J. Differential Equations 250 (2011) 799–812

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Topological dynamics for multidimensional perturbations of maps with covering relations and Liapunov condition ☆

Ming-Chia Li *, Ming-Jiea Lyu

Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan

ARTICLE INFO

Article history:
Received 27 February 2010
Available online 21 July 2010

Keywords:
Topological dynamics
Multidimensional perturbation
Covering relation
Liapunov condition

ABSTRACT

In this paper, we study topological dynamics of high-dimensional systems which are perturbed from a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ of the form $(f(x), g(x, y))$. Assume that f has covering relations determined by a transition matrix A. If g is locally trapping, we show that any small C^0 perturbed system has a compact positively invariant set restricted to which the system is topologically semi-conjugate to the one-sided subshift of finite type induced by A. In addition, if the covering relations satisfy a strong Liapunov condition and g is a contraction, we show that any small C^1 perturbed homeomorphism has a compact invariant set restricted to which the system is topologically conjugate to the two-sided subshift of finite type induced by A. Some other results about multidimensional perturbations of f are also obtained. The strong Liapunov condition for covering relations is adapted with modification from the cone condition in Zgliczyński (2009) [11]. Our results extend those in Juang et al. (2008) [1], Li et al. (2008) [2], Li and Malkin (2006) [3], Misiurewicz and Zgliczyński (2001) [4] by considering a larger class of maps f and their multidimensional perturbations, and by concluding conjugacy rather than entropy. Our results are applicable to both the logistic and Hénon families.

☆ Partially supported NSC 99-2115-M-009-004-MY2.

* Corresponding author. Fax: +866 3 5131223.
E-mail address: mcli@math.nctu.edu.tw (M.-C. Li).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.06.019

1. Introduction

Following [2], we study perturbations from a continuous map f on a phase space, say $\mathbb{R}^m$, to continuous maps G on a high-dimensional space, say $\mathbb{R}^m \times \mathbb{R}^k$ or $\mathbb{R}^m$ such that G is a small perturbation of the singular map F which is one of the following forms:

(i) $F(x) = f(x) \in \mathbb{R}^m$;
(ii) $F(x, y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^k$;
(iii) $F(x, y) = (f(x), g(x, y)) \in \mathbb{R}^m \times \mathbb{R}^k$ and $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^k$ homeomorphic to the closed unit ball in $\mathbb{R}^k$, where $int(S)$ denotes the interior of S; and
(iv) $F(x, y) = (f(x), g(y)) \in \mathbb{R}^m \times \mathbb{R}^k$, where g is a contraction on the closed unit ball in $\mathbb{R}^k$ and has the unique fixed point in the interior of the unit ball.

The maps G in cases (ii)–(iv) are considered as multidimensional perturbations of f due to bigger dimension of phase space, while G in case (i) is a usual perturbation of f and they have the same phase space. The singular maps F in cases (ii)–(iv) can be considered as the skew product $(f(x), q(x, y))$ with different strength on trapping region of $q(x, y)$: vertical contraction $q(x, y) = g(x)$ for case (ii), locally trapping $q(\mathbb{R}^m \times S) \subset int(S)$ for case (iii), and horizontal contraction for $q(x, y) = g(y)$.

Consider a one-parameter family G_λ of continuous maps such that $G_0 = F$. If f is an interval map with positive topological entropy, it was showed that for all λ close to 0, the map G_λ has positive topological entropy, in [4] for case (ii) with $g = 0$, and in [2] for cases (ii) and (iii). If f has a snap-back repeller, the same result about G_λ is also concluded in [2] for cases (ii) and (iii).

In this paper, we assume that f has covering relations determined by a transition matrix A (see Definition 3) and show that for cases (i)–(iii), if G is C^0 close to F, then G has an isolated invariant set to which the restriction G is topologically semi-conjugate to the one-sided subshift of finite type σ_A^+ and hence the topological entropy of G is greater than the logarithm of the spectral radius of A (see Theorems 4–6). In addition, if the covering relations satisfy the strong Liapunov condition (see Definition 8), then we conclude that if a homeomorphism G is C^1 close to F, then G has an isolated invariant set to which the restriction of G is topologically conjugate to σ_A for cases (i) and (iv) provided that F is a homeomorphism (see Theorems 10 and 11), and for case (ii) provided that G is perturbed from F along a one-parameter continuous family $\{F_\lambda\}$ such that $F = F_0$ and $G = F_\lambda$ with small $|\lambda| \neq 0$ (see Theorem 12). In particular, one can apply the last result to the Hénon-like family $F_\lambda(x, y) = (f(x) + p(\lambda, x, y), q(\lambda, x, y))$, where f is the logistic map $f(x) = \mu x(1 - x)$ with $\mu > 4$, p and q are C^1 continuous functions of (λ, x, y) such that F_λ is a homeomorphism for $\lambda \neq 0$, and $h(0, x, y) = 0$ for all (x, y) and $q(0, x, y_1) = q(0, x, y_2)$ for all x, y_1 and y_2. The map f has covering relations which are determined by the 2×2 matrix with all entries one and satisfy the strong Liapunov condition (see Example 9). Thus for sufficiently small $|\lambda| \neq 0$, the map F_λ has an isolated invariant set on which F_λ is topologically conjugate to the 2-shift. By setting $p(\lambda, x, y) = \lambda y$ and $q(\lambda, x, y) = x$, the family F_λ becomes the original Hénon family.

Our results fit well with the study of multidimensional perturbations directioned by Young [8] in a sense of topological chaos. The results from [1,3] about difference equations can be applied to multidimensional perturbations of one-dimensional maps.

The methodology we use is based on the concept of covering relations and the cone condition, introduced by Zgliczyński in [9–11]. With covering relations determined by a transition matrix A, a continuous map restricted to an isolated positively invariant set is topologically semi-conjugate to an one-sided subshift of finite type induced by A (see Proposition 15). Adding the Liapunov condition, a homeomorphism restricted to an isolated invariant set is topologically conjugate to a two-sided subshift of finite type induced by A (see Proposition 16). Furthermore, with a help of the strong Liapunov condition adapted from the cone condition, the conjugacy result as above can be derived for multidimensional perturbations in a sense of C^1 topology (see Sections 4.4–4.6). From Propositions 15 and 16, one also gets the answer on how close G should be to F in order to apply Theorems 4–6 and 10–12: as long as the covering relations and the Liapunov condition for F inheriting from those for f works for G.

In comparison to stability of hyperbolic invariant sets (refer to [6, Theorem X.7.4]), the covering relation with the Liapunov condition is a topological way of detecting an isolated chaos. Furthermore, such a chaotic set is stable under perturbations as well as our multidimensional perturbations if the strong Liapunov condition is satisfied.

2. Definitions and statement of the main results

First, we introduce some notations and definitions. For a positive integer m, let $\mathbb{R}^m$ denote the space of all m-tuples of real numbers, $|\cdot|$ be the Euclidean norm on $\mathbb{R}^m$, and let $\|\cdot\|$ denote the operator-norm on the space of linear maps on $\mathbb{R}^m$ induced by $|\cdot|$. For $x \in \mathbb{R}^m$ and $r > 0$, we denote $B_m(x,r) = \{z \in \mathbb{R}^m\colon\ |z-x| < r\}$; for the particular case when $x = 0$ and $r = 1$, we write $B_m = B_m(0,1)$, that is, the open unit ball in $\mathbb{R}^m$. Moreover, for a subset S of $\mathbb{R}^n$, let $\overline{S}$ and ∂S denote the closure and boundary of S, respectively. The *topological entropy* of a continuous map f, denoted by $h_{\text{top}}(f)$, on $\mathbb{R}^m$ is the supremum of topological entropies of f restricted to compact invariant sets; refer to [6].

We briefly give some definitions following [12].

Definition 1. (See [12, Definition 6].) An *h-set* in $\mathbb{R}^m$ is a quadruple consisting of the following data:

- a nonempty compact subset M of $\mathbb{R}^m$,
- a pair of numbers $u(M), s(M) \in \{0, 1, \ldots, m\}$ with $u(M) + s(M) = m$, and
- a homeomorphism $c_M : \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^{u(M)} \times \mathbb{R}^{s(M)}$ with $c_M(M) = \overline{B_{u(M)}} \times \overline{B_{s(M)}}$.

For simplicity, we will denote such an h-set by M and call c_M the *coordinate chart* of M; furthermore, we use the following notations:

$$M_c = \overline{B_{u(M)}} \times \overline{B_{s(M)}}, \qquad M_c^- = \partial B_{u(M)} \times \overline{B_{s(M)}}, \qquad M_c^+ = \overline{B_{u(M)}} \times \partial B_{s(M)},$$
$$M^- = c_M^{-1}\big(M_c^-\big), \quad \text{and} \quad M^+ = c_M^{-1}\big(M_c^+\big).$$

A covering relation between two h-sets is defined as follows.

Definition 2. (See [12, Definition 7].) Let M, N be h-sets in $\mathbb{R}^m$ with $u(M) = u(N) = u$ and $s(M) = s(N) = s$, $f : M \to \mathbb{R}^u \times \mathbb{R}^s$ be a continuous map, and $f_c = c_N \circ f \circ c_M^{-1} : M_c \to \mathbb{R}^u \times \mathbb{R}^s$. We say M f*-covers* N, denoted by

$$M \overset{f}{\Longrightarrow} N,$$

if the following conditions are satisfied:

1. there exists a homotopy $h : [0,1] \times M_c \to \mathbb{R}^u \times \mathbb{R}^s$ such that

$$h(0,x) = f_c(x) \quad \text{for } x \in M_c,$$
$$h\big([0,1], M_c^-\big) \cap N_c = \emptyset,$$
$$h\big([0,1], M_c\big) \cap N_c^+ = \emptyset;$$

2. there exists a map $\varphi : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$h(1,p,q) = \big(\varphi(p), 0\big) \quad \text{for } p \in \overline{B_u} \text{ and } q \in \overline{B_s},$$
$$\varphi(\partial B_u) \subset \mathbb{R}^u \backslash \overline{B_u}; \quad \text{and}$$

3. there exists a nonzero integer w such that the local Brouwer degree of φ at 0 in B_u, denoted by $\deg(\varphi, B_u, 0)$, is w; refer to [12, Appendix] and [7, Chapter 3] for its definition and properties.

Next, we define covering relations determined by a transition matrix. By a *transition matrix*, it means that a square matrix satisfies (i) all entries are either zero or one, and (ii) all row sums and column sums are greater than or equal to one. For a transition matrix A, let $\rho(A)$ denote the spectral radius of A. Then $\rho(A) \geqslant 1$ and moreover, if A is irreducible and not a permutation, then $\rho(A) > 1$. Let Σ_A^+ (resp. Σ_A) be the space of all allowable one-sided (resp. two sided) sequences for the matrix A with a usual metric, and let $\sigma_A^+ : \Sigma_A^+ \to \Sigma_A^+$ (resp. $\sigma_A : \Sigma_A \to \Sigma_A$) be the one-sided (resp. two sided) subshift of finite type for A. Then $h_{\text{top}}(\sigma_A^+) = h_{\text{top}}(\sigma_A) = \log(\rho(A))$. Refer to [6] for more background.

Definition 3. Let $A = [a_{ij}]_{1 \leqslant i,j \leqslant \eta}$ be a transition matrix and f be a continuous map on $\mathbb{R}^m$. We say that f has *covering relations* determined by A if the following conditions are satisfied:

1. there are η pairwisely disjoint h-sets $\{M_i\}_{i=1}^{\eta}$ in $\mathbb{R}^m$;
2. if $a_{ij} = 1$ then the covering relation $M_i \overset{f}{\Longrightarrow} M_j$ holds.

It is easy to see that the logistic maps $f(x) = \mu x(1-x)$ with $\mu > 4$ has covering relations determined by the 2×2 matrix with all entries one on intervals $[-\epsilon, 1/2 - \delta]$ and $[1/2 + \delta, 1 + \epsilon]$ as h-sets, where $0 < \epsilon < \mu/4 - 1$ and $0 < \delta < [(\mu/4 - 1 - \epsilon)\mu^{-1}]^{1/2}$.

Now, we state the following result about perturbations of a map.

Theorem 4. *Let f be a continuous map on $\mathbb{R}^m$ having covering relations determined by a transition matrix A. If g is a continuous map on $\mathbb{R}^m$ with $|g - f|$ small enough, then there exists a compact subset Λ_g of $\mathbb{R}^m$ such that Λ_g is positively invariant for g and the restriction of g to Λ_g, denoted by $g|\Lambda_g$, is topologically semi-conjugate to σ_A^+, and therefore $h_{\text{top}}(g) \geqslant \log(\rho(A))$.*

If the singular map F depends only on the phase variable of f, we have the following result about multidimensional perturbations.

Theorem 5. *Let $F(x, y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous map having covering relations determined by a transition matrix A, and $g : \mathbb{R}^m \to \mathbb{R}^k$ is a continuous function. If G is a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ with $|G - F|$ small enough, then there exists a compact subset Λ_G of $\mathbb{R}^{m+k}$ such that Λ_G is positively invariant for G and $G|\Lambda_G$ is topologically semi-conjugate to σ_A^+.*

For the case when the singular map is a skew product locally trapping along the second variable, we have the following.

Theorem 6. *Let $F(x, y) = (f(x), g(x, y)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous map having covering relations determined by a transition matrix A, and $g : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function such that $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^k$ homeomorphic to the closed unit ball in $\mathbb{R}^k$. If G is a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ with $|G - F|$ small enough, then there exists a compact subset Λ_G of $\mathbb{R}^{m+k}$ such that Λ_G is positively invariant for G and $G|\Lambda_G$ is topologically semi-conjugate to σ_A^+.*

Next, we slightly modify the Liapunov condition for a covering relation given by Zgliczyński in [11, Definition 11] and furthermore, we define the strong Liapunov condition. We define a quadratic form on a h-set K in $\mathbb{R}^m$ to be of the form

$$Q_K(x, y) = P_K(x) - R_K(y) \quad \text{for all } (x, y) \in \mathbb{R}^{u(K)} \times \mathbb{R}^{s(K)}, \tag{1}$$

where $P_K : \mathbb{R}^{u(K)} \to \mathbb{R}$ and $R_K : \mathbb{R}^{s(K)} \to \mathbb{R}$ are positive definite quadratic forms. Note that a quadratic form on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector z in $\mathbb{R}^n$ can be computed by an expression of the form $Q(z) = z^T S z$, where S is an $n \times n$ symmetric matrix and z^T denotes the transpose of z; refer to [5].

Definition 7. Let Q_M and Q_N be quadratic forms on h-sets M and N, respectively, as in (1). We say that a covering relation $M \overset{f}{\Longrightarrow} N$ satisfies the *Liapunov condition* (resp. the *strong Liapunov condition*) with respect to the pair (Q_M, Q_N) if there exists $\theta \geqslant 0$ (resp. $\theta > 0$) such that for any $u, v \in M_c$ with $u \neq v$,

$$Q_N\big(f_c(u) - f_c(v)\big) - Q_M(u - v) > \theta |u - v|^2.$$

As a Liapunov function, a sequence of quadratic forms has scalar values strictly monotone along the difference of two orbits. More precisely, consider covering relations $M_i \overset{f}{\Longrightarrow} M_{i+1}$ satisfying the Liapunov condition with respect to the pair $(Q_{M_i}, Q_{M_{i+1}})$ of quadratic forms for all $i \geqslant 0$. If u, v are two points such that $f^i(u), f^i(v) \in M_{i,c}$ and $f^i(u) \neq f^i(v)$ for all $i \geqslant 0$, then the sequence $\{Q_{M_i}(f_c^i(u) - f_c^i(v))\}_{i=0}^{\infty}$ is strictly increasing. This property will play an import role while we prove conjugacy results (see the proof of Lemma 18 below).

In the following, we define covering relations with the Liapunov and strong Liapunov conditions determined by a transition matrix.

Definition 8. Let $A = [a_{ij}]_{1 \leqslant i,j \leqslant \eta}$ be a transition matrix and f be a continuous map on $\mathbb{R}^m$. We say that f has *covering relations with the Liapunov condition* (*resp. the strong Liapunov condition*) determined by A if the following conditions are satisfied;

1. there are η pairwisely disjoint h-sets $\{M_i\}_{i=1}^{\eta}$ in $\mathbb{R}^m$; and on each M_i there exists a quadratic form Q_{M_i} as in (1);
2. if $a_{ij} = 1$ then the covering relation $M_i \overset{f}{\Longrightarrow} M_j$ holds and satisfies the Liapunov condition (resp. the strong Liapunov condition) with respect to the pair (Q_{M_i}, Q_{M_j}); and
3. if $a_{ij} = 1$ then the coordinate charts c_{M_i} and c_{M_j} are C^1 diffeomorphisms.

Example 9. Let us show that the logistic map $f(x) = \mu x(1 - x)$ with $\mu > 4$ has covering relations with the strong Liapunov condition determined by the 2×2 matrix with all entries one. Set (i) h-sets $M_1 = [-\epsilon, 1/2 - \delta]$ and $M_2 = [1/2 + \delta, 1 + \epsilon]$, where $0 < 2\epsilon < \mu/4 - 1$ and $0 < \delta < [(\mu/4 - 1 - \epsilon)\mu^{-1}]^{1/2}$; (ii) the coordinate charts $\bar{u} = c_{M_1}(u) = \alpha^{-1}(\int_{-\epsilon}^{u} \rho(t)\,dt - \int_{u}^{1/2-\delta} \rho(t)\,dt)$ and $\bar{u} = c_{M_2}(u) = \alpha^{-1}(\int_{1/2+\delta}^{u} \rho(t)\,dt - \int_{u}^{1+\epsilon} \rho(t)\,dt)$, where $\rho(t) = [(t + 2\epsilon)(1 + 2\epsilon - t)]^{-1}$ for $t \in (-2\epsilon, 1+2\epsilon)$, and $\alpha = \int_{-\epsilon}^{1/2-\delta} \rho(t)\,dt = \int_{1/2+\delta}^{1+\epsilon} \rho(t)\,dt$; and (iii) quadratic forms $Q_{M_1}(\bar{u}) = Q_{M_2}(\bar{u}) = \bar{u}^2$. With a help of the Schwarz lemma and the idea of the Poincaré norm, in Proposition 4.10 of [6], it is shown that there exists $\lambda > 1$ such that if $u, f(u) \in M_1 \cup M_2$, then $\rho(f(u))|f'(u)| \geqslant \lambda \rho(u)$. Let C_1 be a positive constant such that $\rho(t) \geqslant C_1$ for all $t \in M_1 \cup M_2$. Then for any $u, v \in M_1 \cup M_2$ we have $|\int_v^u \rho(t)\,dt| \geqslant C_1 |u - v|$. Since $c'_{M_i}(u) = 2\alpha^{-1}\rho(u)$, there exists $C_2 > 0$ such that $|c'_{M_i}(u)| \leqslant C_2$ for all $u \in M_i$ and $i = 1, 2$. Therefore, the strong Liapunov condition holds

$$\begin{aligned}
&Q_{M_i}\big(f_c(\bar{u}) - f_c(\bar{v})\big) - Q_{M_j}(\bar{u} - \bar{v}) \\
&= \big(c_{M_i} \circ f(u) - c_{M_i} \circ f(v)\big)^2 - \big(c_{M_j}(u) - c_{M_j}(v)\big)^2 \\
&= \left(2\alpha^{-1} \int_{f(v)}^{f(u)} \rho(t)\,dt\right)^2 - \left(2\alpha^{-1} \int_{v}^{u} \rho(t)\,dt\right)^2 \geqslant 4\alpha^{-2}\big(\lambda^2 - 1\big)\left(\int_{v}^{u} \rho(t)\,dt\right)^2 \\
&\geqslant 4\alpha^{-2}\big(\lambda^2 - 1\big)C_1|u - v|^2 \geqslant 4\alpha^{-2}\big(\lambda^2 - 1\big)C_1 C_2^{-2}|\bar{u} - \bar{v}|^2.
\end{aligned}$$

The Liapunov condition is for detection of chaos (see Proposition 16 below), while the strong Liapunov condition is for stability of chaos under small C^1 perturbations as follows.

Theorem 10. *Let $f:\mathbb{R}^m \to \mathbb{R}^m$ be a C^1 homeomorphism having covering relations with the strong Liapunov condition determined by a transition matrix A. If g is a C^1 homeomorphism on $\mathbb{R}^m$ with $|g-f| + \|Dg - Df\|$ small enough, then there exists a compact subset Λ_g of $\mathbb{R}^m$ such that Λ_g is invariant for g and $g|\Lambda_g$ is topologically conjugate to σ_A.*

For small C^1 perturbations of a direct product contracting along the second variable, we have the following result.

Theorem 11. *Let $F(x,y) = (f(x), g(y)) \in \mathbb{R}^m \times \mathbb{R}^k$ be a C^1 homeomorphism for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f:\mathbb{R}^m \to \mathbb{R}^m$ has covering relations with the strong Liapunov condition determined by a transition matrix A, and $g:\mathbb{R}^k \to \mathbb{R}^k$ is a contraction on the closed unit ball $\overline{B_k}$ such that $g(\overline{B_k}) \subset B_k$. If G is a C^1 homeomorphism on $\mathbb{R}^{m+k}$ with $|G-F| + \|DG - DF\|$ small enough, then there exists a compact subset Λ_G of $\mathbb{R}^{m+k}$ such that Λ_G is invariant for G and $G|\Lambda_G$ is topologically conjugate to σ_A.*

Finally, for a one-parameter family of maps with the singular map F depends only on the phase variable of f, we have the following result.

Theorem 12. *Let F_λ be a one-parameter family of maps on $\mathbb{R}^m \times \mathbb{R}^k$ satisfying* (i) *$F_\lambda(x,y)$ is C^1 continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where λ is a parameter*; (ii) *F_λ is a homeomorphism on $\mathbb{R}^m \times \mathbb{R}^k$ provided $\lambda \neq 0$; and* (iii) *$F_0(x,y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^k$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, where $f:\mathbb{R}^m \to \mathbb{R}^m$ has covering relations with the strong Liapunov condition determined by a transition matrix A, and $g:\mathbb{R}^m \to \mathbb{R}^k$. Then for each λ sufficiently close to 0, there exists a compact subset Λ_λ of $\mathbb{R}^{m+k}$ such that if $\lambda \neq 0$ then Λ_λ is invariant for F_λ and $F_\lambda|\Lambda_\lambda$ is topologically conjugate to σ_A, while Λ_0 is positively invariant for F_0 and $F_0|\Lambda_0$ is topologically semi-conjugate to σ_A^+.*

3. Preliminary results

First, we have that a closed loop of covering relations implies existence of a periodic point.

Theorem 13. *(See [12, Theorem 9].) Let $\{f_i\}_{i=1}^k$ be a collection of continuous maps on $\mathbb{R}^m$ and $\{M_i\}_{i=1}^k$ be a collection of h-sets in $\mathbb{R}^m$ such that $M_{k+1} = M_1$ and $M_i \overset{f_i}{\Longrightarrow} M_{i+1}$ for $1 \leqslant i \leqslant k$. Then there exists a point $x \in int(M_1)$ such that*

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in int(M_{i+1}) \quad \text{for } i = 1, \ldots, k;$$

and

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x.$$

Next, we show that a covering relation is persistent under C^0 small perturbations, which slightly extends Theorem 14 of [12] by dropping the Lipschitz condition of c_M.

Proposition 14. *Let M and N be h-sets in $\mathbb{R}^m$ with $u(M) = u(N) = u$ and $s(M) = s(N) = s$, and let $f, g: M \to \mathbb{R}^m$ be continuous. Assume that the covering relation $M \overset{f}{\Longrightarrow} N$ holds. Then there exists $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in M$ then the covering relation $M \overset{g}{\Longrightarrow} N$ holds.*

Proof. By using Theorem 13 of [12], there exists $\varepsilon > 0$ such that if $|f_c(x) - g_c(x)| < \varepsilon$ for all $x \in M_c$ then $M \overset{g}{\Longrightarrow} N$. Since M is compact, there exists $r > 0$ such that $f(M) \subset \overline{B_m(0,r)}$. If $|f(x) - g(x)| < 1$ for all $x \in M$, then $g(M) \subset \overline{B_m(0,r+1)}$. By uniform continuity of c_N on $\overline{B_m(0,r+1)}$, there exists $\delta' > 0$ such that if $z, z' \in \overline{B_m(0,r+1)}$ and $|z - z'| < \delta'$ then $|c_N(z) - c_N(z')| < \varepsilon$. Let $\delta = \min\{\delta', 1\}$. If $|f(x) - g(x)| < \delta$ for all $x \in M$ then

$$\max_{x \in M_c}\bigl|f_c(x) - g_c(x)\bigr| = \max_{x \in M}\bigl|c_N\bigl(f(x)\bigr) - c_N\bigl(g(x)\bigr)\bigr| < \varepsilon.$$

Thus $M \overset{g}{\Longrightarrow} N$. $\square$

In the following, we state two results: the first one says that a continuous map having covering relations determined by a transition matrix is topologically semi-conjugate to a one-sided subshift of finite type, and the second one says that a homeomorphism having covering relations with the Liapunov condition determined by a transition matrix is topologically conjugate to a two-sided subshift of finite type. The proof of the results is postponed to subsequent subsections.

Proposition 15. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map which has covering relations determined by a transition matrix A. Then there exists a compact subset Λ of $\mathbb{R}^m$ such that Λ is maximal positively invariant for f in the union of the h-sets (with respect to A) and $f|\Lambda$ is topologically semi-conjugate to σ_A^+.*

Proposition 16. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism which has covering relations with the Liapunov condition determined by a transition matrix A. Then there exists a compact subset Λ of $\mathbb{R}^m$ such that Λ is maximal invariant for f in the interior of the union of the h-sets (with respect to A) and $f|\Lambda$ is topologically conjugate to σ_A.*

The set Λ in the above theorem is an isolated invariant set for f with the interior of the union of the h-sets as its isolating neighborhood.

3.1. Proof of Proposition 15

For convenience, we denote by $\{M_i\}_{i=1}^{\eta}$ the h-sets with covering relations for f determined by A as in Definition 3, and write $\underline{s} = (s_0, s_1, \ldots)$ for $\underline{s} \in \Sigma_A^+$. Define

$$\Lambda_n = \bigcup_{\underline{s} \in \Sigma_A^+}\left(\bigcap_{i=0}^{n} f^{-i}(M_{s_i})\right) \quad \text{for } n \geqslant 0, \quad \text{and} \quad \Lambda = \bigcap_{n \geqslant 0} \Lambda_n.$$

Then Λ is the set of all points whose forward orbits, following allowable sequences in Σ_A^+, stay in $\bigcup_{i=1}^{\eta} M_i$. Thus Λ is maximal positively invariant set for f in $\bigcup_{i=1}^{\eta} M_i$ with respect to A. Since each M_i is compact and f is continuous, the set $\bigcap_{i=0}^{n} f^{-i}(M_{s_i})$ is compact for all $n \geqslant 0$ and $\underline{s} \in \Sigma_A^+$. Since the number of sets M_i's is η and the intersection $\bigcap_{i=0}^{n} f^{-i}(M_{s_i})$ involves only the first $n+1$ digits of $\underline{s} \in \Sigma_A^+$, that is, $(s_0, s_1, \ldots, s_n)$, there are at most η^{n+1} sets $\bigcap_{i=0}^{n} f^{-i}(M_{s_i})$ for all $\underline{s} \in \Sigma_A^+$, although the set Σ_A^+ itself might be uncountable. Thus the set Λ_n is a union of finitely many compact sets and hence is compact for all $n \geqslant 0$. Therefore, Λ is compact.

For semi-conjugacy, we define $h : \Lambda \to \Sigma_A^+$ by $h(z) = \underline{s}$ for $z \in \Lambda$, where $f^n(z) \in M_{s_n}$ for all $n \geqslant 0$. By the pairwise disjointness of M_i's and the definition of Λ, the map h is well defined. It is easy to show that $\sigma_A \circ h = h \circ f$. Next, we show that h is continuous on Λ. Let $z \in \Lambda$, $h(z) = \underline{s}$ and $\{z_n\}_{n=1}^{\infty}$ be a sequence in Λ such that $z_n \to z$ as $n \to \infty$. Since M_i's are pairwisely disjoint and compact, there exists $n_0 \in \mathbb{N}$ such that $z_n \in M_{s_0}$ for all $n \geqslant n_0$. By the continuity of f, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geqslant n_0$ and $f(z_n) \in M_{s_1}$ for all $n \geqslant n_1$. By using the same process inductively, we get that for each

$i \geqslant 0$, there exist $n_i \in \mathbb{N}$ such that $f^j(z_n) \in M_{s_j}$ for all $n \geqslant n_i$ and $0 \leqslant j \leqslant i$. This proves that $h(z_n) \to \underline{s}$ as $n \to \infty$. Therefore, h is continuous on Λ.

To prove that h is onto, we need the following lemma.

Lemma 17. *For any $\underline{s} \in \Sigma_A$, the intersection $\bigcap_{n \geqslant 0} f^{-n}(M_{s_n})$ is nonempty.*

Proof. Let $\underline{s} \in \Sigma_A$. First, we prove that the intersection $\bigcap_{i=0}^{n} f^{-i}(M_{s_i})$ is nonempty for all $n \geqslant 0$ by applying Theorem 13 to a closed loop of covering relations. Let $n \geqslant 0$. Then we have the loop of covering relations $M_{s_0} \overset{f}{\Longrightarrow} M_{s_1} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} M_{s_n}$. The loop becomes closed by adding a covering relation $M_{s_n} \overset{g}{\Longrightarrow} M_{s_0}$ with a homotopy $h : [0,1] \times M_{s_n,c} \to \mathbb{R}^u \times \mathbb{R}^s$, where $u = u(M_{s_n})$, $s = s(M_{s_n})$, $g_c : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u \times \mathbb{R}^s$ is defined by $g_c(p,q) = (2p, 0)$ for all $(p,q) \in \mathbb{R}^u \times \mathbb{R}^s$, $g = c_{M_{s_0}}^{-1} \circ g_c \circ c_{M_{s_n}}$, and $h(t,p,q) = g_c(p,q)$ for $t \in [0,1]$ and $(p,q) \in M_{s_n,c}$. It follows from Theorem 13 that there exists $z \in int(M_{s_0})$ such that $f^i(z) \in int(M_{s_i})$ for $0 \leqslant i \leqslant n$. Thus $z \in \bigcap_{i=0}^{n} f^{-i}(M_{s_i})$. Therefore, the intersection $\bigcap_{i=0}^{n} f^{-i}(M_{s_i})$ is nonempty for all $n \geqslant 0$. Since $\{\bigcap_{i=0}^{n} f^{-i}(M_{s_i})\}_{n=0}^{\infty}$ is a nested sequence of nonempty compact subsets of $\mathbb{R}^m$, the set $\bigcap_{n \geqslant 0} f^{-n}(M_{s_n})$ is nonempty. □

Finally, we show that h is onto. Let $\underline{s} \in \Sigma_A$. Then there exists $z \in \bigcap_{n \geqslant 0} f^{-n}(M_{s_n})$ from Lemma 17. By the definitions of Λ and h, we have that $z \in \Lambda$ and $h(z) = \underline{s}$. This proves that h is onto. We have finished the proof of Proposition 15.

3.2. Proof of Proposition 16

We denote by $\{M_i\}_{i=1}^{\eta}$ the h-sets with covering relations and the Liapunov condition for f determined by A as in Definition 8, and write $\underline{s} = (\ldots, s_{-1}, s_0, s_1, \ldots)$ for $\underline{s} \in \Sigma_A$. Define

$$\Lambda_n = \bigcup_{\underline{s} \in \Sigma_A} \left(\bigcap_{i=-n}^{n} f^{-i}(M_{s_i}) \right) \quad \text{for } n \geqslant 0, \quad \text{and} \quad \Lambda = \bigcap_{n \geqslant 0} \Lambda_n.$$

Define $h : \Lambda \to \Sigma_A$ by $h(z) = \underline{s}$ for $z \in \Lambda$, where $f^n(z) \in M_{s_n}$ for all $n \in \mathbb{Z}$. By using the same argument as in the proof of Proposition 15, we have that Λ is a maximal compact invariant set for f in $\bigcup_{i=1}^{\eta} M_i$ with respect to A and h is a topological semi-conjugacy. Moreover, the covering relations for f on h-sets implies that any boundary point of a h-set cannot have a full orbit staying in h-sets. Therefore, Λ is maximal invariant for f in $\bigcup_{i=1}^{\eta} int(M_i)$ with respect to A.

To prove that h is one-to-one, we need the following lemma, which is guaranteed by the Liapunov condition.

Lemma 18. *For any $\underline{s} \in \Sigma_A$, the intersection $\bigcap_{n \in \mathbb{Z}} f^{-n}(M_{s_n})$ consists of a single point.*

Proof. Let $\underline{s} \in \Sigma_A$. Then, similar to the proof of Lemma 17, we have that the intersection $\bigcap_{n \in \mathbb{Z}} f^{-n}(M_{s_n})$ is nonempty. Next, we show the uniqueness of the intersection by contradiction. Assume that $u, v \in \bigcap_{n \in \mathbb{Z}} f^{-n}(M_{s_n})$ with $u \neq v$. Since f is a homeomorphism, $f^n(u)$ and $f^n(v)$ are different points in the same h-set M_{s_n} for all $n \in \mathbb{Z}$. By covering relations with the Liapunov condition, we have that for all $n \in \mathbb{Z}$,

$$Q_{M_{s_{n+1}}}\big(c_{M_{s_{n+1}}} \circ f^{n+1}(u) - c_{M_{s_{n+1}}} \circ f^{n+1}(v)\big) > Q_{M_{s_n}}\big(c_{M_{s_n}} \circ f^n(u) - c_{M_{s_n}} \circ f^n(v)\big). \tag{2}$$

That is, the value of $Q_{M_{s_n}}$ at the point $c_{M_{s_n}} \circ f^n(u) - c_{M_{s_n}} \circ f^n(v)$ is strictly increasing as $n \in \mathbb{Z}$ increases. It follows that there exits $j \in \mathbb{Z}$ such that $Q_{M_{s_j}}(c_{M_{s_j}} \circ f^j(u) - c_{M_{s_j}} \circ f^j(v)) \neq 0$.

First, we consider the case when

$$Q_{M_{s_j}}\bigl(c_{M_{s_j}}\circ f^j(u)-c_{M_{s_j}}\circ f^j(v)\bigr)>0. \tag{3}$$

By using the compactness of the union $\bigcup_{i=1}^{\eta} M_i$, sequentially twice for two sequences, both sequences $\{f^{n+j}(u)\}_{n=0}^{\infty}$ and $\{f^{n+j}(v)\}_{n=0}^{\infty}$ have convergent subsequences, say $\{f^{n(i)+j}(u)\}_{i=0}^{\infty}$ and $\{f^{n(i)+j}(v)\}_{i=0}^{\infty}$, with the limits, say $\bar{u}$ and $\bar{v}$ in $\bigcup_{i=1}^{\eta} M_i$, respectively. By the fact that M_i's are pairwisely disjoint and compact, and $f^n(u), f^n(v)\in M_{s_n}$ for all $n\in\mathbb{Z}$, there exists $\alpha\in\mathbb{N}$ such that $f^{n(i)+j}(u)$, $f^{n(i)+j}(v)$, $\bar{u}$ and $\bar{v}$ are all in the same h-set, namely $M_{s_{n(\alpha)+j}}$, for all $i\geqslant\alpha$. By the continuity of f, the points $f(\bar{u})$ and $f(\bar{v})$ are limits of sequences $\{f^{n(i)+j+1}(u)\}_{i=0}^{\infty}$ and $\{f^{n(i)+j+1}(v)\}_{i=0}^{\infty}$, respectively. Again by the same fact as above, there exists an integer $\beta\geqslant\alpha$ such that $f^{n(i)+j+1}(u)$, $f^{n(i)+j+1}(v)$, $f(\bar{u})$ and $f(\bar{v})$ are all in the same h-set, namely $M_{s_{n(\beta)+j+1}}$, for all $i\geqslant\beta$. For convenience, we denote $N_0=M_{s_{n(\alpha)+j}}$ and $N_1=M_{s_{n(\beta)+j+1}}$.

By (2), we get that for all $i\geqslant\beta$,

$$Q_{N_0}\bigl(c_{N_0}\circ f^{n(i)+j}(u)-c_{N_0}\circ f^{n(i)+j}(v)\bigr)>Q_{M_{s_j}}\bigl(c_{M_{s_j}}\circ f^j(u)-c_{M_{s_j}}\circ f^j(v)\bigr).$$

By letting $i\to\infty$, it follows from the continuity of Q_{N_0} and c_{N_0} that

$$Q_{N_0}\bigl(c_{N_0}(\bar{u})-c_{N_0}(\bar{v})\bigr)\geqslant Q_{M_{s_j}}\bigl(c_{M_{s_j}}\circ f^j(u)-c_{M_{s_j}}\circ f^j(v)\bigr).$$

Thus from (3), we have $Q_{N_0}(c_{N_0}(\bar{u})-c_{N_0}(\bar{v}))>0$ and hence $\bar{u}\neq\bar{v}$. Since $f(\bar{u}), f(\bar{v})\in N_1$, the Liapunov condition implies that

$$Q_{N_1}\bigl(c_{N_1}\circ f(\bar{u})-c_{N_1}\circ f(\bar{v})\bigr)>Q_{N_0}\bigl(c_{N_0}(\bar{u})-c_{N_0}(\bar{v})\bigr).$$

Because that $f^{n(i)+j+1}(u)$ and $f^{n(i)+j+1}(v)$ converge to $f(\bar{u})$ and $f(\bar{v})$, respectively, and both Q_{N_1} and c_{N_1} are continuous, we obtain that for some large γ,

$$Q_{N_1}\bigl(c_{N_1}\circ f^{n(\gamma)+j+1}(u)-c_{N_1}\circ f^{n(\gamma)+j+1}(v)\bigr)>Q_{N_0}\bigl(c_{N_0}(\bar{u})-c_{N_0}(\bar{v})\bigr). \tag{4}$$

By using (2), we get that for all $i>\gamma+1$,

$$Q_{N_0}\bigl(c_{N_0}\circ f^{n(i)+j}(u)-c_{N_0}\circ f^{n(i)+j}(v)\bigr)>Q_{N_1}\bigl(c_{N_1}\circ f^{n(\gamma)+j+1}(u)-c_{N_1}\circ f^{n(\gamma)+j+1}(v)\bigr).$$

Letting $i\to\infty$, it follows from the continuity of Q_{N_0} and c_{N_0} that

$$Q_{N_0}\bigl(c_{N_0}(\bar{u})-c_{N_0}(\bar{v})\bigr)\geqslant Q_{N_1}\bigl(c_{N_1}\circ f^{n(\gamma)+j+1}(u)-c_{N_1}\circ f^{n(\gamma)+j+1}(v)\bigr).$$

Together with (4), it leads to a contradiction.

For the case when $Q_{M_{s_j}}(c_{M_{s_j}}\circ f^j(u)-c_{M_{s_j}}\circ f^j(v))<0$, by working on the backward orbits of u and v, that is, replacing n and $n(i)$ by $-n$ and $-n(i)$ in the above argument, a contradiction occurs too.

Therefore, the intersection $\bigcap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ consisting of a single point. We have done the proof of the desired result. □

By using Lemma 18, we can easily prove that h is one-to-one. Indeed, let $\underline{s}\in\Sigma_A$ and $h(z_1)=h(z_2)=\underline{s}$ for $z_1,z_2\in\Lambda$. Then $z_1,z_2\in\bigcap_{n\in\mathbb{Z}} f^{-n}(M_{s_n})$ and hence $z_1=z_2$.

Because that the sets Λ and Σ_A are compact and h is a continuous and one-to-one function, it follows that h is a homeomorphism. This completes the proof of Proposition 16.

4. Proof of theorems

Throughout this section, we write $A = [a_{ij}]_{1 \leqslant i,j \leqslant \eta}$ and denote by $\{M_i\}_{i=1}^{\eta}$ the pairwisely disjoint h-sets with covering relations for f determined by A. For the case when the covering relations on $\{M_i\}_{i=1}^{\eta}$ satisfy the Liapunov or strong Liapunov condition, we denote by Q_{M_i} be the corresponding quadratic form on M_i for all i.

4.1. Proof of Theorem 4

Since the dimension of A is η, there are at most η^2 choices of the covering relations $M_i \overset{f}{\Longrightarrow} M_j$. By Proposition 14, if $g : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous map with $|g - f|$ small enough, then g has covering relations on h-sets $\{M_i\}_{i=1}^{\eta}$ determined by A. By applying Proposition 15 to the map g, there exists a compact subset Λ_g of $\mathbb{R}^m$ such that Λ_g is positively invariant for g and $g|\Lambda_g$ is topologically semi-conjugate to the one-sided subshift of finite type σ_A^+. Therefore, $h_{\text{top}}(g) \geqslant h_{\text{top}}(g|\Lambda_g) \geqslant h_{\text{top}}(\sigma_A^+) = \log(\rho(A))$.

4.2. Proof of Theorem 5

Let $M = \bigcup_{i=1}^{\eta} M_i$. Since g is continuous and M is compact, there exists $r > 0$ such that $g(M) \subset B_k(0,r)$. For $i \in \{1, \ldots, \eta\}$, we define h-sets M_i' in $\mathbb{R}^m \times \mathbb{R}^k$ by $M_i' = M_i \times \overline{B_k(0,r)}$, with $u(M_i') = u(M_i)$, $s(M_i') = s(M_i) + n$ and $c_{M_i'}(x, y) = (c_{M_i}(x), y/r)$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$. Suppose $a_{ij} = 1$. Then $M_i \overset{f}{\Longrightarrow} M_j$ implies $M_i' \overset{F}{\Longrightarrow} M_j'$ by defining a homotopy $H : [0,1] \times \overline{B_m} \times \overline{B_k} \to \mathbb{R}^{m+k}$ as follows

$$H(t,x,y) = \left(h(t,x), \frac{1-t}{r} g\bigl(c_{M_i}^{-1}(x)\bigr)\right),$$

where h is the homotopy for the covering relation $M_i \overset{f}{\Longrightarrow} M_j$. This shows that F has covering relations on $\{M_i'\}_{i=1}^{\eta}$ determined by A. By applying Theorem 4 to the map F on $\mathbb{R}^m \times \mathbb{R}^k$, we get that if G is a continuous map on $\mathbb{R}^m \times \mathbb{R}^k$ with $|G - F|$ small enough, then there exists a compact subset Λ_G of $\mathbb{R}^{m+k}$ such that Λ_G is positively invariant for g and $g|\Lambda_g$ is topologically semi-conjugate to the one-sided subshift of finite type σ_A^+.

4.3. Proof of Theorem 6

Define $\hat{F} = (id, c) \circ F \circ (id, c)^{-1}$, where id denotes the identity map on $\mathbb{R}^m$ and c is a homeomorphism form S to $\overline{B_k}$. Then the conclusion follows from the above argument.

4.4. Proof of Theorem 10

Suppose $a_{ij} = 1$. Then $M_i \overset{f}{\Longrightarrow} M_j$ holds. By Proposition 14, if $|g - f|$ is small enough, then $M_i \overset{g}{\Longrightarrow} M_j$ holds. Assume that such a map g is C^1. Before proving that $M_i \overset{g}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition, let us have some observations. Since $M_i \overset{f}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition, there exists $\theta_{i,j} > 0$ such that for $x, y \in M_{i,c}$ with $x \neq y$,

$$Q_{M_j}\bigl(f_c(x) - f_c(y)\bigr) > Q_{M_i}(x - y) + \theta_{i,j}|x - y|^2. \tag{5}$$

For $\alpha = i, j$, let S_α be the $m \times m$ symmetric matrix such that $Q_{M_\alpha}(z) = z^T S_\alpha z$ for $z \in \mathbb{R}^m$. Since f, g and c_{M_i} are C^1, for $x, y \in M_{i,c}$, we can define

$$E_{x,y} = \int_0^1 Df_c\big(y + t(x-y)\big)\,dt \quad \text{and} \quad C_{x,y} = \int_0^1 Dg_c\big(y + t(x-y)\big)\,dt.$$

Then $\|E_{x,y} - C_{x,y}\| \leqslant \|Df_c - Dg_c\|$. Since both f_c and g_c are C^1 on the compact set $M_{i,c}$, there exists $\beta_i > 0$ such that $\|E_{x,y}\| + \|C_{x,y}\| < \beta_i$ for all $x, y \in M_{i,c}$. Thus

$$\begin{aligned}\big\|E_{x,y}^T S_j E_{x,y} - C_{x,y}^T S_j C_{x,y}\big\| &\leqslant \big\|E_{x,y}^T S_j E_{x,y} - C_{x,y}^T S_j E_{x,y}\big\| + \big\|C_{x,y}^T S_j E_{x,y} - C_{x,y}^T S_j C_{x,y}\big\| \\ &\leqslant \beta_i \|S_j\| \|Df_c - Dg_c\|. \end{aligned} \tag{6}$$

Now we check the strong Liapunov condition for $M_i \overset{g}{\Longrightarrow} M_j$. Let $u, v \in M_{i,c}$ with $u \neq v$. By the mean value theorem for integrals, we have that $f_c(u) - f_c(v) = E_{u,v}(u-v)$ and $g_c(u) - g_c(v) = C_{u,v}(u-v)$. Thus

$$Q_{M_j}\big(f_c(u) - f_c(v)\big) - Q_{M_j}\big(g_c(u) - g_c(v)\big) = (u-v)^T\big(E_{u,v}^T S_j E_{u,v} - C_{u,v}^T S_j C_{u,v}\big)(u-v).$$

From (6), we obtain that

$$\big|Q_{M_j}\big(f_c(u) - f_c(v)\big) - Q_{M_j}\big(g_c(u) - g_c(v)\big)\big| \leqslant \beta_i \|S_j\| \|Df_c - Dg_c\| |u-v|^2.$$

Imposing (5), we get that

$$\begin{aligned} Q_{M_j}\big(g_c(u) - g_c(v)\big) &\geqslant Q_{M_j}\big(f_c(u) - f_c(v)\big) - \big|Q_{M_j}\big(f_c(u) - f_c(v)\big) - Q_{M_j}\big(g_c(u) - g_c(v)\big)\big| \\ &> Q_{M_i}(u-v) + \theta_{i,j}|u-v|^2 - \beta_i \|S_j\| \|Df_c - Dg_c\| |u-v|^2 \\ &= Q_{M_i}(u-v) + \big(\theta_{i,j} - \beta_i \|S_j\| \|Df_c - Dg_c\|\big)|u-v|^2. \end{aligned}$$

Finally, we denote

$$\hat{\theta}_{i,j} = \theta_{i,j} - \beta_i \|S_j\| \|Df_c - Dg_c\|.$$

Then $\hat{\theta}_{i,j}$ is independent of $u, v \in M_{i,c}$. Since c_{M_α} is C^1 diffeomorphism and M_α is compact for $\alpha = i, j$, we have that $\|Df_c - Dg_c\|$ approaches to zero as $\|Df - Dg\|$ tends to zero. Therefore, if $\|Df - Dg\|$ is small enough, then $\hat{\theta}_{i,j} > 0$ and hence $M_i \overset{g}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition.

Since there are at most η^2 choices of pairs (i, j), from the above, we get that if g is a C^1 continuous map with $|g - f| + \|Dg - Df\|$ small enough, then g has covering relations with the strong Liapunov condition determined by A. In addition, if such maps g are C^1 homeomorphisms, then we have the desired result, by applying Proposition 16 and the fact that the strong Liapunov condition implies the Liapunov condition.

4.5. Proof of Theorem 11

Suppose $a_{ij} = 1$. Then the covering relation $M_i \overset{f}{\Longrightarrow} M_j$ holds. First, we show that there is a corresponding covering relation for F on h-sets. For $\alpha = i, j$, let $M'_\alpha = M_\alpha \times \overline{B_k}$. Then each M'_α is an h-set with $c_{M'_\alpha}(x, y) = (c_{M_\alpha}(x), y)$, $u(M'_\alpha) = u(M_\alpha)$, and $s(M'_\alpha) = s(M_\alpha) + k$. Define a homotopy $H : [0, 1] \times \overline{B_m} \times \overline{B_k} \to \mathbb{R}^{m+k}$ by

$$H(t, x, y) = \big(h(t, x), (1 - t)g(y)\big),$$

where h is the homotopy for the covering relation $M_i \overset{f}{\Longrightarrow} M_j$. Then for all $x \in \overline{B_m}$ and $y \in \overline{B_k}$, we have

$$H(0, x, y) = \big(h(0, x), g(y)\big) = \big(c_{M_j} \circ f \circ c^{-1}_{M_i}(x), g(y)\big) = F_c(x, y),$$
$$H(1, x, y) = \big(h(1, x), 0\big).$$

Thus, we have that $M'_i \overset{F}{\Longrightarrow} M'_j$ follows from $M_i \overset{f}{\Longrightarrow} M_j$.

Next, we show that the strong Liapunov condition is satisfied for $M'_i \overset{F}{\Longrightarrow} M'_j$. For $\alpha = i, j$, define the quadratic form $Q_{M'_\alpha}(x, y) = Q_{M_\alpha}(x) - |y|^2$. Let $(x_1, y_1), (x_2, y_2) \in M'_{i,c}$ with $(x_1, y_1) \neq (x_2, y_2)$. Since $M_i \overset{f}{\Longrightarrow} M_j$ satisfies the strong Liapunov condition, there exists $\theta_{i,j} > 0$ such that $Q_{M_j}(f_c(x_1) - f_c(x_2)) > Q_{M_i}(x_1 - x_2) + \theta_{i,j}|x_1 - x_2|^2$ if $x_1 \neq x_2$. Since g is a contraction on $\overline{B_k}$, there exists $0 < \gamma < 1$ such that

$$\big|g(y_1) - g(y_2)\big| \leqslant \gamma|y_1 - y_2|.$$

Thus no matter what x_1 is equal to x_2 or not, we get that

$$\begin{aligned}
&Q_{M'_j}\big(F_c\big((x_1, y_1)\big) - F_c\big((x_2, y_2)\big)\big) - Q_{M'_i}\big((x_1, y_1) - (x_2, y_2)\big)\\
&\quad = Q_{M'_j}\big(\big(f_c(x_1) - f_c(x_2), g(y_1) - g(y_2)\big)\big) - Q_{M'_i}\big((x_1 - x_2, y_1 - y_2)\big)\\
&\quad = Q_{M_j}\big(f_c(x_1) - f_c(x_2)\big) - \big|g(y_1) - g(y_2)\big|^2 - Q_{M_i}(x_1 - x_2) + |y_1 - y_2|^2\\
&\quad \geqslant \theta_{i,j}|x_1 - x_2|^2 + \big(1 - \gamma^2\big)|y_1 - y_2|^2\\
&\quad > \hat{\theta}_{i,j}\big|(x_1, y_1) - (x_2, y_2)\big|^2,
\end{aligned}$$

where $\hat{\theta}_{i,j} = \min\{\theta_{i,j}, 1 - \gamma^2\}/2 > 0$. Thus $M'_i \overset{F}{\Longrightarrow} M'_j$ satisfies the strong Liapunov condition.

Since the number of pairs (i, j) is finite, F has covering relations with the strong Liapunov condition determined by A. From Theorem 10, the desired result follows.

4.6. Proof of Theorem 12

By the continuity of g on the compact union $\bigcup_{i=1}^{\eta} M_i$, there exists $r > 0$ such that $g(\bigcup_{i=1}^{\eta} M_i) \subset B_k(r)$. For each $\alpha \in \{1, 2, \dots, \eta\}$, since g and $c^{-1}_{M_\alpha}$ are C^1, the composition $g \circ c^{-1}_{M_\alpha}$ satisfies the Lipschitz condition on the compact set $M_{\alpha,c}$, i.e., there exists $L_\alpha > 0$ such that for all $x_1, x_2 \in M_{\alpha,c}$,

$$\big|g \circ c^{-1}_{M_\alpha}(x_1) - g \circ c^{-1}_{M_\alpha}(x_2)\big| \leqslant L_\alpha|x_1 - x_2|.$$

For $i, j \in \{1, 2, \ldots, \eta\}$ with $a_{ij} = 1$, we have that $M_i \overset{f}{\Longrightarrow} M_j$ holds and satisfies the strong Liapunov condition. Thus there exists $\theta_{i,j} > 0$ such that $Q_{M_j}(f_c(x_1) - f_c(x_2)) > Q_{M_i}(x_1 - x_2) + \theta_{i,j}|x_1 - x_2|^2$ if $x_1, x_2 \in M_{i,c}$ with $x_1 \neq x_2$. Take a real number $\hat{\theta}$ such that $0 < \hat{\theta} < \min\{\theta_{i,j}/(1 + L_i^2/r^2)\colon\ i, j \in \{1, 2, \ldots, \eta\},\ a_{ij} = 1\}$.

Suppose $a_{ij} = 1$. For $\alpha \in \{i, j\}$, let $M'_\alpha = M_\alpha \times \overline{B_k(r)}$. Then each M'_α is an h-set with $c_{M'_\alpha}(x, y) = (c_{M_\alpha}(x), y/r)$, $u(M'_\alpha) = u(M_\alpha)$, and $s(M'_\alpha) = s(M_\alpha) + k$. Define a quadratic form on M'_α by $Q_{M'_\alpha}(x, y) = Q_{M_\alpha}(x) - \hat{\theta}|y|^2$. Then $M'_i \overset{F_0}{\Longrightarrow} M'_j$ holds for a homotopy $H : [0, 1] \times \overline{B_m} \times \overline{B_k} \to \mathbb{R}^{m+k}$ defined by

$$H(t, x, y) = \left(h(t, x), \frac{1-t}{r} g\big(c_{M_i}^{-1}(x)\big)\right),$$

where h is the homotopy for the covering relation $M_i \overset{f}{\Longrightarrow} M_j$. Furthermore, we check the strong Liapunov condition. Let $(x_1, y_1), (x_2, y_2) \in M'_{i,c}$ with $(x_1, y_1) \neq (x_2, y_2)$. Then

$$\begin{aligned}
& Q_{M'_j}\big(F_{0,c}(x_1, y_1) - F_{0,c}(x_2, y_2)\big) - Q_{M'_i}\big((x_1, y_1) - (x_2, y_2)\big) \\
&= Q_{M'_j}\left(f_c(x_1) - f_c(x_2), \frac{g \circ c_{M_i}^{-1}(x_1) - g \circ c_{M_i}^{-1}(x_2)}{r}\right) - Q_{M'_i}\big((x_1 - x_2, y_1 - y_2)\big) \\
&= Q_{M_j}\big(f_c(x_1) - f_c(x_2)\big) - \frac{\hat{\theta}}{r^2}\big|g \circ c_{M_i}^{-1}(x_1) - g \circ c_{M_i}^{-1}(x_2)\big|^2 - Q_{M_i}(x_1 - x_2) + \hat{\theta}|y_1 - y_2|^2 \\
&\geqslant \theta_{i,j}|x_1 - x_2|^2 - \frac{L_i^2}{r^2}\hat{\theta}|x_1 - x_2|^2 + \hat{\theta}|y_1 - y_2|^2 \\
&\geqslant \hat{\theta}\big(|x_1 - x_2|^2 + |y_1 - y_2|^2\big) \\
&> \frac{\hat{\theta}}{2}\big|(x_1, y_1) - (x_2, y_2)\big|^2.
\end{aligned}$$

Therefore, $M'_i \Longrightarrow^{F_0} M'_j$ satisfies the strong Liapunov condition.

By the finiteness of numbers of pairs (i, j), F_0 has covering relations with the strong Liapunov condition determined by A. By Proposition 15, there exists a compact subset Λ_0 of $\mathbb{R}^{m+k}$ such that Λ_0 is positively invariant for F_0 and $F_0|\Lambda_0$ is topologically semi-conjugate to σ_A^+. Since $F_\lambda(x, y)$ is C^1 in the triple (λ, x, y) of variables, by using the same argument as in the proof of Theorem 10, there exists $\lambda_0 > 0$ such that for all λ with $|\lambda| < \lambda_0$, the map F_λ has covering relation with the strong Liapunov condition determined by A. Since F_λ is a homeomorphism on $\mathbb{R}^{m+k}$ provided $\lambda \neq 0$, by Proposition 16, if $0 < |\lambda| < \lambda_0$ then there exists a compact subset Λ_λ of $\mathbb{R}^{m+k}$ such that Λ_λ is invariant for F_λ and $F_\lambda|\Lambda_\lambda$ is topologically conjugate to σ_A. We have finished the proof of the theorem.

References

[1] J. Juang, M.-C. Li, M. Malkin, Chaotic difference equations in two variables and their multidimensional perturbations, Nonlinearity 21 (2008) 1019–1040.

[2] M.-C. Li, M.-J. Lyu, P. Zgliczyński, Topological entropy for multidimensional perturbations of snap-back repellers and one-dimensional maps, Nonlinearity 21 (2008) 2555–2567.

[3] M.-C. Li, M. Malkin, Topological horseshoes for perturbations of singular difference equations, Nonlinearity 19 (2006) 795–811.

[4] M. Misiurewicz, P. Zgliczyński, Topological entropy for multidimensional perturbations of one-dimensional maps, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11 (2001) 1443–1446.

[5] O.T. O'Meara, Introduction to Quadratic Forms, Springer, Berlin, 2000.

[6] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, CRC Press, Boca Raton, FL, 1999.

[7] J.T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach Science Publishers, New York, 1969.
[8] L.-S. Young, Chaotic phenomena in three settings: large, noisy and out of equilibrium, Nonlinearity 21 (2008) T245–T252.
[9] P. Zgliczyński, Fixed point index for iterations, topological horseshoe and chaos, Topol. Methods Nonlinear Anal. 8 (1996) 169–177.
[10] P. Zgliczyński, Computer assisted proof of chaos in the Rössler equations and in the Hénon map, Nonlinearity 10 (1997) 243–252.
[11] P. Zgliczyński, Covering relation, cone conditions and the stable manifold theorem, J. Differential Equations 246 (2009) 1774–1819.
[12] P. Zgliczyński, M. Gidea, Covering relations for multidimensional dynamical systems, J. Differential Equations 202 (2004) 32–58.