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The super laceability of the hypercubes

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Abstract

A k-*container* $C(u, v)$ of a graph G is a set of k disjoint paths joining u to v. A k-container $C(u, v)$ is a k^*-*container* if every vertex of G is incident with a path in $C(u, v)$. A bipartite graph G is k^*-*laceable* if there exists a k^*-container between any two vertices u, v from different partite set of G. A bipartite graph G with connectivity k is *super laceable* if it is i^*-laceable for all $i \leqslant k$. A bipartite graph G with connectivity k is f-*edge fault-tolerant super laceable* if $G - F$ is i^*-laceable for any $1 \leqslant i \leqslant k - f$ and for any edge subset F with $|F| = f < k - 1$. In this paper, we prove that the hypercube graph Q_r is super laceable. Moreover, Q_r is f-edge fault-tolerant super laceable for any $f \leqslant r - 2$.

Keywords: Hypercube; Hamiltonian; Hamiltonian laceable; Connectivity; Fault tolerance

1. Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation we basically follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices a and b are *adjacent* if $(a, b) \in E$. Let E' be a subset of E. We use $G - E'$ to denote the graph with vertex set V and edge set $E - E'$. A *path* is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, in which all the vertices $v_0, v_1, \ldots, v_k$ are distinct except possibly $v_0 = v_k$. We also write the path $P = \langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, P, v_k\rangle$. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$. A path is a *Hamiltonian path* if its vertices are distinct and span V. A graph G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two vertices of G. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A *Hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. Let $G = (V, E)$ be a graph with connectivity $\kappa(G) = \kappa$. It follows from

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Menger's theorem [5] that there are l *internal node-disjoint paths* joining any two vertices u and v for $l \leqslant \kappa$. A *k-container* $C(u, v)$ of a graph G is a set of k internal node-disjoint paths joining u to v. We use $V(C(u, v))$ to denote the set of vertices incident with some path in $C(u, v)$. Connectivity and container are impotent concepts to measure the fault tolerance of a networks [3].

In this paper, we discuss another type of container. A k-container $C(u, v)$ is a *k^*-container* if $V(C(u, v)) = V(G)$. A graph G is *k^*-connected* if there exists a k^*-container between any two distinct vertices u, v. In particular, G is 1^*-connected if and only if it is Hamiltonian connected. Moreover, G is 2^*-connected if it is Hamiltonian. Since any 1^*-connected graph with more than 3 vertices is Hamiltonian, it is 2^*-connected. The study of k^*-connected graph is motivated by the globally 3^*-connected graphs proposed by Albert, Aldred and Holton [1]. We define a graph G to be *super connected* if G is i^*-connected for any i with $1 \leqslant i \leqslant \kappa(G)$.

A graph G is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 such that every edge joins a vertex of V_1 and a vertex of V_2. A *k^*-laceable* graph is a bipartite graph such that there exists a k^*-container between any two vertices from different partite sets. Obviously, any k^*-laceable graph with $k \geqslant 1$ has bipartition of equal size. A 1^*-laceable graph is also known as *Hamiltonian laceable graph*. Since any 1^*-laceable graph with more than 3 vertices is Hamiltonian, it is 2^*-laceable. A bipartite graph G is *super laceable* if G is i^*-laceable for any i with $1 \leqslant i \leqslant \kappa(G)$.

In this paper, we prove that the hypercube Q_r is super laceable for all r. We further discuss the corresponding fault-tolerant property. Assume that $r \geqslant 2$. Let $F \subseteq E(Q_r)$ with $|F| = f \leqslant (r-2)$. Obviously, $\kappa(Q_r - F) \geqslant r - f$. We prove that $Q_r - F$ is i^*-laceable for any i with $1 \leqslant i \leqslant r - f$.

Let G be super laceable graph with connectivity k and let F be any edge subset of G with $|F| = f$. We say that G is *f-edge fault-tolerant super laceable* if $G - F$ is i^*-laceable for any i with $1 \leqslant i \leqslant r - f$ and for any edge subset F with $|F| = f$. The *edge fault-tolerant super laceability* of G is defined as the largest f such that G is f edge fault-tolerant super laceable. Let F be any edge set incident with some vertex x in G such that $|F| = r - 1$. Obviously, $deg_{G-F}(x) = 1$ and there is no Hamiltonian path joining any two vertices u and v with $x \notin \{u, v\}$. Hence $G - F$ is not Hamiltonian laceable. Thus the edge fault-tolerant super laceability is at most $r - 2$. We shall prove that the edge fault-tolerant super laceability of Q_r is $r - 2$ if $r \geqslant 2$.

In the following, we give the definition of the hypercubes Q_r and some basic properties of Q_r. In Section 3, we prove that Q_r is r^*-laceable. In Section 4, we prove that the edge fault-tolerant super laceability of Q_r is $r - 2$ if $r \geqslant 2$. In particular, Q_r is super laceable for any r.

2. Preliminaries

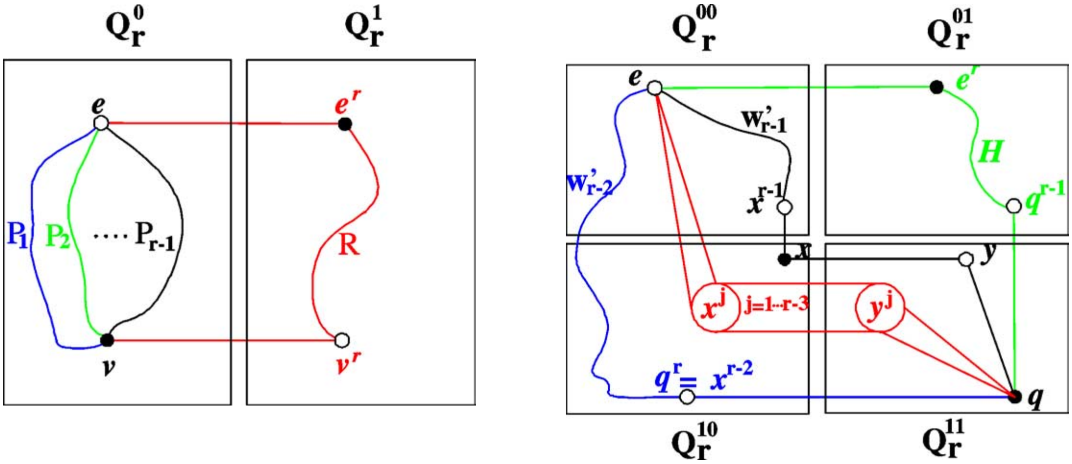
Let $\mathbf{u} = u_1u_2 \ldots u_{r-1}u_r$ be an r-bit binary strings. For $1 \leqslant i \leqslant r$, we use $\mathbf{u}^i$ to denote the *ith neighbor* of $\mathbf{u}$, i.e., the binary string $v_1v_2 \ldots v_{r-1}v_r$ such that $v_i = 1 - u_i$ and $v_k = u_k$ if $k \neq i$. The *Hamming weight* of $\mathbf{u}$, denoted by $w(\mathbf{u})$, is the number of i such that $u_i = 1$. The *r-dimensional hypercube*, denoted by Q_r, consists of all r-bit binary strings as its vertices. Two vertices $\mathbf{u}$ and $\mathbf{v}$ are adjacent if and only if $\mathbf{v} = \mathbf{u}^i$ for some i. Obviously, Q_r is an r-regular graphs with 2^r vertices. Moreover, Q_r is a bipartite graph with bipartition $\{\mathbf{u} \mid w(\mathbf{u}) \text{ is odd}\}$ and $\{\mathbf{u} \mid w(\mathbf{u}) \text{ is even}\}$. We will use black vertices to denote those vertices of odd weight and white vertices to denote those vertices of even weight. We set Q_r^i be the subgraph of Q_r induced by $\{\mathbf{u} \in V(Q_r) \mid u_r = i\}$ for $i = 0, 1$. Obviously, Q_r^i is isomorphic to Q_{r-1} for $i = 0, 1$. It is well known that Q_r is vertex transitive. Furthermore, the permutation on the coordinate of Q_r and the componentwise complement operations are graph isomorphisms. Let $\mathbf{e} = 00 \ldots 000 \in V(Q_r)$ and $q = 11 \ldots 111$ be the antipodal point of $\mathbf{e}$.

The topological properties of Q_r has been studied extensively in recent years. Readers can refer [4] for a survey on the properties of hypercubes. The following theorem is proven by Tsai et al. [6].

Theorem 1 [6]. *Let F be any edge subset of Q_r with $|F| \leqslant r - 2$. Then $Q_r - F$ is 1^*-laceable.*

3. Q_r is r^*-laceable

Theorem 2. *Q_r is r^*-laceable for $r \geqslant 1$.*



(a) (b)

Fig. 1. Illustration for Theorem 2.

Proof. Obviously, this theorem is true for $r = 1$ and $r = 2$. Assume $r \geqslant 3$. Since Q_r is vertex transitive, we only need to find an r^*-container joining $\mathbf{e}$ to any black vertex $\mathbf{v}$ of Q_r.

Case 1: $\mathbf{v} \neq \mathbf{q}$. Without loss of generality, we assume that $\mathbf{v} \in Q_r^0$. By induction, there exists an $(r-1)^*$-container $\{P_1, P_2, \ldots, P_{r-1}\}$ of Q_r^0 joining $\mathbf{e}$ to $\mathbf{v}$. By Theorem 1, there exists a Hamiltonian path R of Q_r^1 joining the black vertex $\mathbf{e}^r$ to the white vertex $\mathbf{v}^r$. We set P_r as $\langle \mathbf{e}, \mathbf{e}^r, R, \mathbf{v}^r, \mathbf{v}\rangle$. Obviously, $\{P_1, P_2, \ldots, P_{r-1}, P_r\}$ is an r^*-container of Q_r joining $\mathbf{e}$ to $\mathbf{v}$. See Fig. 1(a) for illustration.

Case 2: $\mathbf{v} = \mathbf{q}$. Since $w(\mathbf{v})$ is odd, r is odd and $r \geqslant 3$. Let Q_r^{ij} be the subgraph of Q_r induced by $\{\mathbf{u} \in V(Q_r) \mid u_{r-1} = i \text{ and } u_r = j\}$ for $0 \leqslant i, j \leqslant 1$. Obviously, Q_r^{ij} is isomorphic to Q_{r-2} for $0 \leqslant i, j \leqslant 1$. Let $\mathbf{y} = \mathbf{q}^{r-2} \in Q_r^{11}$ be the $(r-2)$th neighbor of $\mathbf{q}$. Let $\mathbf{x} = \mathbf{y}^r$ be the rth neighbor of $\mathbf{y}$. Obviously, $\mathbf{x}^k$ is adjacent to $\mathbf{y}^k$ for all $1 \leqslant k \leqslant r-3$ and $\mathbf{q}^r = \mathbf{x}^{r-2}$. By induction, there exists an $(r-1)^*$-container $\{W_1, W_2, \ldots, W_{r-1}\}$ of $Q_r^{00} \cup Q_r^{10} = Q_r^0$ joining $\mathbf{e}$ to the black vertex $\mathbf{x}$ where $W_k = \langle \mathbf{e}, W'_k, \mathbf{x}^k, \mathbf{x}\rangle$ for $1 \leqslant k \leqslant r-1$. There exists an $(r-2)^*$-container $\{R_1, R_2, \ldots, R_{r-2}\}$ of Q_r^{11} joining the white vertex $\mathbf{y}$ to $\mathbf{q}$. Since $\mathbf{y}$ and $\mathbf{q}$ are adjacent, one of these paths is the $\langle \mathbf{y}, \mathbf{q}\rangle$. Without loss of generality, we assume that $R_k = \langle \mathbf{y}, \mathbf{y}^k, R'_k, \mathbf{q}\rangle$ for $1 \leqslant k \leqslant r-3$. We set $P_k = \langle \mathbf{e}, W'_k, \mathbf{x}^k, \mathbf{y}^k, R'_k, \mathbf{q}\rangle$ for $1 \leqslant k \leqslant r-3$, $P_{r-2} = \langle \mathbf{e}, W'_{r-2}, \mathbf{q}^r, \mathbf{q}\rangle$, and $P_{r-1} = \langle \mathbf{e}, W'_{r-1}, x^{r-1}, \mathbf{x}, \mathbf{y}, \mathbf{q}\rangle$. By Theorem 1, there exists a Hamiltonian path H of Q_r^{01} joining the black vertex $\mathbf{e}^r$ to the white vertex $\mathbf{q}^{r-1}$. Let $P_r = \langle \mathbf{e}, \mathbf{e}^r, H, \mathbf{q}^{r-1}, \mathbf{q}\rangle$. Obviously, $\{P_1, P_2, \ldots, P_r\}$ form an r^*-container of Q_r joining $\mathbf{e}$ to $\mathbf{q}$. See Fig. 1(b) for illustration. □

4. The edge fault-tolerant super laceability of Q_r

Lemma 1. *Assume that $r \geqslant 2$. Let $\mathbf{u}$ and $\mathbf{x}$ be any two distinct white vertices, $\mathbf{v}$ and $\mathbf{y}$ be any two distinct black vertices in Q_r, there exist two disjoint paths P_1 and P_2 such that*

(1) *P_1 joins $\mathbf{u}$ to $\mathbf{v}$.*
(2) *P_2 joins $\mathbf{x}$ to $\mathbf{y}$, and*
(3) *$P_1 \cup P_2$ spans Q_r.*

Proof. We prove this lemma by induction on r. Obviously, the lemma is true for $r = 2$. Assume that $r \geqslant 3$. Without loss of generality, we assume that $\mathbf{u} \in V(Q_r^0)$ and $\mathbf{y} \in V(Q_r^1)$.

Case 1: $\mathbf{x} \in V(Q_r^1)$ and $\mathbf{v} \in V(Q_r^0)$. By Theorem 1, we can set P_1 as the Hamiltonian path of Q_r^0 joining $\mathbf{u}$ to $\mathbf{v}$. Again, we can set P_2 as the Hamiltonian path of Q_r^1 joining $\mathbf{x}$ to $\mathbf{y}$.

Case 2: $\mathbf{x} \in V(Q_r^0)$ and $\mathbf{v} \in V(Q_r^1)$. By induction, there exist two disjoint paths R_1 and R_2 of Q_r^0 such that

(1) R_1 joins $\mathbf{u}$ to some black vertex $\mathbf{a}$,

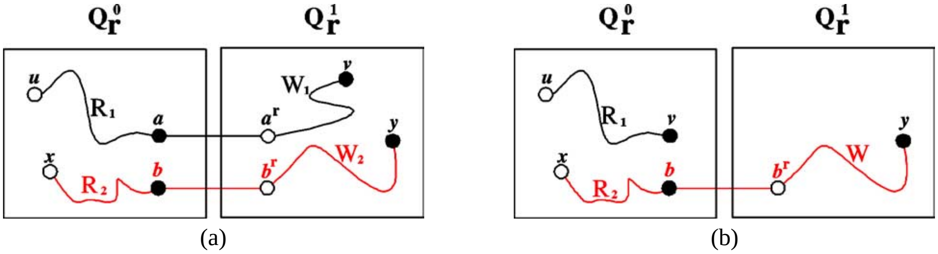


Fig. 2. Illustration for Lemma 1.

(2) R_2 joins $\mathbf{x}$ to some black vertex $\mathbf{b}$ distinct from $\mathbf{a}$, and
(3) $R_1 \cup R_2$ spans Q_r^0.

Again, there exist two disjoint paths W_1 and W_2 of Q_r^1 such that

(1) W_1 joins the white vertex $\mathbf{a}^r$ to $\mathbf{v}$,
(2) W_2 joins the white vertex $\mathbf{b}^r$ to $\mathbf{y}$, and
(3) $W_1 \cup W_2$ spans Q_r^1.

We can set P_1 as $\langle \mathbf{u}, R_1, \mathbf{a}, \mathbf{a}^r, W_1, \mathbf{v}\rangle$ and P_2 as $\langle \mathbf{x}, R_2, \mathbf{b}, \mathbf{b}^r, W_2, \mathbf{y}\rangle$. See Fig. 2(a) for illustration.

Case 3: $\mathbf{x}, \mathbf{v} \in V(Q_r^0)$ or $V(Q_r^1)$. Without loss of generality, we assume that $\mathbf{x}, \mathbf{v} \in V(Q_r^0)$. By induction, there exist two disjoint paths R_1 and R_2 of Q_r^0 such that

(1) R_1 joins $\mathbf{u}$ to $\mathbf{v}$,
(2) R_2 joins $\mathbf{x}$ to some black vertex $\mathbf{b}$, and
(3) $R_1 \cup R_2$ spans Q_r^0.

By Theorem 1, there exists a Hamiltonian path W joining the white vertex $\mathbf{b}^r$ to $\mathbf{y}$. We set P_1 as R_1 and set P_2 as $\langle \mathbf{x}, R_2, \mathbf{b}, \mathbf{b}^r, W, \mathbf{y}\rangle$. See Fig. 2(b) for illustration.

Obviously, P_1 and P_2 are the desired paths. □

Theorem 3. *Let F be any edge subset of Q_r with $|F| = f \leqslant r-2$ and $r \geqslant 2$. Then $Q_r - F$ is $(r-f)^*$-laceable.*

Proof. It is easy checked by brute force that this theorem is true for $r = 2, 3$. Suppose $f = r-2$. By Theorem 1, $Q_r - F$ is 1^*-laceable. Since any 1^*-laceable graph G with $|V(G)| \geqslant 3$ is 2^*-laceable, $Q_r - F$ is also 2^*-laceable. Suppose $f = 0$. By Theorem 2, Q_r is r^*-laceable. Hence we only need to prove the theorem for $1 \leqslant f \leqslant r-3$. Since Q_r is vertex transitive, we only need to find an $(r-f)^*$-container of $Q_r - F$ between the white vertex $\mathbf{e}$ and any black vertex $\mathbf{v}$. We prove our claims according to the locations of v and the faulty edges as follows.

Case 1: $\mathbf{v} \neq \mathbf{q}$. Without loss of generality, we assume that $\mathbf{v} \in Q_r^0$. Let F^i denote the set of edges of F in Q_r^i for $0 \leqslant i \leqslant 1$. Since $f \leqslant r-3 = (r-1)-2$, by induction there exists an $(r-f)^*$-container $C = \{P_1, P_2, \ldots, P_{r-f}\}$ of $Q_r^0 - F^0$ joining $\mathbf{e}$ to $\mathbf{v}$. An edge $(\mathbf{a}, \mathbf{b})$ is an *adjacent pair* in C if $(\mathbf{a}, \mathbf{b}) \in P_k$ for some k. An adjacent pair $(\mathbf{a}, \mathbf{b})$ is *healthy* if $(\mathbf{a}, \mathbf{a}^r)$ and $(\mathbf{b}, \mathbf{b}^r) \notin F$. A faulty edge from Q_r^0 to Q_r^1 can destroy at most 2-healthy pairs. Since there are at least 2^{r-1} adjacent pairs in C and $2^{r-1} > 2(r-3) \geqslant 2|F|$ for $r \geqslant 4$, there exists at least one healthy pair $(\mathbf{a}, \mathbf{b})$ in C. Without loss of generality, we may assume that $(\mathbf{a}, \mathbf{b}) \in P_{r-f}$ and write $P_{r-f} = \langle \mathbf{e}, R_1, \mathbf{a}, \mathbf{b}, R_2, \mathbf{v}\rangle$. By Theorem 1, there exists a Hamiltonian path W of $Q_r^1 - F^1$ joining $\mathbf{a}^r$ to $\mathbf{b}^r$. We set $P'_{r-f} = \langle \mathbf{e}, R_1, \mathbf{a}, \mathbf{a}^r, W, \mathbf{b}^r, \mathbf{b}, R_2, \mathbf{v}\rangle$. Obviously, $\{P_1, P_2, \ldots, P_{r-f-1}, P'_{r-f}\}$ is an $(r-f)^*$-container of $Q_r - F$ joining $\mathbf{e}$ to $\mathbf{v}$. See Fig. 3(a) for illustration.

Case 2: $\mathbf{v} = \mathbf{q}$. Since $\mathbf{v}$ is a black vertex, we may assume r is an odd with $r \geqslant 5$. An edge $(\mathbf{x}, \mathbf{y})$ is an i-*dimensional* edge if $\mathbf{y} = \mathbf{x}^i$. Let F_i denote the subsets in F of dimension i. Without loss of generality, we assume that $|F_1| \geqslant |F_2| \geqslant \cdots \geqslant |F_r|$. Since $|F| = f \leqslant r-3$, $|F_r| = |F_{r-1}| = 0$. Let F^{ij} be the subsets of F in Q_r^{ij}. Since Q_r^{10} and Q_r^{01} are symmetric with respect to Q_r and $F^{00} \cup F^{10} \cup F^{01} \cup F^{11} = F$, we may assume that $|F^{10}| \leqslant \lfloor f/2 \rfloor$.

Let $\mathbf{u} = u_1u_2 \ldots u_{r-1}u_r \in V(Q_r)$. We use $\mathbf{u}[i, j]$ to denote $u_1u_2 \ldots u_{r-3}u_{r-2}ij$. In other words, $\mathbf{u}[i, j]$ is the *mirror image* of $\mathbf{u}[i', j']$ if $i \neq i'$ or $j \neq j'$.

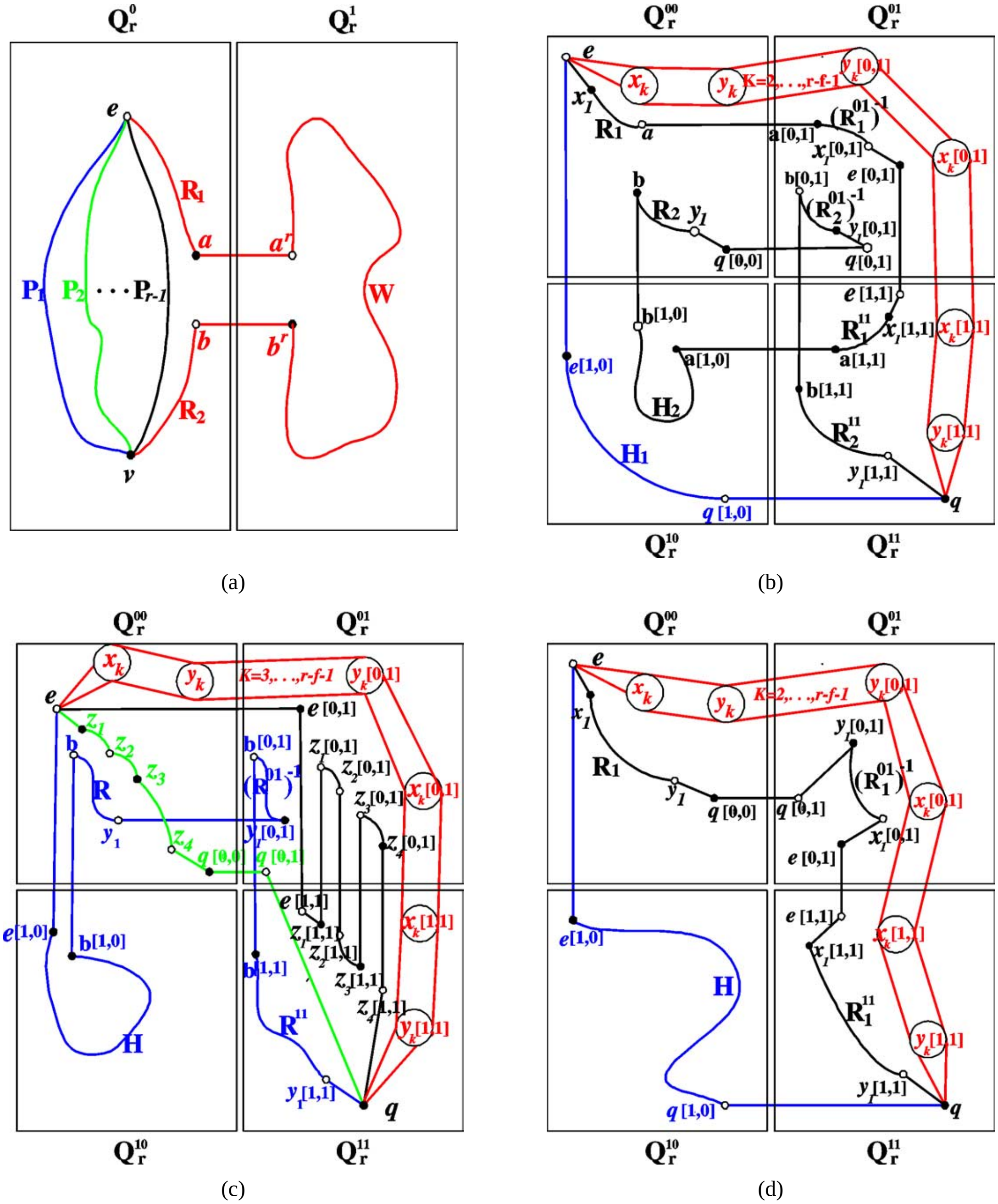


Fig. 3. Illustration for Theorem 3.

Case 2.1: $|F^{10}| = 0$. Pick any element $(\mathbf{a}', \mathbf{b}')$ in F and set $\widetilde{F} = F - (\mathbf{a}', \mathbf{b}')$. Let $\widetilde{F}^{ij}$ be the subsets of $\widetilde{F}$ in Q_r^{ij}. Obviously, $|\widetilde{F}^{00}| \leqslant f - 1$. Since $\dim(Q_r^{00}) = r - 2$ and $(r-2)-(f-1) = r - f - 1 \geqslant 2$, by induction there exists an $(r-f-1)^*$-container $C = \{W_1, W_2, \ldots, W_{r-f-1}\}$ of $Q_{r-2}^{00} - \widetilde{F}^{00}$ joining $\mathbf{e}$ to $\mathbf{q}[0,0]$. Suppose the edge $(\mathbf{a}', \mathbf{b}')$ or its mirror image is in C. We set $\mathbf{a} = \mathbf{a}'[0,0]$, $\mathbf{b} = \mathbf{b}'[0,0]$ and assume

$(\mathbf{a}, \mathbf{b}) \in E(W_1)$. Suppose the edge $(\mathbf{a}', \mathbf{b}')$ or its mirror image is not in C. We pick any edge in $E(W_1)$ and name it $(\mathbf{a}, \mathbf{b})$.

Suppose $\mathbf{a} \neq \mathbf{e}[0,0]$ and $\mathbf{b} \neq \mathbf{q}[0,0]$. We can write $W_1 = \langle \mathbf{e}, \mathbf{x}_1, R_1, \mathbf{a}, \mathbf{b}, R_2, \mathbf{y}_1, \mathbf{q}[0,0]\rangle$, $W_k = \langle \mathbf{e}, \mathbf{x}_k, Z_k, \mathbf{y}_k, \mathbf{q}[0,0]\rangle$ for $2 \leqslant k \leqslant r-f-1$. Let W_k^{01} and W_k^{11} be the mirror images of W_1 in Q_r^{01} and Q_r^{11}, respectively. Obviously, $C^{01}(\mathbf{e}[0,1], \mathbf{q}[0,1]) = \{W_1^{01}, W_2^{01}, \ldots, W_{r-f-1}^{01}\}$ and $C^{11}(\mathbf{e}[1,1], \mathbf{q}[1,1]) = \{W_1^{11}, W_2^{11}, \ldots, W_{r-f-1}^{11}\}$ are $(r-f-1)^*$-containers of $Q_{r-2}^{01} - \widetilde{F}^{01}$ and $Q_{r-2}^{11} - \widetilde{F}^{11}$, respectively. By Lemma 1, there exist two disjoint paths H_1 and H_2 of Q_r^{10} such that

(1) H_1 joins the black vertex $\mathbf{e}[1,0]$ to the white vertex $\mathbf{q}[1,0]$,
(2) H_2 joins $\mathbf{a}[1,0]$ to $\mathbf{b}[1,0]$, and
(3) $H_1 \cup H_2$ spans Q_r^{10}.

We set $\{P_1, P_2, \ldots, P_{r-f}\}$ as

$$\begin{aligned} P_1 = \big\langle & \mathbf{e}, \mathbf{x}_1, R_1, \mathbf{a}, \mathbf{a}[0,1], \big(R_1^{01}\big)^{-1}, \mathbf{x}_1[0,1], \mathbf{e}[0,1], \\ & \mathbf{e}[1,1], \mathbf{x}_1[1,1], R_1^{11}, \mathbf{a}[1,1], \mathbf{a}[1,0], H_2, \\ & \mathbf{b}[1,0], \mathbf{b}, R_2, \mathbf{y}_1, \mathbf{q}[0,0], \mathbf{q}[0,1], \mathbf{y}_1[0,1], \\ & \big(R_2^{01}\big)^{-1}, \mathbf{b}[0,1], \mathbf{b}[1,1], R_2^{11}, \mathbf{y}_1[1,1], \mathbf{q}\big\rangle; \end{aligned}$$

$$\begin{aligned} P_k = \big\langle & \mathbf{e}, \mathbf{x}_k, Z_k, \mathbf{y}_k, \mathbf{y}_k[0,1], \big(Z_k[0,1]\big)^{-1}, \\ & \mathbf{x}_k[0,1], \mathbf{x}_k[1,1], Z_k[1,1], \mathbf{y}_k[1,1], \mathbf{q}\big\rangle \end{aligned}$$

for $2 \leqslant k \leqslant r-f-1$; and

$$P_{r-f} = \big\langle \mathbf{e}, \mathbf{e}[1,0], H_1, \mathbf{q}[1,0], \mathbf{q}\big\rangle.$$

Obviously, $\{P_1, P_2, \ldots, P_{r-f}\}$ is an $(r-f)^*$-container of $Q_r - F$ joining $\mathbf{e}$ to $\mathbf{q}$. See Fig. 3(b) for illustration.

Remark 1. Suppose $\mathbf{a} = \mathbf{e}$. Since $r \geqslant 5$, W_i contains at least 4 vertices for $1 \leqslant i \leqslant r-f-1$. We write $W_1 = \langle \mathbf{e}, \mathbf{b}, R, \mathbf{y}_1, \mathbf{q}[0,0]\rangle$, $W_2 = \langle \mathbf{e}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{2t-1}, \mathbf{z}_{2t}, \mathbf{q}[0,0]\rangle$ for some $t \geqslant 1$, and $W_k = \langle \mathbf{e}, \mathbf{x}_k, Z_k, \mathbf{y}_k, \mathbf{q}[0,0]\rangle$ for $3 \leqslant k \leqslant r-f-1$. By Theorem 1, there exists a Hamiltonian path H of Q_r^{10} joining the black vertex $\mathbf{e}[1,0]$ to the white vertex $\mathbf{b}[1,0]$. We revise the previously paths P_1, P_2 and P_{r-f} as follows:

$$P_1 = \big\langle \mathbf{e}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{2t-1}, \mathbf{z}_{2t}, \mathbf{q}[0,0], \mathbf{q}[0,1], \mathbf{q}\big\rangle;$$

$$\begin{aligned} P_2 = \big\langle & \mathbf{e}, \mathbf{e}[0,1], \mathbf{e}[1,1], \mathbf{z}_1[1,1], \mathbf{z}_1[0,1], \mathbf{z}_2[0,1], \\ & \mathbf{z}_2[1,1], \mathbf{z}_3[1,1], \mathbf{z}_3[0,1], \mathbf{z}_4[0,1], \ldots, \\ & \mathbf{z}_{2t-1}[1,1], \mathbf{z}_{2t-1}[0,1], \mathbf{z}_{2t}[0,1], \\ & \mathbf{z}_{2t}[1,1], \mathbf{q}\big\rangle; \quad \text{and} \end{aligned}$$

$$\begin{aligned} P_{r-f} = \big\langle & \mathbf{e}, \mathbf{e}[1,0], H, \mathbf{b}[1,0], \mathbf{b}, R, \mathbf{y}_1, \mathbf{y}_1[0,1], \\ & \big(R^{01}\big)^{-1}, \mathbf{b}[0,1], \mathbf{b}[1,1], R^{11}, \mathbf{y}_1[1,1], \mathbf{q}\big\rangle. \end{aligned}$$

Obviously, $\{P_1, P_2, \ldots, P_{r-f}\}$ is an $(r-f)^*$-container of $Q_r - F$ joining $\mathbf{e}$ to $\mathbf{q}$. See Fig. 3(c) for illustration $t = 2$.

Remark 2. Suppose $\mathbf{b} = \mathbf{q}[0,0]$. Since $\mathbf{q}[0,0]$ is symmetric with respect to $\mathbf{e}$, we use the similar reason to find an $(r-f)^*$-container of $Q_r - F$ joining $\mathbf{e}$ to $\mathbf{q}$.

Case 2.2: $|F^{10}| \neq 0$. Obviously, $|F^{00}| \leqslant f-1$. By induction, there exists an $(r-f-1)^*$-container $C = \{W_1, W_2, \ldots, W_{r-f-1}\}$ of $Q_r^{00} - F^{00}$ joining $\mathbf{e}$ to $\mathbf{q}[0,0]$ and we can write $W_k = \langle \mathbf{e}, \mathbf{x}_k, R_k, \mathbf{y}_k, \mathbf{q}[0,0]\rangle$ for $1 \leqslant k \leqslant r-f-1$. By Theorem 1, there exists a Hamiltonian path H of $Q_r^{10} - F^{10}$ joining the black vertex $\mathbf{e}[1,0]$ to the white vertex $\mathbf{q}[1,0]$. We revise the previously defined paths P_1 and P_{r-f} as follows:

$$\begin{aligned} P_1 = \big\langle & \mathbf{e}, \mathbf{x}_1, R_1, \mathbf{y}_1, \mathbf{q}[0,0], \mathbf{q}[0,1], \mathbf{y}_1[0,1], \\ & \big(R_1^{01}\big)^{-1}, \mathbf{x}_1[0,1], \mathbf{e}[0,1], \mathbf{e}[1,1], \mathbf{x}_1[1,1], R_1^{11}, \\ & \mathbf{y}_1[1,1], \mathbf{q}\big\rangle; \quad \text{and} \end{aligned}$$

$$P_{r-f} = \big\langle \mathbf{e}, \mathbf{e}[1,0], H, \mathbf{q}[1,0], \mathbf{q}\big\rangle.$$

Obviously, $\{P_1, P_2, \ldots, P_{r-f}\}$ is an $(r-f)^*$-container of $Q_r - F$ joining $\mathbf{e}$ to $\mathbf{q}$. See Fig. 3(d) for illustration. □

Theorem 4. *Q_r is super laceable for any positive integer r. Moreover, Q_r is f-edge fault-tolerant super laceable for any $f \leqslant r-2$.*

Proof. By Theorem 1, Q_r is 1^*-laceable. By Theorem 2, Q_r is r^*-laceable. Assume that $r \geqslant 3$ and $2 \leqslant i \leqslant r-1$. We arbitrarily choose an faulty edge set F with $|F| = r-i$. By Theorem 3, $Q_r - F$ is i^*-laceable. Thus Q_r is i^*-laceable.

Therefore, Q_r is super laceable for any positive integer r. For similar reason, Q_r is f-edge fault-tolerant super laceable for any $f \leqslant r-2$. □

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