is the yield index, which is defined as the proportion of conforming items. Suppose a proportion conforming interpretation is the primary concern, the most natural measure is the proportion itself called the yield, which we refer to as Y defined as:

$$Y = \int_{LSL}^{USL} \mathrm{d}F(x) \tag{5}$$

where $F(x)$ is the cumulative distribution function of the measured characteristic X, USL and LSL are the upper and the lower specification limits, respectively.

The disadvantage of yield measure is that it does not distinguish among the products that fall inside of the specification limits.

Loss Index

To remedy for that, the quadratic loss function is considered to distinguish the products that fall inside of the specification limits by increasing the penalty as the departure from the target increases. However, the quadratic loss function itself does not provide comparison with the specification limits and depends on the unit of the characteristic. To address these issues, Johnson (1992) developed the relative expected loss L_e for the symmetric case, to provide numerical measures on process performance for industrial applications. Tsui (1997) interpret $L_e = L_{pe} + L_{ot}$, which provides an uncontaminated separation between information concerning the potential relative expected loss (L_{pe}) and the relative off-target squared (L_{ot}). The index L_e is defined as the ratio of the expected quadratic loss and the square of half specification width as follows:

$$L_e = \int_{-\infty}^{\infty} \left[\frac{(x-T)^2}{d^2} \right] \mathrm{d}F(x) = \left(\frac{\sigma}{d} \right)^2 + \left(\frac{\mu - T}{d} \right)^2 \tag{6}$$

where μ is the process mean, σ is the process standard deviation, $d = (USL - LSL)/2$ is the half specification width, USL and LSL are the upper and the lower specification limits, and T is the target value. Define $L_{pe} = (\sigma/d)^2$ and $L_{ot} = [(\mu - T)/d]^2$, then L_e can be expressed as $L_e = L_{pe} + L_{ot}$. Since L_{pe} measures the process variation relative to the specification tolerance, it has been referred to as the potential relative expected loss index. On the other hand, L_{ot} measures the relative process departure and has been referred to as the relative off-target squared index. We note that the mathematical relationship $L_e = (3C_{pm})^{-2}$, $L_{pe} = (3C_p)^{-2}$ and $L_{ot} = (1 - C_a)^2$ can be established, where C_{pm}, C_p and C_a (defined as $C_a = 1 - |\mu - T|/d$) are three basic process capability indices considered by Chan *et al.* (1988), Kane (1986) and Pearn *et al.* (1998), respectively. The advantage of L_e over C_{pm} is that the estimator of the former has better statistical properties than that of latter, as the former does not involve a reciprocal transformation of process mean and variance.

In this paper, we consider three loss function indices L_{pe}, L_{ot} and L_e, and investigate the statistical properties of their natural estimators. For L_{pe}, we show that the natural estimator is the UMVUE (uniformly minimum variance unbiased estimator), which is consistent and asymptotically efficient. We also obtain the MLE (maximum likelihood estimator), which has smaller mean squared error than the UMVUE, hence it is more reliable, particularly, for short production run applications. For L_{ot}, we show that the natural estimator is the MLE. We also obtain the UMVUE, which is shown to be more reliable than the MLE for applications with $n \geqslant 4$. We show that the UMVUE is consistent and asymptotically efficient. For L_e, we show that the natural estimator is the MLE and also the UMVUE, which is consistent and asymptotically efficient. In addition, we construct tables of 90%, 95%, and 99% upper confidence limits for L_e based on the UMVUE. We also construct tables of the maximum values of $\hat{L}_e$ under $\mu = T$

for which the process is capable 90%, 95% and 99% of the time. An efficient UMP test based on the UMVUE of L_e is derived. Using the UMP test, a testing procedure is proposed. The estimators we recommend have all the desired statistical properties, and are considered reliable in determining whether a process meets the capability requirement.

Estimating Process Relative Inconsistency Loss

To estimate the process relative inconsistency loss, we consider the natural estimator $\hat{L}_{pe}$ defined as follows, where $S_{n-1} = [\Sigma_{i=1}^{n}(X_i - \bar{X})^2/(n-1)]^{1/2}$ is the conventional estimator of the process standard deviation σ,

$$\hat{L}_{pe} = \frac{1}{n-1}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{d^2} = \frac{S_{n-1}^2}{d^2} \tag{7}$$

The natural estimator $\hat{L}_{pe}$ can be rewritten as:

$$\hat{L}_{pe} = \frac{L_{pe}}{n-1}\frac{(n-1)\hat{L}_{pe}}{L_{pe}} = \frac{L_{pe}}{n-1}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{\sigma^2} \tag{8}$$

If the process follows the normal distribution, then $\hat{L}_{pe}$ is distributed as $[L_{pe}/(n-1)]\chi_{n-1}^2$, where χ_{n-1}^2 is a chi-squared distribution with $(n-1)$ degrees of freedom. The expected value, the variance, and the mean squared error of $\hat{L}_{pe}$ can be obtained as follows:

$$\mathrm{E}(\hat{L}_{pe}) = \left(\frac{L_{pe}}{n-1}\right)\mathrm{E}(\chi_{n-1}^2) = L_{pe} \tag{9}$$

$$\mathrm{Var}(\hat{L}_{pe}) = \left(\frac{L_{pe}}{n-1}\right)^2\mathrm{Var}(\chi_{n-1}^2) = 2(n-1)\left(\frac{L_{pe}}{n-1}\right)^2 = \frac{2L_{pe}^2}{n-1} \tag{10}$$

$$\mathrm{MSE}(\hat{L}_{pe}) = \mathrm{E}(\hat{L}_{pe} - L_{pe})^2 = \mathrm{Var}(\hat{L}_{pe}) + [\mathrm{E}(\hat{L}_{pe}) - L_{pe}]^2 = \frac{2L_{pe}^2}{n-1} \tag{11}$$

If the process characteristic is normally distributed, an $100(1-\alpha)\%$ upper confidence limit on L_{pe} can be established in terms of the estimator $\hat{L}_{pe}$ as $[(n-1)\hat{L}_{pe}/\chi_{n-1}^2(\alpha)]$, where $\chi_{n-1}^2(\alpha)$ is the (lower) αth percentile of the χ_{n-1}^2 distribution. A capability testing can then be conducted. In addition, we can show that the natural estimator $\hat{L}_{pe}$ is the UMVUE of L_{pe}, which is consistent. We can also show that $\sqrt{n}(\hat{L}_{pe} - L_{pe})$ converges to $N(0, 2L_{pe}^2)$ in distribution, and that $\hat{L}_{pe}$ is asymptotically efficient (see Proposition 1). These results follow from the well-known properties of the sample variance (because $\hat{L}_{pe}$ is just a constant multiplied by the sample variance). Thus, in real-world applications using $\hat{L}_{pe}$, which has all the desired statistical properties, as an estimate of L_{pe} would be reasonable.

Table 1. Recommended range of L_{pe} for various precision requirements

Range	Precision Requirement
$0.06 \leqslant L_{pe} \leqslant 0.11$	Capable
$0.05 \leqslant L_{pe} \leqslant 0.06$	Satisfactory
$0.04 \leqslant L_{pe} \leqslant 0.05$	Good
$0.03 \leqslant L_{pe} \leqslant 0.04$	Excellent
$L_{pe} \leqslant 0.03$	Super

Proposition 1

If the process characteristic is normally distributed, then:

(a) $\hat{L}_{pe}$ is the UMVUE of L_{pe};
(b) $\hat{L}_{pe}$ is consistent;
(c) $\sqrt{n}(\hat{L}_{pe} - L_{pe})$ converges to $N(0, 2L_{pe}^2)$ in distribution;
(d) $\hat{L}_{pe}$ is asymptotically efficient.

We note that by multiplying the UMVUE $\hat{L}_{pe}$ by the constant $c_n = (n-1)/n$, we obtain the MLE of L_{pe}. We can show that the MLE $\tilde{L}_{pe}$ is consistent, and is asymptotically unbiased. We can show that $\sqrt{n}(\tilde{L}_{pe} - L_{pe})$ converges to $N(0, 2L_{pe}^2)$ in distribution, and that $\tilde{L}_{pe}$ is asymptotically efficient. Since $c_n < 1$, then $\tilde{L}_{pe} = c_n \hat{L}_{pe}$ underestimates the index L_{pe}, but it has smaller variance. In fact, we may calculate:

$$\mathrm{MSE}(\tilde{L}_{pe}) = [(2n-1)/n^2](L_{pe})^2 \tag{12}$$

$$\mathrm{MSE}(\hat{L}_{pe}) - \mathrm{MSE}(\tilde{L}_{pe}) = \{(3n-1)/[n^2(n-1)]\}(L_{pe})^2 > 0, \text{ for all } n \tag{13}$$

Therefore, the MLE $\tilde{L}_{pe}$ has smaller mean squared error than the UMVUE $\hat{L}_{pe}$, hence it is more reliable, particular for short production run applications. We consider some commonly used values of $L_{pe} = 0.11$, 0.06, 0.05, 0.04 and 0.03, equivalent to $C_p = 1.00$, 1.33, 1.50, 1.67 and 2.00, covering the widespread range of the precision requirements for most applications (see Table 1).

The square root of the relative mean squared error is a direct measurement, which presents the expected relative error of the estimation from the true L_{pe}. We note that for UMVUE $\tilde{L}_{pe}$. $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2} = [2/(n-1)]^{1/2}$, which is a function of the sample size n only. Therefore, $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2}$ values are the same for all L_{pe} values. For example, with $n = 300$ we have $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2} = 0.0818$. Thus, for $n = 300$, we expect that the average error of $\hat{L}_{pe}$ would be no greater than 8.18% of the true L_{pe}. We note that $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2} = [(2n-1/n^2]^{1/2}$, which is also a function of the sample size n only. Thus, $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2}$ values are the same for all L_{pe} values.

Figure 1 plots $[\mathrm{MSE_R}(\hat{L}_{pe})]^{1/2}$ with $L_{pe} = 0.11$, 0.06, 0.05, 0.04, 0.03, versus sample size $n = 2(1)120$ and Figure 2 plots $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{1/2}$ with $L_{pe} = 0.11$, 0.06, 0.05, 0.04, 0.03, versus sample size $n = 1(1)120$. The sensitivity of the square root of the relative mean squared error for both estimators due to the process relative inconsistency loss L_{pe}, as well as sample size n can then be easily understood.

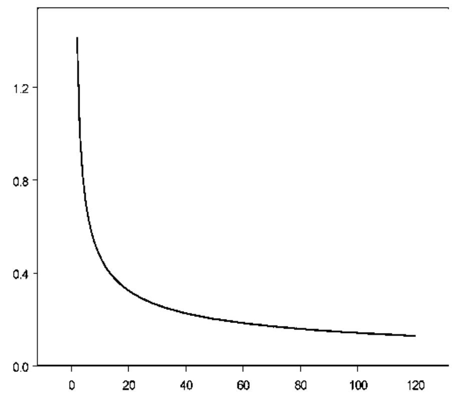

Figure 1. Plots of $[\mathrm{MSE_R}(\tilde{L}_{pe})]^{0.5}$ with $L_{pe}=0.11$, 0.06, 0.05, 0.04, 0.03 (overlap), versus sample size $n=2(1)120$

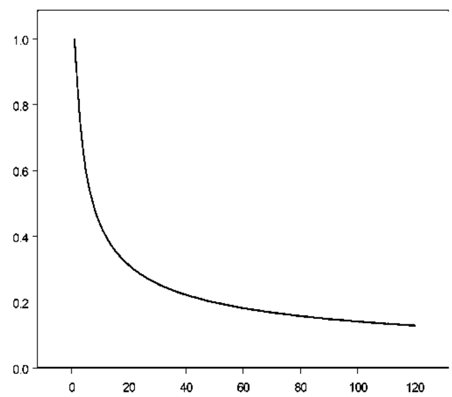

Figure 2. Plots of $[\mathrm{MSE_R}(\hat{L}_{pe})]^{0.5}$ with $L_{pe}=0.11$, 0.06, 0.05, 0.04, 0.03 (overlap), versus sample size $n=1(1)120$

For short run applications (such as accepting a supplier providing short production runs in QS-9000 certification), the difference between the two relative errors is considered significant for sample size $n \leqslant 35$, and we strongly recommend using the MLE $\tilde{L}_{pe}$ rather than the UMVUE $\hat{L}_{pe}$. For other applications with sample size $n > 35$, the difference between the two estimators is negligible (less than 0.52%).

Estimating Process Relative Off-Target Loss

To estimate the relative off-target loss, we consider the natural estimator $\hat{L}_{ot}$ defined as the following, where $\bar{X}=\Sigma_{i=1}^{n} X_i/n$ is the conventional estimator of the process mean μ. We note that the estimator $\hat{L}_{ot}$ can also be written as a function of L_{pe}:

$$\hat{L}_{ot} = \frac{(\bar{X}-T)^2}{d^2} = \frac{L_{pe}}{n}\frac{n\hat{L}_{ot}}{L_{pe}} = \frac{L_{pe}}{n}\frac{n(\bar{X}-T)^2}{\sigma^2} \tag{14}$$

If the process characteristic is normally distributed, then the estimator $\hat{L}_{ot}$ is distributed as $[L_{pe}/n]\chi_1^2(\delta)$, where $\chi_1^2(\delta)$ is a non-central chi-squared distribution with one degree of freedom and non-centrality parameter $\delta = n(\mu-T)^2/\sigma^2$. The expected value, the variance, and the mean squared error of $\hat{L}_{ot}$, therefore, can be calculated as:

$$\mathrm{E}(\hat{L}_{ot}) = \left(\frac{L_{pe}}{n}\right)\mathrm{E}[\chi_1^2(\delta)] = \left(\frac{L_{pe}}{n}\right)(1+\delta) = \frac{L_{pe}}{n} + L_{ot} \tag{15}$$

$$\mathrm{Var}(\hat{L}_{ot}) = \left(\frac{L_{pe}}{n}\right)^2\mathrm{Var}[\chi_1^2(\delta)] = \left(\frac{L_{pe}}{n}\right)^2(2+4\delta) = \frac{4L_{pe}L_{ot}}{n} + \frac{2L_{pe}^2}{n} \tag{16}$$

$$\mathrm{MSE}(\hat{L}_{ot}) = \mathrm{Var}(\hat{L}_{ot}) + [\mathrm{E}(\hat{L}_{ot}) - L_{ot}]^2 = \frac{4L_{pe}L_{ot}}{n} + \frac{3L_{pe}^2}{n^2} \tag{17}$$

If the process characteristic is normally distributed, an $100(1-\alpha)\%$ upper confidence limit on L_{ot} can be expressed in terms of the estimator $\hat{L}_{ot}$ as $(\delta\hat{L}_{ot}/\chi_1^2(\alpha, \delta)$, where $\chi_1^2(\alpha, \delta)$ is the (lower) αth percentile of the $\chi_1^2(\delta)$ distribution. A capability testing can then be conducted. In practice, we note that parameter δ is unknown and should be estimated by the sample data. Since $\bar{X}$ is the MLE of μ, then by the invariance property of MLE, the natural estimator $\hat{L}_{ot}$ is the MLE of L_{ot}. Noting that $\mathrm{E}(\hat{L}_{ot}) = L_{ot} + (L_{pe}/n)$, and $\mathrm{E}(\hat{L}_{pe}) = L_{pe}$, the corrected estimator $\tilde{L}_{ot} = \hat{L}_{ot} - (\hat{L}_{pe}/n)$ must be unbiased for L_{ot}. We can show that $\tilde{L}_{ot}$ is the UMVUE of L_{ot}, which is consistent. We can also show that $\sqrt{n}(\tilde{L}_{ot} - L_{ot})$ converges to $N(0,\, 4L_{pe}L_{ot})$ in distribution, and $\tilde{L}_{ot}$ is asymptotically efficient (see Proposition 2 for proofs). Thus, in real-world applications, using the UMVUE $\tilde{L}_{ot}$, which has all the desired statistical properties, as an estimate of L_{ot} would be reasonable.

Proposition 2

If the process characteristic is normally distributed, then:

(a) $\tilde{L}_{ot}$ is the UMVUE of L_{ot};
(b) $\tilde{L}_{ot}$ is consistent;
(c) $\sqrt{n}(\tilde{L}_{ot} - L_{ot})$ converges to $N(0,\, 4L_{pe}L_{ot})$ in distribution;
(d) $\tilde{L}_{ot}$ is asymptotically efficient.

Proof

(a) Note that $(\bar{X}, S_{n-1}^2)$ is sufficient and complete for (μ, σ^2), and that the unbiased estimator $\tilde{L}_{ot}$ is a function of $(\bar{X}, S_{n-1}^2)$ only. By the Lehmann–Scheffé Theorem, $\tilde{L}_{ot}$ is the UMVUE of L_{ot}.

(b) For every $\varepsilon > 0$,

$$P(|\tilde{L}_{ot} - L_{ot}| > \varepsilon) < E(\tilde{L}_{ot} - L_{ot})^2/\varepsilon^2 \tag{18}$$

Since:

$$E(\tilde{L}_{ot} - L_{ot})^2 = [4L_{pe}L_{ot}/n] + [2L_{pe}^2/(n^2 - n)] \tag{19}$$

converges to zero, then $\tilde{L}_{ot}$ must be consistent.

(c) Under general conditions, $\sqrt{n}(\hat{L}_{ot} - L_{ot})$ converges to $N\,(0, \sigma_{ot}^2)$ in distribution, where $\sigma_{ot}^2 = 4(\mu - T)^2\sigma^2/d^4$. Under normality assumption, $\sqrt{n}(\hat{L}_{ot} - L_{ot})$ converges to $N(0, 4L_{pe}L_{ot})$ in distribution. Since $\sqrt{n}(\tilde{L}_{ot} - \hat{L}_{ot})$ converges to zero in probability, then by Slutsky's Theorem,

$$\sqrt{n}(\tilde{L}_{ot} - L_{ot}) = \sqrt{n}(\tilde{L}_{ot} - \hat{L}_{ot}) + \sqrt{n}(\hat{L}_{ot} - L_{ot}) \tag{20}$$

converges to $N(0, 4L_{pe}L_{ot})$ in distribution.

(d) Under normality assumption, the information matrix can be calculated as follows. Since the information lower bound is achieved, then $\tilde{L}_{ot}$ must be asymptotically efficient:

$$I(\theta) = I(\mu, \sigma) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} \dfrac{\partial L_{ot}}{\partial \mu} & \dfrac{\partial L_{ot}}{\partial \sigma^2} \end{bmatrix} \frac{I^{-1}(\theta)}{n} \begin{bmatrix} \dfrac{\partial L_{ot}}{\partial \mu} \\ \dfrac{\partial L_{ot}}{\partial \sigma^2} \end{bmatrix} = \frac{4L_{pe}L_{ot}}{n} \tag{22}$$

We note that the MLE $\hat{L}_{ot}$ has smaller variance than the UMVUE $\tilde{L}_{ot}$. However, we can show that $\mathrm{MSE}(\tilde{L}_{ot}) = 4L_{pe}L_{ot}/n + \{2/[n(n-1)]\}(L_{pe})^2$, and so $\mathrm{MSE}(\tilde{L}_{ot}) - \mathrm{MSE}(\hat{L}_{ot}) = \{(3-n)/[n^2(n-1)]\}(L_{pe})^2$, which is greater than 0 for $n = 2$, equal to 0 for $n = 3$, and less than 0 for $n \geqslant 4$. Therefore, the UMVUE $\tilde{L}_{ot}$ has smaller mean squared error than the MLE $\hat{L}_{ot}$, and is more reliable for applications with $n \geqslant 4$. Figure 3 plots the relative error, $[\mathrm{MSE_R}(\tilde{L}_{ot})]^{1/2}$, of the UMVUE $\tilde{L}_{ot}$ for $L_{pe} = 0.11$, 0.06, 0.05, 0.04, 0.03, versus sample size $n = 2(1)120$ ($L_{ot} = 0.25$ is fixed). This value of L_{ot} is equivalent to $C_a = 0.50$. The relative errors, $[\mathrm{MSE_R}(\tilde{L}_{ot})]^{1/2}$, for other values of L_{ot} and sample size n are available from the authors. We note that if the process is perfectly centred, then $L_{ot} = 0.00$ (equivalently, $C_a = 1.00$). For example, for $L_{pe} = 0.11$, $L_{ot} = 0.25$, and $n = 300$ we have $[\mathrm{MSE_R}(\tilde{L}_{ot})]^{1/2} = 0.0770$. Thus, the average error (average relative deviation) of $\tilde{L}_{ot}$ would be no greater than 7.70% of the true L_{ot}.

Figure 4 plots the relative error, $[\mathrm{MSE_R}(\hat{L}_{ot})]^{1/2}$, of the MLE $\hat{L}_{ot}$ for $L_{pe} = 0.11$, 0.06, 0.05, 0.04, 0.03, versus sample size $n = 1(1)120$ ($L_{ot} = 0.25$ is fixed). Tables of $[\mathrm{MSE_R}(\hat{L}_{ot})]^{1/2}$ for other values of L_{ot} are available from the authors. We note that for $n \leqslant 30$, the difference between the two relative errors (percentage of deviations) is significant, and we recommend using the UMVUE $\tilde{L}_{ot}$ rather than

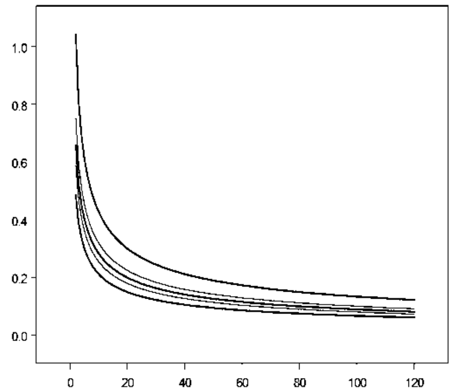

Figure 3. Plots of $[\text{MSE}_R(\hat{L}_{ot})]^{0.5}$ with $L_{pe}=0.11, 0.06, 0.05, 0.04, 0.03$ (top to bottom), versus sample size $n=2(1)120$

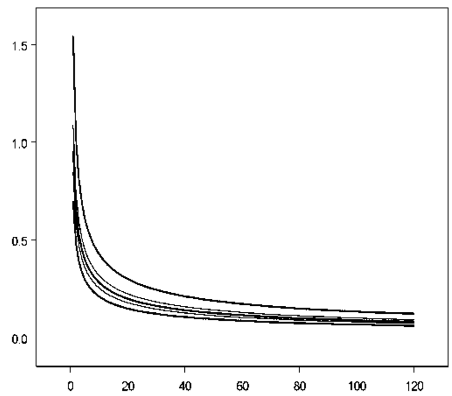

Figure 4. Plots of $[\text{MSE}_R(\hat{L}_{ot})]^{0.5}$ with $L_{pe}=0.11, 0.06, 0.05, 0.04, 0.03$ (top to bottom), versus sample size $n=1(1)120$

the MLE $\hat{L}_{ot}$. However, for $n>30$, the difference between the two is negligible (less than 0.04%), and using either of the two estimators is equally reliable.

Estimating Process Expected Relative Loss

To estimate the process expected relative loss (a combined measure of process relative inconsistency loss and process relative off-target loss), we consider the nature estimator $\hat{L}_e$ defined as the following, where $\bar{X}=\Sigma_{i=1}^{n}\text{X}_i/n$, which can also be written as a function of L_{pe}:

$$\hat{L}_e=\frac{1}{n}\sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{d^2}+\frac{(\bar{X}-T)^2}{d^2}=\frac{L_{pe}}{n}\frac{n\hat{L}_e}{L_{pe}}=\frac{L_{pe}}{n}\sum_{i=1}^{n}\frac{(X_i-T)^2}{\sigma^2} \qquad (23)$$

If the process characteristic is normally distributed, then the estimator $\hat{L}_e$ is distributed as $[L_{pe}/n]\chi_n^2(\delta)$, where $\chi_n^2(\delta)$ is a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nL_{ot}/L_{pe}$. The expected value, the variance, and the mean squared error of $\hat{L}_{pe}$ can be calculated as follows:

$$\mathrm{E}(\hat{L}_e) = \left(\frac{L_{pe}}{n}\right)\mathrm{E}[\chi_n^2(\delta)] = \left(\frac{L_{pe}}{n}\right)(n+\delta) = L_{pe} + L_{ot} = L_e \tag{24}$$

$$\mathrm{Var}(\hat{L}_e) = \left(\frac{L_{pe}}{n}\right)^2 \mathrm{Var}[\chi_n^2(\delta)] = \left(\frac{L_{pe}}{n}\right)^2 (2n + 4\delta) = \frac{2L_{pe}}{n}(L_{ot} + L_e) \tag{25}$$

$$\mathrm{MSE}(\hat{L}_e) = \mathrm{Var}(\hat{L}_e) + [\mathrm{E}(\hat{L}_e) - L_e]^2 = \frac{2L_{pe}}{n}(L_{ot} + L_e) \tag{26}$$

If the process characteristic follows the normal distribution, then we can show that $\hat{L}_e$ is the MLE, which is also the UMVUE of L_e. We can also show that $\hat{L}_e$ is consistent, $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0,\ 2L_{pe}L_{ot\,+\,2}L_{pe}L_e)$ in distribution, and $\hat{L}_e$ is asymptotically efficient (see Proposition 3 for proofs). Since the estimator has all the desired statistical properties, in practice using $\hat{L}_e$ to estimate process expected relative loss would be reasonable.

Proposition 3

If the process characteristic is normally distributed, then:

(a) $\hat{L}_e$ is the MLE of L_e;
(b) $\hat{L}_e$ is the UMVUE of L_e;
(c) $\hat{L}_e$ is consistent;
(d) $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0,\ 2L_{pe}L_{ot} + 2L_{pe}L_e)$ in distribution;
(e) $\hat{L}_e$ is asymptotically efficient.

Proof

(a) Since $(\bar{X},\ S_n^2)$ is the MLE of $(\mu,\ \sigma^2)$, where $S_n^2 = \Sigma_{i=1}^n (X_i - \bar{X})^2/n$, and $\hat{L}_e = (S_n^2/d^2) + [(\bar{X} - T)^2/d^2]$, then by the invariance property of MLE, $\hat{L}_e$ is the MLE of L_e.
(b) We note that $(\bar{X},\ S_n^2)$ is sufficient and complete for $(\mu,\ \sigma^2)$. Since the unbiased estimator $\hat{L}_e$ is a function of $(\bar{X},\ S_n^2)$ only, then by the Lehmann–Scheffé Theorem, $\hat{L}_e$ is the UMVUE.
(c) Under general conditions, $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0,\ \sigma_e^2)$ in distribution, where $\sigma_e^2 = [4(\mu - T)^2\sigma^2/d^4] + [4\mu_3(\mu - T)/d^4] + [(\mu_4 - \sigma^4)/d^4]$. Therefore, $\sqrt{n}(\hat{L}_e - L_e)$ converges to $N(0,\ 2L_{pe}L_{ot} + 2L_{pe}L_e)$ in distribution if the process is normal. Parts (c) and (e) directly follow from part (a), since MLEs must be consistent and asymptotically efficient.

Testing Process Capability Based on Process Loss

Under normality assumption, $n\hat{L}_e/(L_e - L_{ot})$ is distributed as $\chi_n^2(\delta)$, a non-central chi-squared distribution with n degrees of freedom and non-centrality parameter $\delta = n(\mu - T)^2/\sigma^2 = nL_{ot}/L_{pe}$. Let $U = U(X_1, X_2, \ldots, X_n)$ be a statistic calculated from the sample data satisfying $\mathrm{P}(L_e \leqslant U) = 1 - \alpha$, where the confidence level $1 - \alpha$ does not depend on L_e. Then, U is an $100(1-\alpha)\%$ upper confidence limit for L_e. We note that:

$$\begin{aligned} \mathrm{P}(L_e \leqslant U) &= \mathrm{P}(L_e - L_{ot} \leqslant U - L_{ot}) = \mathrm{P}(n\hat{L}_e/(L_e - L_{ot}) \geqslant n\hat{L}_e/(U - L_{ot})) \\ &= \mathrm{P}(\chi_n^2(\delta) \geqslant n\hat{L}_e/(U - L_{ot})) = 1 - \alpha \end{aligned} \tag{27}$$

Thus, $n\hat{L}_e/(U - L_{ot}) = \chi_n^2(\alpha, \delta)$, where $\chi_n^2(\alpha, \delta)$ is the (lower) αth percentile of the $\chi_n^2(\delta)$ distribution. A $100(1-\alpha)\%$ upper confidence limit on L_e can be expressed, in terms of $\hat{L}_e$, as $U = L_{ot} + [n\hat{L}_e/\chi_n^2(\alpha, \delta)]$. On the other hand, to test H_0: $L_e \geqslant C$(*incapable*) versus $H_1: L_e < C$ (*capable*), we claim that the process is capable for at least $100(1-\alpha)\%$ of the time if $\hat{L}_e \leqslant c_0$. We can show that the critical value $c_0 = [\chi_n^2(\alpha, \delta) \cdot C]/(n + \delta)$, where C is the capability requirement preset. Then, $c_0 = [\chi_n^2(\alpha, \delta) \cdot C]/(n + \delta)$, is the maximum value of the estimated expected relative loss $\hat{L}_e$ in order that the process is considered capable at least $100(1-\alpha)\%$ of the time.

By letting $\xi = (\mu - T)/\sigma$, we have $\delta = n(\mu - T)^2/\sigma^2 = n\xi^2$. The formula for calculating critical value c_0 can be written as $c_0 = [\chi_n^2(\alpha, n\xi^2) \cdot C]/[n(1 + \xi^2)]$, which is easy to understand and straightforward to apply. But, since the process measurement μ and σ must be estimated from the sampled data to obtain the characteristic parameter ξ, a great degree of uncertainty may be introduced to capability assessments due to sampling errors. Johnson (1992) suggested estimating μ and σ by $\bar{X}$ and S_n, respectively, to obtain the upper confidence limit $[(n + n\xi^2)/\chi_n^2(\alpha, n\xi^2)]\hat{L}_e$ (which is equivalent to our expression $U = L_{ot} + [n\hat{L}_e/\chi_n^2(\alpha, \delta)]$) for L_e. Such an approach introduces additional sampling errors from estimating ξ, and would be less reliable. Consequently, any decisions made would provide less quality assurance to the customers.

To eliminate the need for further estimating the characteristic parameter $\xi = (\mu - T)/\sigma$, we examine the sensitivity of the critical value c_0 against the parameter ξ. The results indicate that the critical value c_0 is increasing in ξ and reaches its minimum at $\xi = 0$ (hence $\mu = T$) in all cases. Figures 5–8 plot the curves of the critical value c_0 versus the parameter $\xi = 0(0.05)3.00$, $n = 30$, 50, 70, 100, 150, 200 with confidence level $\gamma = 0.95$, for $L_e = 0.03$, 0.04, 0.06 and 0.11, respectively. Hence, for practical purposes we may calculate the critical value c_0 by setting $\hat{\xi} = \xi = 0$ for given L_e, n and γ, without having to further estimate the parameter ξ. Thus, based on such an approach, the γ confidence level can be ensured and the decisions made are indeed more reliable.

Uniformly Most Powerful Test

For testing hypotheses about L_e, $H_0: L_e \geqslant C$ (*incapable*) versus $H_1: L_e < C$ (*capable*), we define a test as $\phi^*(x) = 1$ (reject H_0) if $\hat{L}_e < c_0$, and $\phi^*(x) = 0$ otherwise, is the uniformly most powerful (UMP) test of level α under $\xi = 0$ (hence $\mu = T$),

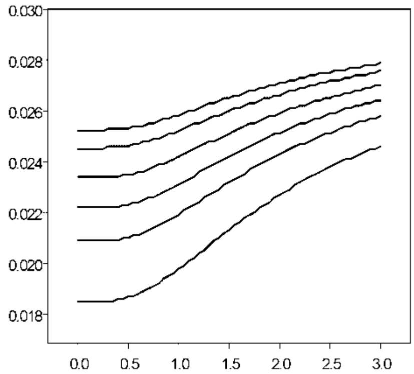

Figure 5. Plots of c_0 versus $|\xi|$ for $L_e = 0.03$, $n = 30$, 50, 70, 100, 150, 200 (bottom to top)

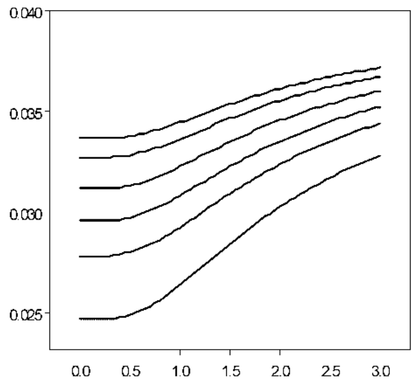

Figure 6. Plots of c_0 versus $|\xi|$ for $L_e = 0.04$, $n = 30$, 50, 70, 100, 150, 200 (bottom to top)

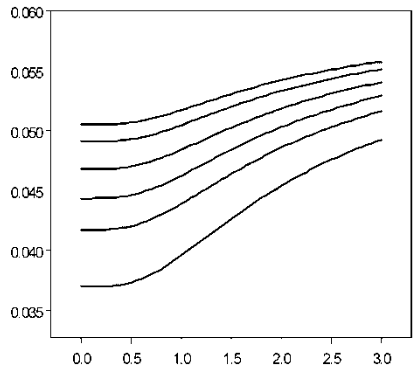

Figure 7. Plots of c_0 versus $|\xi|$ for $L_e = 0.06$, $n = 30$, 50, 70, 100, 150, 200 (bottom to top)

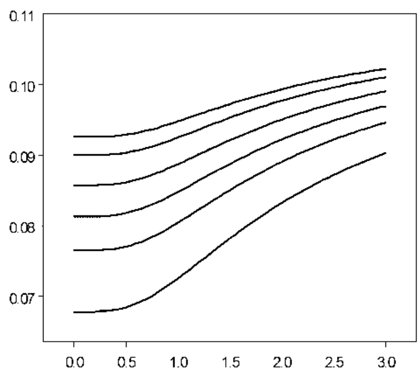

Figure 8. Plots of c_0 versus $|\xi|$ for $L_e = 0.11$, $n = 30$, 50, 70, 100, 150, 200 (bottom to top)

where c_0 is determined by $\mathrm{E}_C[\phi^*(X)] = \alpha$. The proof is shown as follows. For the test, the power function is:

$$\beta(L_e, \phi^*) = \mathrm{E}_{L_e}[\phi^*(X)] = \mathrm{P}_{L_e}[\chi_n^2 < (nc_0)/L_e] \tag{28}$$

For $\alpha(c_0) = \alpha$, $c_0 = [\chi_n^2(\alpha) \cdot C]/n$, where $\chi_n^2(\alpha)$ is the (lower) αth percentile of the χ_n^2 distribution. From the probability density function of $\hat{L}_e$, we define $\lambda(x)$ as:

$$\lambda(x) = f_{\hat{L}_e}(x, L_e')/f_{\hat{L}_e}(x, L_e) = (L_e/L_e')^{n/2} \exp\left[\frac{n}{2}\left(\frac{1}{L_e} - \frac{1}{L_e'}\right) \cdot x\right] \tag{29}$$

Since, for $L_e' > L_e > 0$, the ratio $\lambda(x)$ is an increasing function of x, then $\{f_{\hat{L}_e}(x, L_e) | L_e > 0\}$ has monotone likelihood ratio (MLR) property in L_e. Therefore, the test ϕ^* must be the UMP test.

Making Decisions

Tables 2(a), 3(a) and 4(a) give 90%, 95% and 99% upper confidence limits for L_e under $\mu = T$ with n given, and $\hat{L}_e$ calculated from the sample data. On the other hand, we note that $\hat{L}_e = \chi_n^2(\alpha, \delta)(U - L_{ot})/n$ depends on U and L_{ot}. In the special case where $\mu = T$ and U equals the recommended maximum value for L_e, the probability that $L_e \leqslant U$ would be either 1 or 0 if L_e were known. In practice, since L_e is unknown, we take a random sample of size n and calculate $\hat{L}_e$. Tables 2(b), 3(b) and 4(b) give critical values of $\hat{L}_e$ in the case $\mu = T$, for the process to be considered capable (i.e. $L_e \leqslant C$) 90%, 95% and 99% of the time. The following example illustrates the use of these tables. To determine whether the process meets the capability requirement, we first determine C, and the α-risk. Then, we calculate the estimator $\hat{L}_e$ from the sample. From the appropriate table, we find the critical value c_0 based on the α-risk, capability requirement C, and sample size n. If the estimated value $\hat{L}_e$ is less than the critical value c_0, then we conclude that the process meets the preset capability requirement.

Table 2(a). The 90% upper confidence limits for L_{pe} under $\mu = T$, with given $\hat{L}_{pe}$

	Sample size n													
$\hat{L}_{pe}$	5	10	15	20	25	30	35	40	45	50	55	60	65	70
0.03	0.0862	0.0571	0.0488	0.0446	0.0422	0.0405	0.0392	0.0382	0.0375	0.0369	0.0363	0.0359	0.0355	0.0351
0.04	0.1242	0.0822	0.0702	0.0643	0.0607	0.0583	0.0565	0.0551	0.0540	0.0531	0.0523	0.0517	0.0511	0.0506
0.05	0.1533	0.1015	0.0867	0.0794	0.0749	0.0719	0.0697	0.0680	0.0666	0.0655	0.0646	0.0638	0.0631	0.0625
0.06	0.1941	0.1285	0.1097	0.1005	0.0948	0.0910	0.0882	0.0861	0.0843	0.0829	0.0817	0.0807	0.0798	0.0791
0.11	0.3450	0.2284	0.1950	0.1786	0.1686	0.1618	0.1568	0.1530	0.1499	0.1474	0.1453	0.1435	0.1419	0.1406
	Sample size n													
$\hat{L}_{pe}$	80	90	100	110	120	130	140	150	160	170	180	190	200	300
0.03	0.0346	0.0341	0.0337	0.0334	0.0331	0.0329	0.0327	0.0325	0.0323	0.0322	0.0320	0.0319	0.0318	0.0310
0.04	0.0498	0.0491	0.0486	0.0481	0.0477	0.0474	0.0470	0.0468	0.0465	0.0463	0.0461	0.0459	0.0458	0.0446
0.05	0.0615	0.0606	0.0600	0.0594	0.0589	0.0585	0.0581	0.0577	0.0574	0.0572	0.0569	0.0567	0.0565	0.0551
0.06	0.0778	0.0767	0.0759	0.0752	0.0745	0.0740	0.0735	0.0731	0.0727	0.0724	0.0720	0.0718	0.0715	0.0697
0.11	0.1383	0.1364	0.1349	0.1336	0.1325	0.1315	0.1307	0.1299	0.1292	0.1286	0.1281	0.1276	0.1271	0.1239

Table 2(b). The critical value of $\hat{L}_{pe}$ under $\mu = T$ for which the process is capable 90% of the time

	Sample size n													
C	5	10	15	20	25	30	35	40	45	50	55	60	65	70
0.03	0.0089	0.0135	0.0158	0.0173	0.0183	0.0191	0.0197	0.0202	0.0206	0.0209	0.0212	0.0215	0.0217	0.0220
0.04	0.0129	0.0195	0.0228	0.0249	0.0264	0.0275	0.0283	0.0291	0.0296	0.0302	0.0306	0.0310	0.0313	0.0316
0.05	0.0159	0.0240	0.0281	0.0307	0.0325	0.0339	0.0350	0.0359	0.0366	0.0372	0.0378	0.0382	0.0387	0.0390
0.06	0.0201	0.0304	0.0356	0.0389	0.0412	0.0429	0.0443	0.0454	0.0463	0.0471	0.0478	0.0484	0.0489	0.0494
0.11	0.0358	0.0541	0.0633	0.0691	0.0732	0.0763	0.0787	0.0807	0.0823	0.0838	0.0850	0.0860	0.0870	0.0878
	Sample size n													
C	80	90	100	110	120	130	140	150	160	170	180	190	200	300
0.03	0.0223	0.0226	0.0229	0.0231	0.0233	0.0235	0.0236	0.0238	0.0239	0.0240	0.0241	0.0242	0.0243	0.0249
0.04	0.0321	0.0326	0.0329	0.0329	0.0335	0.0338	0.0340	0.0342	0.0344	0.0346	0.0347	0.0348	0.0350	0.0359
0.05	0.0397	0.0402	0.0407	0.0407	0.0414	0.0417	0.0420	0.0422	0.0425	0.0427	0.0428	0.0430	0.0432	0.0443
0.06	0.0502	0.0509	0.0515	0.0520	0.0524	0.0528	0.0531	0.0534	0.0537	0.0540	0.0542	0.0544	0.0546	0.0561
0.11	0.0893	0.0905	0.0915	0.0924	0.0932	0.0939	0.0945	0.0950	0.0955	0.0960	0.0964	0.0968	0.0971	0.0997

Table 3(a). The 95% upper confidence limits for L_{pe} under $\mu = T$, with given $\hat{L}_{pe}$

	Sample size n													
C	5	10	15	20	25	30	35	40	45	50	55	60	65	70
0.03	0.1212	0.0705	0.0574	0.0512	0.0475	0.0451	0.0433	0.0419	0.0408	0.0400	0.0392	0.0386	0.0381	0.0376
0.04	0.1746	0.1015	0.0826	0.0737	0.0684	0.0649	0.0623	0.0604	0.0588	0.0575	0.0565	0.0556	0.0548	0.0541
0.05	0.2156	0.1253	0.1020	0.0910	0.0845	0.0801	0.0769	0.0745	0.0726	0.0710	0.0679	0.0686	0.0676	0.0668
0.06	0.2728	0.1586	0.1291	0.1152	0.1069	0.1014	0.0974	0.0943	0.0919	0.0899	0.0882	0.0868	0.0856	0.0846
0.11	0.4850	0.2820	0.2295	0.2048	0.1901	0.1803	0.1731	0.1677	0.1633	0.1598	0.1569	0.1544	0.1522	0.1503
	Sample size n													
C	80	90	100	110	120	130	140	150	160	170	180	190	200	300
0.03	0.0368	0.0362	0.0356	0.0352	0.0348	0.0345	0.0342	0.0340	0.0337	0.0335	0.0333	0.0332	0.0330	0.0278
0.04	0.0530	0.0521	0.0513	0.0507	0.0502	0.0497	0.0493	0.0489	0.0486	0.0483	0.0480	0.0478	0.0475	0.0401
0.05	0.0654	0.0643	0.0634	0.0626	0.0619	0.0613	0.0608	0.0604	0.0600	0.0596	0.0593	0.0590	0.0587	0.0495
0.06	0.0828	0.0814	0.0802	0.0792	0.0784	0.0776	0.0770	0.0764	0.0759	0.0754	0.0750	0.0746	0.0743	0.0626
0.11	0.1472	0.1447	0.1426	0.1408	0.1393	0.1380	0.1369	0.1358	0.1349	0.1341	0.1334	0.1327	0.1321	0.1114

Table 3(b). The critical value of $\hat{L}_{pe}$ under $\mu = T$ for which the process is capable 95% of the time

	Sample size n													
C	5	10	15	20	25	30	35	40	45	50	55	60	65	70
0.03	0.0064	0.0109	0.0134	0.0151	0.0162	0.0171	0.0178	0.0184	0.0189	0.0193	0.0197	0.0200	0.0203	0.0205
0.04	0.0092	0.0158	0.0194	0.0217	0.0234	0.0247	0.0257	0.0265	0.0272	0.0278	0.0283	0.0288	0.0292	0.0296
0.05	0.0113	0.0195	0.0239	0.0268	0.0289	0.0304	0.0317	0.0327	0.0336	0.0343	0.0350	0.0355	0.0360	0.0365
0.06	0.0143	0.0246	0.0303	0.0339	0.0365	0.0385	0.0401	0.0414	0.0425	0.0435	0.0443	0.0450	0.0456	0.0462
0.11	0.0255	0.0438	0.0538	0.0603	0.0649	0.0685	0.0713	0.0736	0.0756	0.0773	0.0787	0.0800	0.0811	0.0821
	Sample size n													
C	80	90	100	110	120	130	140	150	160	170	180	190	200	300
0.03	0.0210	0.0213	0.0216	0.0219	0.0222	0.0224	0.0226	0.0227	0.0229	0.0230	0.0231	0.0233	0.0234	0.0242
0.04	0.0302	0.0307	0.0312	0.0316	0.0319	0.0322	0.0325	0.0327	0.0329	0.0331	0.0333	0.0335	0.0337	0.0348
0.05	0.0373	0.0379	0.0385	0.0390	0.0394	0.0398	0.0401	0.0404	0.0407	0.0409	0.0411	0.0414	0.0416	0.0429
0.06	0.0472	0.0480	0.0487	0.0493	0.0498	0.0503	0.0507	0.0511	0.0515	0.0518	0.0521	0.0523	0.0526	0.0543
0.11	0.0839	0.0853	0.0866	0.0877	0.0886	0.0895	0.0902	0.0909	0.0915	0.0921	0.0926	0.0930	0.0935	0.0966

Table 4(a). The 99% upper confidence limits for L_{pe} under $\mu = T$, with given $\hat{L}_{pe}$

	Sample size n													
$\hat{L}_{pe}$	5	10	15	20	25	30	35	40	45	50	55	60	65	70
0.03	0.2506	0.1086	0.0797	0.0673	0.0603	0.0557	0.0525	0.0501	0.0483	0.0468	0.0455	0.0445	0.0436	0.0428
0.04	0.3608	0.1564	0.1147	0.0968	0.0868	0.0802	0.0756	0.0722	0.0695	0.0673	0.0655	0.0640	0.0627	0.0616
0.05	0.4455	0.1930	0.1417	0.1196	0.1071	0.0991	0.0934	0.0891	0.0858	0.0831	0.0809	0.0790	0.0775	0.0761
0.06	0.5638	0.2443	0.1793	0.1513	0.1356	0.1254	0.1182	0.1128	0.1086	0.1052	0.1024	0.1000	0.0980	0.0963
0.11	1.0023	0.4343	0.3187	0.2690	0.2410	0.2229	0.2101	0.2005	0.1930	0.1870	0.1820	0.1778	0.1743	0.1721

	Sample size n													
$\hat{L}_{pe}$	80	90	100	110	120	130	140	150	160	170	180	190	200	300
0.03	0.0415	0.0405	0.0396	0.0389	0.0383	0.0378	0.0374	0.0370	0.0366	0.0363	0.0360	0.0358	0.0355	0.0339
0.04	0.0598	0.0583	0.0571	0.0561	0.0552	0.0545	0.0538	0.0533	0.0527	0.0523	0.0519	0.0515	0.0511	0.0488
0.05	0.0738	0.0720	0.0705	0.0692	0.0682	0.0673	0.0665	0.0657	0.0651	0.0645	0.0640	0.0636	0.0631	0.0602
0.06	0.0934	0.0911	0.0892	0.0876	0.0863	0.0851	0.0841	0.0832	0.0824	0.0817	0.0810	0.0804	0.0799	0.0762
0.11	0.1660	0.1619	0.1586	0.1558	0.1534	0.1513	0.1495	0.1479	0.1465	0.1452	0.1441	0.1430	0.1421	0.1355

Table 4(b). The critical value of $\hat{L}_{pe}$ under $\mu = T$ for which the process is capable 99% of the time

	Sample size n													
C	5	10	15	20	25	30	35	40	45	50	55	60	65	70
0.03	0.0031	0.0071	0.0097	0.0115	0.0128	0.0138	0.0147	0.0154	0.0160	0.0165	0.0170	0.0174	0.0177	0.0180
0.04	0.0044	0.0102	0.0139	0.0165	0.0184	0.0199	0.0212	0.0222	0.0230	0.0238	0.0244	0.0250	0.0255	0.0260
0.05	0.0055	0.0126	0.0172	0.0204	0.0228	0.0246	0.0261	0.0274	0.0284	0.0293	0.0301	0.0309	0.0315	0.0321
0.06	0.0069	0.0160	0.0218	0.0258	0.0288	0.0312	0.0331	0.0346	0.0360	0.0371	0.0381	0.0390	0.0398	0.0406
0.11	0.0123	0.0284	0.0387	0.0459	0.0512	0.0554	0.0588	0.0616	0.0640	0.0660	0.0678	0.0694	0.0708	0.0721
	Sample size n													
C	80	90	100	110	120	130	140	150	160	170	180	190	200	300
0.03	0.0186	0.0191	0.0195	0.0198	0.0201	0.0204	0.0206	0.0209	0.0211	0.0213	0.0214	0.0216	0.0217	0.0228
0.04	0.0268	0.0274	0.0280	0.0285	0.0290	0.0294	0.0297	0.0300	0.0303	0.0306	0.0308	0.0311	0.0313	0.0328
0.05	0.0330	0.0339	0.0346	0.0352	0.0358	0.0363	0.0367	0.0371	0.0375	0.0378	0.0381	0.0384	0.0386	0.0405
0.06	0.0418	0.0429	0.0438	0.0446	0.0453	0.0459	0.0464	0.0469	0.0474	0.0478	0.0482	0.0486	0.0489	0.0512
0.11	0.0744	0.0762	0.0778	0.0793	0.0805	0.0816	0.0826	0.0835	0.0843	0.0850	0.0857	0.0863	0.0869	0.0911

Table 5. Recommended estimator of the loss indices for different sample size

Loss Indices	Definition	UMVUE	MLE	Estimator Recommended
L_{pe}	$\left(\frac{\sigma}{d}\right)^2$	$\frac{S_{n-1}^2}{d^2}$	$\frac{S_n^2}{d^2}$	$n \leqslant 35$: MLE $n > 35$: Difference is negligible ($<0.52\%$)
L_{ot}	$\left(\frac{\mu - T}{d}\right)^2$	$\frac{(\bar{X}-T)^2}{d^2} - \frac{S_{n-1}^2}{nd^2}$	$\frac{(\bar{X}-T)^2}{d^2}$	$n \leqslant 30$: UMVUE $n > 30$: Difference is negligible ($<0.04\%$)
L_e	$\frac{\sigma^2 + (\mu - T)^2}{d^2}$	$\frac{S_n^2 + (\bar{X}-T)^2}{d^2}$	$\frac{S_n^2 + (\bar{X}-T)^2}{d^2}$	—

An Example of Testing L_e

A practice that is becoming increasingly common in industry is to require a supplier to demonstrate process capability as part of the contractual agreement. Suppose a customer has told his supplier that, in order to qualify for business with his company, the supplier must demonstrate that his process capability L_e is less than 0.06. This problem may be formulated as a hypothesis-testing problem:

$$H_0 : L_e \geqslant 0.06 \ (\textit{incapable})$$

$$H_1 : L_e < 0.06 \ (\textit{capable}).$$

In statistical hypothesis testing, rejection of H_0 is always a strong conclusion. The supplier would like to reject H_0, thereby demonstrating that his process is capable. Moreover, he wants to be sure that if the process capability is below 0.06 there will be a high probability of judging the process capable (say, 0.95). One takes a random sample of size n, and calculates the value of $\hat{L}_e$. Using Table 3(b) based on the random sample of size $n = 50$, for example, we obtain $c_0 = 0.0435$. Thus, if the calculated $\hat{L}_e \leqslant 0.0435$, then we claim that the process is capable at least 95% of the time, or equivalently, at the significant level $\alpha = 0.05$.

Conclusion

Johnson (1992) introduced the relative expected loss $L_e = L_{pe} + L_{ot}$, which provides an uncontaminated separation between information concerning the relative inconsistency loss (L_{pe}) and the relative off-target loss (L_{ot}). In this paper, we considered the three indices, and investigate the statistical properties of their natural estimators. For the three indices, we obtained their UMVUEs and MLEs. For each index, we compare the reliability of the two estimators based on their relative errors (square root of the relative mean squared error). We summarize the definitions of the process loss indices L_{pe}, L_{ot} and L_e, accompanied with different estimators corresponding to these indices (see Table 5). Which estimator should be preferred for what sample sizes is also suggested. In addition, we constructed 90%, 95% and 99% upper confidence limits, and the maximum values of $\hat{L}_e$ for which the process is capable. The results obtained in this paper are useful for practitioners in choosing good estimators and making reliable decisions on judging process capability.

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