

## CONSIDERING THE EFFECT OF BODY FORCE FOR REGIONAL LAND DISPLACEMENTS

RUEY-HSING LU

*Department of Civil Engineering, National Chiao-Tung University, Taiwan, ROC*

AND

HUND-DER YEH

*Institute of Environmental Engineering, National Chiao-Tung University, 75, Po-Ai Street, Hsinchu, Taiwan 30039, ROC*

### SUMMARY

In this study, a general mathematical model was developed for land displacements that occur as a result of the pressure decline in confined and/or phreatic aquifers. Two equations were developed by employing the equilibrium (force balance) equation and the flow equation in a deforming aquifer system. Both of these equations were integrated over the thickness of the aquifer system once a regional model had been formulated. The change of the body force in saturated and unsaturated zones of aquifers was considered in the derivation of the equilibrium equation. It was assumed that no external loading or force acting on the aquifer system was present. Two coupled equations expressed in terms of the averaged dilation and pressure were then obtained. The pumping data sets given in the papers of Bear and Corapcioglu<sup>1,2</sup> in 1981 and 1983 were analysed. Drawdown and displacements in a confined aquifer were demonstrated to be identical to those estimated by Bear and Corapcioglu.<sup>1</sup> In the case of a phreatic aquifer, however, the results estimated by the present approach were slightly different from those obtained by Corapcioglu and Bear.<sup>2</sup> The discrepancies in the results are possibly due to errors in the equations presented by Corapcioglu and Bear.<sup>2</sup> The present approach was able to avoid several assumptions and complex procedures used by Bear and Corapcioglu,<sup>1,2</sup> especially in the case of a phreatic aquifer, by taking into account the change in the body force.

### INTRODUCTION

The occurrence of land subsidence has recently become a common and extremely serious problem in several cities around the world. Several factors cause land subsidence to occur, including crustal isostatic adjustments, volcanic/tectonic activity, earthquakes, soil extraction, surface loading, hydrocompaction and fluid withdrawal from underground reservoirs.<sup>3</sup> Two factors involving humans are the physical loading of soil layers and the change of hydraulic equilibrium in a system comprised of aquifers and aquitards. The latter, a result of heavy groundwater withdrawal, has in particular posed serious land subsidence problems in several cities. Concern regarding such problems has been demonstrated by many investigators and a substantial amount of relevant research has recently been conducted.

Several models that require laboratory and/or field data have been based upon the consolidation theory introduced by Terzaghi.<sup>4</sup> Models presented by Miller,<sup>5</sup> Bouwer<sup>6</sup> and Zeevaert<sup>7</sup> predicted the amount of land subsidence by applying the results of one-dimensional consolidation tests conducted on core samples of an aquifer system and the decline in an artesian head. A formulation established by Schatz *et al.*<sup>8</sup> estimated the upper bound of surface subsidence on the basis of some assumptions and the depth–porosity relation obtained from downhole logging

data. Empirical relations were employed for predicting future subsidence by Lofgren and Klausning<sup>9</sup> and Gabrysch and Bonnet<sup>10</sup> by means of utilizing the observed relationship between water-level decline and land subsidence. Gambolati *et al.*<sup>11</sup> performed a mathematical simulation of the subsidence at Ravenna by utilizing the mixed finite element, finite difference and integral models. Lewis and Schrefler<sup>12</sup> employed the finite element method to simulate land displacement problems in the areas of soil mechanics and reservoir engineering. Various mathematical models have been formulated by several investigators to compute land subsidence or displacements on the basis of theories and/or physical concepts. Models discussed by Corapcioglu<sup>3</sup> included the semi-infinite elastic solid models (e.g. References 13–16), elastic constitutive approach (e.g. References 17–22), viscoelastic constitutive approach (e.g. References 23–25), plastic stress–strain relation models (e.g. References 26 and 27), and models that combine two or more of the above approaches.

Most of the models mentioned above predict only the vertical displacement (i.e. subsidence). Uncoupled situations allow for the elimination of variables depending on the assumption that only one-dimensional (vertical) displacement occurs. These situations, however, occur only in special circumstances and potentially lead to substantial errors.<sup>28</sup> Horizontal displacement is also substantial and may be, in some situations, of the same order of magnitude as vertical displacement.<sup>29</sup> Coupling the transient flow equation and the equilibrium equation allows pressure, horizontal displacement and vertical displacement to be simultaneously solved. Although this results in more rigorous, often lengthy mathematical and numerical procedures, most real subsidence problems in nature are coupled and should be solved as such.<sup>28</sup> Vertical displacement produced by coupled models has been demonstrated by Lewis *et al.*<sup>28</sup> to be smaller than that produced by uncoupled approaches. An approach that coupled the flow and the equilibrium equations by utilizing the elastic-constitutive relationship was presented by Biot.<sup>17</sup> Recently, several investigators predicted the pressure decline, horizontal displacement and vertical displacement and discussed land displacement behaviour on the basis of Biot's<sup>17</sup> work (e.g. References 1, 2, 21 and 22). Debbbarh<sup>30</sup> solved the fully coupled Biot's<sup>17</sup> equations by using Laplace transforms. He then compared the coupled results with the uncoupled ones obtained under Terzaghi's<sup>4</sup> assumption of three-dimensional flow and vertical stress. In a recent comparative analysis of coupling and uncoupling, Gambolati<sup>31</sup> points out that coupling has been found by Debbbarh's<sup>30</sup> studies to be weak in land subsidence simulations. Nevertheless, Lewis *et al.*<sup>31</sup> advocate the use of a coupled model, since Biot's<sup>17</sup> coupled equations do not require further assumptions regarding either the displacement direction or total stress changes. Biot's<sup>17</sup> work is employed in the present study to propose a more generalized mathematical formulation. A regional formulation is established by considering the effect of body force in the equilibrium equation of stress field. Results for regional land displacements are demonstrated in two case studies.

#### DERIVATION OF MATHEMATICAL MODEL

A unit volume of deforming porous media is first considered. The forces which act on the unit volume must satisfy the equilibrium conditions. In the absence of inertial effects, the equilibrium (force balance) equation of stress field<sup>32</sup> takes the following form:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \mathbf{f}_i = 0, \quad i, j = x, y, z \quad (1)$$

where  $\sigma_{ij} = \sigma_{ji}$  is the total stress tensor and  $\mathbf{f}_i$  is the body force in the  $i$  direction. According to the effective stress concept introduced by Terzaghi,<sup>4</sup> the total stress is composed of two parts: one is

due to the fluid pressure in the pores and the other is due to effective stress in the solid skeleton. The relationship between the stress and the fluid pressure can be expressed as  $\sigma'_{ij} = \sigma_{ij} + (1 - \gamma) P_{\text{avg}} \delta_{ij}$ .  $\sigma'_{ij}$  is the effective stress tensor,  $P_{\text{avg}} = S_o P_o + S_w P_w + S_g P_g$  represents the average of fluid pressure (positive for compression) of multiphases;<sup>12</sup>  $\delta_{ij}$  is the Kronecker delta symbol and  $\gamma = \beta_s/\alpha$  denotes the ratio between the solid compressibility  $\beta_s$  and the bulk porous medium compressibility  $\alpha$ .  $S_o, S_w, S_g, P_o, P_w$  and  $P_g$  are the saturations and the pressures of oil, water and gas (air) phases, respectively. The value of  $\gamma$  is quite small in natural soil<sup>33</sup> and is consequently omitted here. The oil phase is absent in this study. The air pressure is assumed to be atmospheric and can therefore be neglected. The relationship between the stress and the pressure can then be reduced to  $\sigma_{ij} = \sigma'_{ij} - S_w P \delta_{ij}$  once the subscript of  $P_w$  has been omitted. The body force may be expressed as  $\mathbf{f}_i = [\rho n S_w + (1 - n)\rho_s] \mathbf{g}_i$ , where  $\rho$  is the density of water,  $n$  is the porosity,  $\rho_s$  is the density of solid and  $\mathbf{g}_i$  is the gravity acceleration component in the  $i$  direction. Equation (1) may be rewritten in terms of the effective stress and the water pressure as

$$\frac{\partial \sigma'_{ij}}{\partial x_j} + \mathbf{f}_i = \frac{\partial}{\partial x_i} (S_w P), \quad i, j = x, y, z \quad (2)$$

The displacement of media may take place as a result of changes which occur in the effective stress. Media are assumed here to be perfectly elastic. The constitutive relationship between the effective stress and the displacement of the media may be expressed by Hooke's law as<sup>34</sup>

$$\sigma'_{ij} = G \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \varepsilon \delta_{ij}, \quad i, j = x, y, z \quad (3)$$

where  $G$  and  $\lambda$  are known as Lamé's constants,  $u_i$  is the displacement component in the  $i$  direction and  $\varepsilon = \partial u_k / \partial x_k$  and  $k = x, y, z$  is the volume dilation of porous media. Equation (2) can be written in terms of the displacement with the assistance of the constitutive relationship expressed in equation (3). This produces

$$G \nabla^2 u_i + (\lambda + G) \frac{\partial}{\partial x_i} \left( \frac{\partial u_k}{\partial x_k} \right) + [\rho n S_w + (1 - n)\rho_s] \mathbf{g}_i = \frac{\partial}{\partial x_i} (S_w P), \quad i, k = x, y, z \quad (4)$$

Equation (4) contains four unknown variables,  $u_x, u_y, u_z$  and  $P$ , but represents only three equations in the  $x, y$  and  $z$  directions, respectively.

The flow equation of deforming media<sup>35</sup> may be written as

$$\nabla \cdot \mathbf{q}_r + S_w \frac{\partial}{\partial t} \left( \frac{\partial u_k}{\partial x_k} \right) + n \frac{\partial S_w}{\partial t} + n \beta S_w \frac{\partial P}{\partial t} = q, \quad k = x, y, z \quad (5)$$

where  $\mathbf{q}_r$  is Darcy's velocity,  $\beta$  is the compressibility of water and  $q$  is the sink term. The relationship between the saturation  $S_w$  and the pressure  $P$  can be found by the retention curve in unsaturated media.

The general equations of saturated or unsaturated deforming media may then be obtained by coupling equation (4) with equation (5). These equations would then be equivalent to the equations provided by Biot<sup>17,34</sup> if the body force and the unsaturated zone are neglected. The change of water content in an aquifer due to pumping would induce a change in the body force. This change occurs even though the influence may be small in some situations.

## FORMULATION FOR REGIONAL PROBLEMS

Land subsidence may be treated as a regional problem if the horizontal distance of interest is significantly larger than the aquifer thickness. The variations of horizontal displacements in the

vertical direction, along the vertical thickness of an aquifer system, are normally not of interest. The integrated effect over the entire thickness of an aquifer system, being a function of horizontal co-ordinates and time only, is generally employed to predict regional problems.<sup>36</sup> The unsaturated effect may be insignificant and negligible for regional problems. The equilibrium equation, equation (2), is integrated over the thickness of the compressible layer of an aquifer system to obtain

$$\begin{aligned} \frac{\partial}{\partial x_j} (D\bar{\sigma}'_{ij}) + D\bar{\mathbf{f}}_i - \frac{\partial}{\partial x_i} (D\bar{P}) + \left( \sigma'_{ik}|_{F_2} \frac{\partial F_2}{\partial x_k} - \sigma'_{ik}|_{F_1} \frac{\partial F_1}{\partial x_k} \right) \\ - \left( P|_{F_2} \frac{\partial F_2}{\partial x_i} - P|_{F_1} \frac{\partial F_1}{\partial x_i} \right) = 0, \quad i, j = x, y, \quad k = x, y, z \end{aligned} \quad (6)$$

where  $F_1(x, y, z) = z - b_1(x, y)$  and  $F_2(x, y, z) = z - b_2(x, y)$  represent the equations of the bottom and top surfaces, respectively, and  $b_1(x, y)$  and  $b_2(x, y)$  are the elevations of the bottom and top bounding surfaces of the compressible layer, respectively. The overbar denotes the averaged quantity over the thickness of the compressible layer  $D(x, y) = b_2(x, y) - b_1(x, y)$ . Thus,

$$\bar{\chi}(x, y) = \frac{1}{D} \int_{b_1(x, y)}^{b_2(x, y)} \chi(x, y, z) dz,$$

where  $\chi$  denotes any scalar or component of a vector. No additional external force on the top and bottom bounding surfaces of the compressible layer is assumed to exist, i.e.  $[\sigma'_{ij}] \cdot \nabla \mathbf{F} = [\sigma'_{ij} - P\delta_{ij}] \cdot \nabla \mathbf{F} = 0$ . Then, equation (6) may be reduced to

$$\frac{\partial}{\partial x_j} (D\bar{\sigma}'_{ij}) + D\bar{\mathbf{f}}_i - \frac{\partial}{\partial x_i} (D\bar{P}) = 0, \quad i, j = x, y \quad (7)$$

It is assumed here that no variations are present in horizontal displacements along the vertical direction when the averaged constitutive relationship between the effective stress and the displacement is considered. The relationship between components of averaged dilation and displacement may be described as

$$\bar{\varepsilon} = \bar{\varepsilon}_x + \bar{\varepsilon}_y + \bar{\varepsilon}_z = \frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} = \frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\Delta z}{D} \quad (8)$$

The averaged dilation,  $\bar{\varepsilon}$ , represents the average of the volume dilation over the compressible layer in one unit area of soil column. Based on equation (3), the relationship between the averaged effective stress and the averaged displacement may be represented as

$$\bar{\sigma}_{ij} = G \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \bar{\varepsilon} \delta_{ij}, \quad i, j = x, y, z \quad (9)$$

Equation (7) in the  $x$  direction is differentiated with respect to  $x$ , and equation (7) in the  $y$  direction is differentiated with respect to  $y$ . The results are then totalled. The following relationship can be obtained by substituting equations (8) and (9) into this latter result:

$$\begin{aligned} (\lambda + 2G) \nabla'^2 (D\bar{\varepsilon}') + \lambda \nabla'^2 \left[ D \left( \frac{\Delta z}{D} \right) \right] + \nabla' (D\bar{\mathbf{f}}) - \nabla'^2 (D\bar{P}) \\ + 2G \sum_{i, j, i \neq j} \left\{ \frac{\partial}{\partial x_i} \left[ \frac{\partial \bar{u}_j}{\partial x_i} \frac{\partial D}{\partial x_j} \right] - \frac{\partial}{\partial x_i} \left[ \frac{\partial \bar{u}_j}{\partial x_j} \frac{\partial D}{\partial x_i} \right] \right\} = 0, \quad i, j = x, y \end{aligned} \quad (10)$$

where the prime symbol over a vector or an operator denotes that the vector component or operator is only in the  $x, y$  horizontal plane. The last two bracket terms become small and negligible if the thickness of the compressible layer varies smoothly in space at any time. So,

$$(\lambda + 2G)\mathcal{V}'^2(D\bar{\varepsilon}') + \lambda\mathcal{V}'^2[D(\Delta z/D)] + \mathcal{V}'(D\bar{\mathbf{f}}') - \mathcal{V}'^2(D\bar{P}) = 0 \quad (11)$$

The displacements of an aquifer are assumed to be under the conditions of plane total stress, i.e.  $\sigma_{xz} = \sigma_{zx} = 0$  and  $\sigma_{yz} = \sigma_{zy} = 0$ . Integrating the equilibrium equation (1) in the  $z$  direction over the thickness of the compressible layer, one obtains

$$\sigma_{zz} + D\bar{\mathbf{f}}_z = \bar{\sigma}'_{zz} + D\bar{\mathbf{f}}_z - \bar{P} = 0 \quad (12)$$

where  $\bar{\sigma}'_{zz} \approx \sigma_{zz}$  is assumed. Rewriting equation (12) by the averaged constitutive relationship, i.e. equations (8) and (9), results in

$$2G\left(\frac{\Delta z}{D}\right) = \bar{P} - D\bar{\mathbf{f}}_z - \lambda\bar{\varepsilon} \quad (13)$$

Substituting equation (13) into equation (11) yields

$$2(\lambda + G)\mathcal{V}'^2 = (D\bar{\varepsilon}) = 2\mathcal{V}'^2(D\bar{P}) - \mathcal{V}'(D\bar{\mathbf{f}}') - \mathcal{V}'^2(D^2\bar{\mathbf{f}}_z) \quad (14)$$

By integrating the flow equation (5) over the saturated thickness, one then has<sup>1,2</sup>

$$\mathcal{V}' \cdot (B \bar{\mathbf{q}}_r') + B \frac{\partial \bar{\varepsilon}}{\partial t} + \rho g n \beta B \frac{\partial \bar{\phi}}{\partial t} = \bar{q} \quad (15a)$$

and

$$\mathcal{V}' \cdot (B \bar{\mathbf{q}}_r') + B \frac{\partial \bar{\varepsilon}}{\partial t} + (\rho g n \beta B + S_y) \frac{\partial \bar{\phi}}{\partial t} = \bar{q} + N \quad (15b)$$

Equations (15a) and (15b) describe the flow moving in confined aquifers and phreatic aquifers, respectively, where  $N$  is the rate of accretion and represents the surface infiltration or evapotranspiration,  $S_y$  is the specific yield,  $B$  is the saturated thickness of an aquifer ( $B = b_2 - b_1 = D$  for confined aquifers and  $B = h(x, y) - b_1$  for phreatic aquifers, with  $h(x, y)$  representing the elevation of the water table) and  $\bar{\phi} \approx (\bar{P}/\rho g) + \bar{z}$  is the hydraulic head.  $S_y$  represents the volume of water drained by gravity per unit area and unit drop of the phreatic surface. The coupled equations, equations (14) and (15), which include the averaged dilation and pressure, are then constructed. These two equations may be linearized by assuming that  $\Delta z \ll D$  for confined aquifers and both  $\Delta z \ll D$  and  $\delta \ll D$  and  $\delta \ll B$  for phreatic aquifers, where  $\delta = \bar{\phi}^0 - \bar{\phi}$  is the drawdown, and the superscript 0 denotes the initial quantity of a scalar or a vector. These assumptions lead to a constancy of thickness of the compressible layer and saturated thickness for both confined and phreatic aquifers.  $\bar{\mathbf{V}}'_s \cdot \mathcal{V}' D \ll \partial D / \partial t$  is also assumed, where  $\bar{\mathbf{V}}'_s$  is the horizontal vector of displacement velocity. The equilibrium equation then becomes

$$2(\lambda + G)\mathcal{V}'^2(D\bar{\varepsilon}) = 2\rho g\mathcal{V}'^2[D(\bar{\phi} - \bar{z})] - \mathcal{V}'(D\bar{\mathbf{f}}') - \mathcal{V}'^2(D^2\bar{\mathbf{f}}_z) \quad (16)$$

The flow equation for a confined aquifer becomes

$$-\mathcal{V}' \cdot B(\bar{\mathbf{K}}' \cdot \mathcal{V}' \bar{\phi}) + B \frac{\partial \bar{\varepsilon}}{\partial t} + \rho g n \beta B \frac{\partial \bar{\phi}}{\partial t} = \bar{q} \quad (17a)$$

and the flow equation for a phreatic aquifer becomes

$$-\nabla' \cdot B(\bar{K}' \cdot \nabla' \bar{\phi}) + B \frac{\partial \bar{\varepsilon}}{\partial t} + (\rho g n \beta B + S_y) \frac{\partial \bar{\phi}}{\partial t} = \bar{q} + N \quad (17b)$$

where  $\bar{q}'_r = -\bar{K}' \cdot \nabla' \bar{\phi}$  and  $\bar{K}'$  is the averaged hydraulic conductivity tensor in the horizontal plane. Note that  $\partial \bar{z} / \partial t \approx 0$  occurs due to linearization. The operator of  $\nabla'$  can be expressed in either Cartesian co-ordinates or radial co-ordinates. Radial co-ordinates are employed in the following case studies.

## CASE STUDIES AND RESULTS

### *A regional case for single well pumping from a confined aquifer*

A single fully penetrating well of radius  $r_w$  pumping at a constant rate of  $Q_w$  from a horizontal infinite confined aquifer is considered. The formation of the aquifer is homogeneous and isotropic. The magnitude of the vertical displacement is assumed to be rather small as compared with the thickness of the compressible layer. Thus,  $\Delta z(r, t) \ll D$ . The thickness of the aquifer system is assumed to be uniform in space and constant in time, i.e.  $D = B \simeq B^0$ . The initial and boundary conditions for such an aquifer system are as follows:

$$t \leq 0, \quad r \geq r_w, \quad \bar{\phi} = \bar{\phi}^0 \quad \text{and} \quad \bar{\varepsilon}, \bar{u}_r, \Delta z = 0 \quad (18a)$$

$$t > 0, \quad r = r_w, \quad \frac{\partial \bar{\phi}}{\partial r} = \frac{Q_w}{2\pi r_w \bar{K} B} \quad \text{and} \quad \bar{u}_r = 0 \quad (18b)$$

$$t > 0, \quad r \rightarrow \infty, \quad \bar{\phi} = \bar{\phi}^0, \quad \bar{\varepsilon}, \Delta z = 0 \quad \text{and} \quad \bar{u}_r \rightarrow 0 \quad (18c)$$

in which  $\bar{K}' = \bar{K}$  is a constant for a homogeneous and isotropic aquifer. Equation (18a) indicates that the hydraulic head is horizontally distributed and the dilation and displacements of the aquifer are zero at the initial time. Equation (18b) indicates that the pumping flux around the well obeys Darcy's law. The averaged horizontal displacement  $\bar{u}_r$  is always equal to zero at the well. Equation (18c) indicates that the aquifer system far from the well is not influenced by pumping.

If  $\rho$ ,  $\rho_s$  and  $n$  are constants, the body force over the entire soil column in the aquifer is uniform in space and constant in time. Equation (16) then becomes

$$(\lambda + G)\nabla'^2 \bar{\varepsilon} = \rho g \nabla'^2 \bar{\phi} \quad (19)$$

An integral function  $F(r, t)$ , which satisfies  $\nabla'^2 F = 0$  at all times, is introduced. The solution of equation (19) may be written as

$$(\lambda + G)\bar{\varepsilon} = \rho g \bar{\phi} + F(r, t) \quad (20)$$

Let  $F(r, t) = -\rho g \bar{\phi}^0$ , which satisfies the initial condition, equation (18a), and boundary condition, equation (18c). Accordingly,

$$\bar{\phi} = \bar{\phi}^0 + \frac{(\lambda + G)\bar{\varepsilon}}{\rho g} \quad (21)$$

Substituting equation (21) into the flow equation (17a) yields

$$\left( \frac{\rho g}{\lambda + G} + \rho g n \beta \right) \frac{\partial \bar{\phi}}{\partial t} = \bar{K} \nabla'^2 \bar{\phi} \quad (22)$$

The solution to equation (22) with the initial and boundary conditions, equation (18), then becomes

$$s = \bar{\phi}^0 - \bar{\phi}(r, t) = \frac{Q_w}{4\pi\bar{T}} \mathbf{W}(u) \quad (23)$$

where  $u = r^2/4C_v t = r^2 S/4\bar{T}t$ ,  $C_v = \bar{K}/\rho g(n\beta + 1/(\lambda + G)) = \bar{T}/S$  is the hydraulic diffusivity,  $\bar{T} = \bar{K}B$  is the averaged transmissivity,  $S$  is the storage coefficient, and  $\mathbf{W}(u)$  is the well function. The dilation  $\bar{\varepsilon}$  may be obtained from equations (21) and (23) as

$$\bar{\varepsilon}(r, t) = -\frac{\rho g Q_w}{4(\lambda + G)\pi\bar{T}} \mathbf{W}(u) \quad (24)$$

At  $t = 0$ , vertical displacement  $\Delta z = 0$  and dilation  $\bar{\varepsilon} = 0$ , equation (13) can be employed to obtain the following relation:

$$D\bar{\mathbf{f}}_z = D^0\bar{\mathbf{f}}_z^0 = \rho g(\bar{\phi}^0 - \bar{z}) \quad (25)$$

in which  $D\bar{\mathbf{f}}_z$  is a constant in time, since  $\bar{z}$  is a constant in time because of linearization. By substituting equation (25) into equation (13) and utilizing the result of equations (23) and (24), one has

$$\Delta z(r, t) = -\frac{\rho g D Q_w}{8\pi\bar{T}} \frac{1}{\lambda + G} \mathbf{W}(u) \quad (26)$$

which represents the solution for the vertical displacement. From equations (24) and (26), one obtains  $\Delta z/D = \bar{\varepsilon}/2$  and

$$\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} = \frac{\bar{\varepsilon}}{2} = -\frac{\rho g Q_w}{8(\lambda + G)\pi\bar{T}} \mathbf{W}(u) \quad (27)$$

where  $\bar{\varepsilon} = (\partial \bar{u}_r/\partial r) + (\bar{u}_r/r) + (\Delta z/D)$  is in radial co-ordinates. Equation (27) is solved by using the boundary condition of equation (18b) and letting  $u_w \mathbf{W}(u_w) \approx 0$  and  $e^{-u_w} \approx 1$ , where  $u_w = r_w^2/4C_v t$ . One then obtains the following solution for the horizontal displacement:

$$\bar{u}_r(r, t) = -\frac{\rho g C_v t Q_w}{4\pi\bar{T}r} \frac{1}{\lambda + G} [u\mathbf{W}(u) - e^{-u} + 1] \quad (28)$$

The drawdown, horizontal displacement and vertical displacement, i.e. equations (23), (28) and (26), respectively, are demonstrated here by utilizing data from the paper of Bear and Corapcioglu.<sup>1</sup> Consider a confined aquifer with  $\bar{T} = 95 \text{ cm}^2/\text{s}$ ,  $B^0 = D^0 = 142 \text{ m}$ ,  $Q_w = 50 \text{ l/s}$ ,  $\gamma_w = \rho g = 9.806 \times 10^{-3} \text{ N/cm}^3$ ,  $\lambda = 4.475 \times 10^3 \text{ N/cm}^2$ ,  $G = 4.475 \times 10^3 \text{ N/cm}^2$  and  $C_v = 6 \times 10^3 \text{ cm}^2/\text{s}$ . The water compressibility  $\beta$  is neglected. The results are depicted in Figures 1 and 2. The relationship of the drawdown, horizontal displacement and vertical displacement versus the distance from the well after continuous pumping for 3 yrs is shown in Figure 1. The relationship of the drawdown, horizontal displacement and vertical displacement versus time at a distance of 3 km from the pumping well is shown in Figure 2. The horizontal displacement can be seen from Figure 1 to have a maximum value of 16.92 cm at a distance of approximately 8.6 km from the pumping well. The magnitude of the horizontal displacement is of one order of magnitude or more greater than that of the vertical displacement beyond a distance of 2.2 km from the well. The value of the horizontal displacement is quite significant despite the fact that the vertical displacement far from the pumping well is small. Both vertical and horizontal displacements can be observed from Figure 2 to increase with time if the well is pumped continuously.

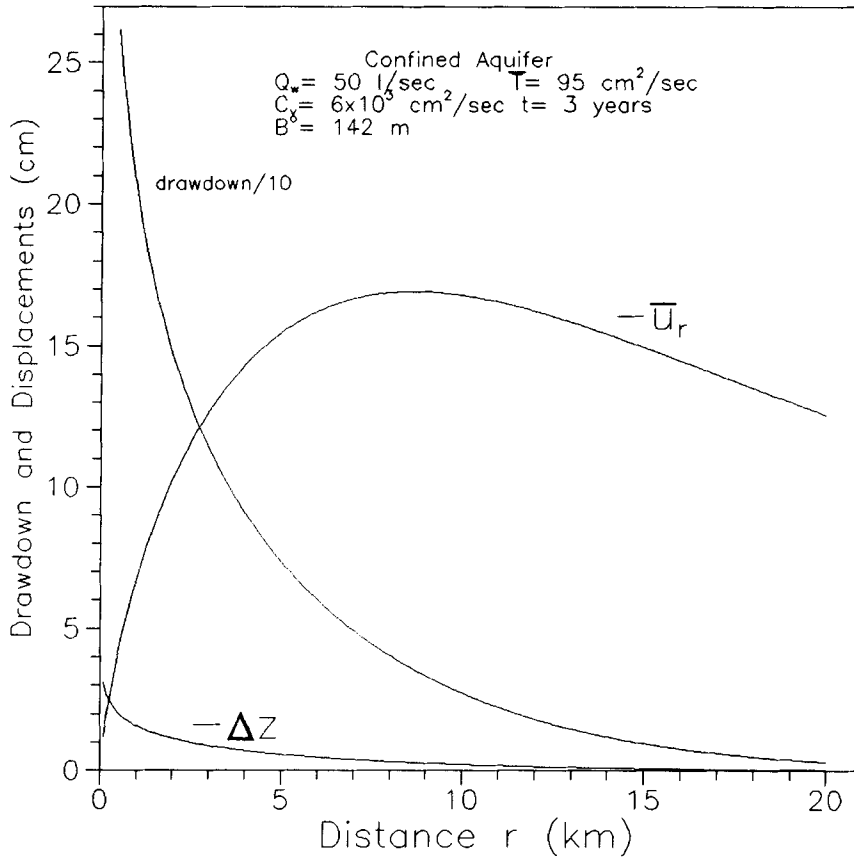


Figure 1. Spatial distribution of the drawdown, horizontal displacement  $\bar{u}_r$ , and vertical displacement  $\Delta z$  after continuous pumping in a confined aquifer for 3 yrs

*A regional case for single well pumping from a phreatic aquifer*

A single fully penetrating well of radius  $r_w$  pumping at a constant rate of  $Q_w$  from a horizontal infinite phreatic aquifer is considered. The formation of the aquifer is homogeneous and isotropic. The drawdown of the water table is assumed to be rather small as compared with the saturated thickness of the aquifer system at any time. The magnitude of the vertical displacement is also assumed to be rather small as compared with the thickness of the compressible layer. Thus,  $s(r, t) \ll B$  and  $\Delta z(r, t) \ll D$ . Based on these two assumptions,  $B$  and  $D$  are considered to be equal and constant in time. Both  $B^0$  and  $D^0$  are again assumed to be uniform in space. The initial and boundary conditions are

$$t \leq 0, \quad r \geq r_w, \quad \bar{\phi} = \bar{\phi}^0 \quad \text{and} \quad s, \bar{e}, \bar{u}_r, \Delta z = 0 \quad (29a)$$

$$t > 0, \quad r = r_w, \quad \frac{\partial \bar{\phi}}{\partial r} = -\frac{\partial s}{\partial r} = \frac{Q_w}{2\pi r_w K B} \quad \text{and} \quad \bar{u}_r = 0 \quad (29b)$$

$$t > 0, \quad r \rightarrow \infty, \quad \bar{\phi} = \bar{\phi}^0, \quad s, \bar{e}, \Delta z = 0 \quad \text{and} \quad \bar{u}_r \rightarrow 0 \quad (29c)$$

When the phreatic aquifer is pumped, the water table may drop as a consequence. The integrated effect of the body force over the entire soil column in the aquifer may then change. In



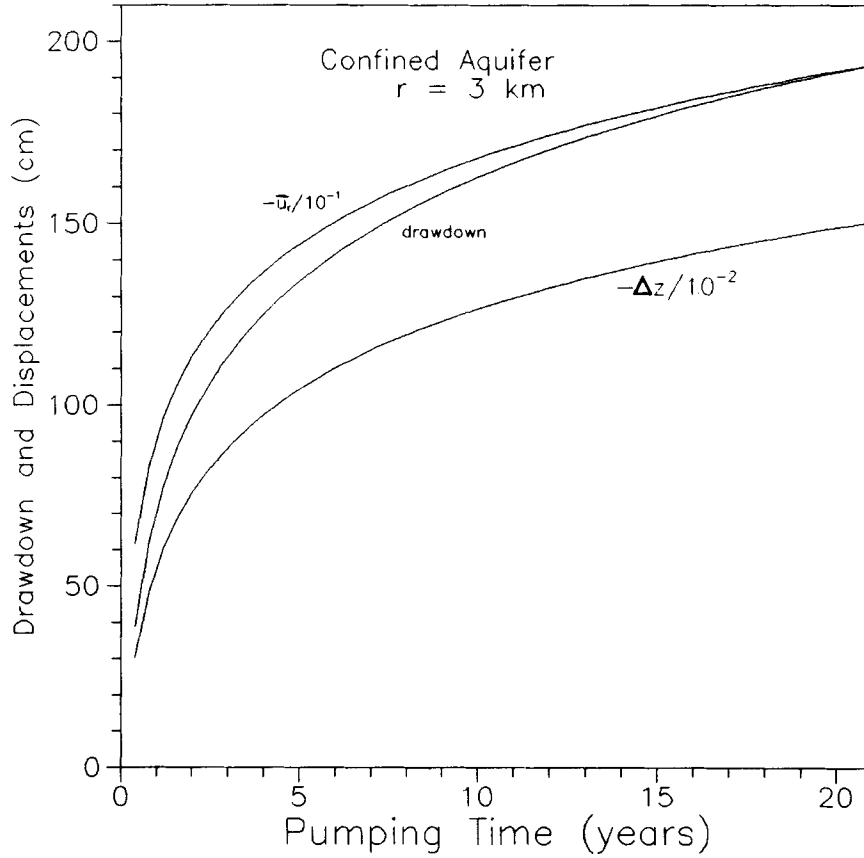


Figure 2. Temporal distribution of the drawdown, horizontal displacement  $\bar{u}_r$ , and vertical displacement  $\Delta z$  at a distance of 3 km from the pumping well in a confined aquifer

one unit area of the soil column, this change will be equal to the pore water weight of the dewatering zone. If  $\rho$ ,  $\rho_s$  and  $n$  are constants, the body force term of equation (16) will become

$$D\bar{f}_z = D^0\bar{f}_z^0 - \rho g S_y \sigma \quad (30)$$

where  $D\bar{f}_z = -D\bar{f}_z\mathbf{k}$  and  $\mathbf{k}$  denotes the unit vector along the positive  $z$  axis. On the basis of equation (30), equation (16) becomes

$$(\lambda + G)\nabla'^2\bar{\epsilon} = \rho g\nabla'^2\bar{\phi} - \frac{1}{2}\rho g S_y \nabla'^2\sigma \quad (31)$$

The integral function  $F(r, t)$ , which satisfies  $\nabla'^2 F = 0$  at all times, is again employed here. The solution of equation (31) may be written as

$$(\lambda + G)\bar{\epsilon} = \rho g\bar{\phi} - \frac{1}{2}\rho g S_y \sigma + F(r, t) \quad (32)$$

Let  $F(r, t) = -\rho g\bar{\phi}^0$ , which satisfies the initial condition, equation (29a), and boundary condition, equation (29c). Accordingly,

$$\left(1 + \frac{S_y}{2}\right)\sigma = -\frac{\lambda + G}{\rho g}\bar{\epsilon} \quad (33)$$

Substituting equation (33) into the flow equation (17b) leads to

$$\left[ B\rho g \left( \frac{1 + S_y/2}{\lambda + G} + n\beta \right) + S_y \right] \frac{\partial \phi}{\partial t} = \bar{T} \nabla^2 \phi \tag{34}$$

With the initial and boundary conditions, equation (29), the solution of equation (34) is

$$\phi(r, t) = \frac{Q_w}{4\pi \bar{T}} \mathbf{W}(u) \tag{35}$$

where

$$u = \frac{r^2}{4t(S_y/\bar{T} + 1/C_v^*)^{-1}}$$

and

$$C_v^* = \frac{\bar{T}}{\rho g B \left[ \left( n\beta + \frac{1}{\lambda + G} \right) + \frac{S_y/2}{\lambda + G} \right]}$$

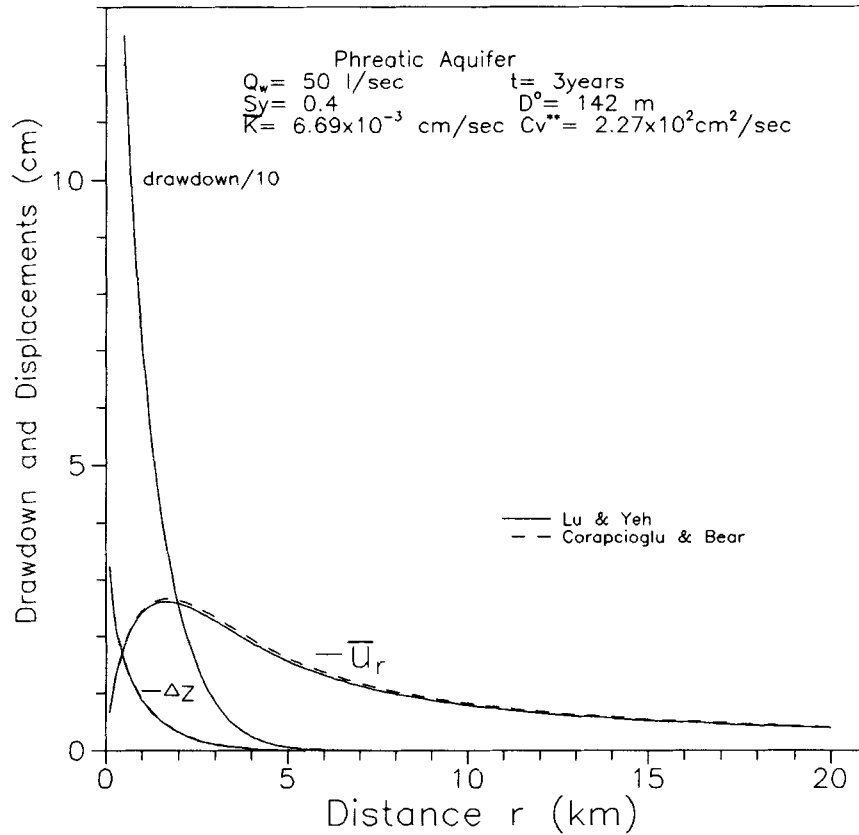


Figure 3. Spatial distribution of the drawdown, horizontal displacement  $\bar{u}_r$ , and vertical displacement  $\Delta z$  after continuous pumping in a phreatic aquifer for 3 yrs

The dilation  $\bar{\epsilon}$  is obtained from equations (33) and (35) as

$$\bar{\epsilon}(r, t) = -\frac{\rho g(2 + S_y)Q_w}{8(\lambda + G)\pi T} \mathbf{W}(u) \tag{36}$$

At  $t = 0$ , vertical displacement  $\Delta z = 0$  and dilation  $\bar{\epsilon} = 0$ , equation (13) can be applied to obtain the following relation:

$$D^0 \bar{f}_z^0 = \rho g \bar{\phi}^0 - \rho g \bar{z} \tag{37}$$

where  $\bar{z}$  is constant in time because of linearization. Substituting equations (30) and (37) into equation (13) and utilizing the result of equations (35) and (36), one obtains the following solution for the vertical displacement:

$$\Delta z(r, t) = -\frac{\rho g D Q_w}{8\pi T} \left[ \frac{1}{\lambda + G} + \frac{(\lambda + 2G)S_y}{2G(\lambda + G)} \right] \mathbf{W}(u) \tag{38}$$

From equations (35), (36) and (38), one obtains  $\Delta z/D = (\bar{\epsilon}/2) - (\rho g S_y/4G)\bar{\phi}$  and

$$\frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} = \frac{\bar{\epsilon}}{2} + \frac{\rho g S_y}{4G} \bar{\phi} \tag{39}$$

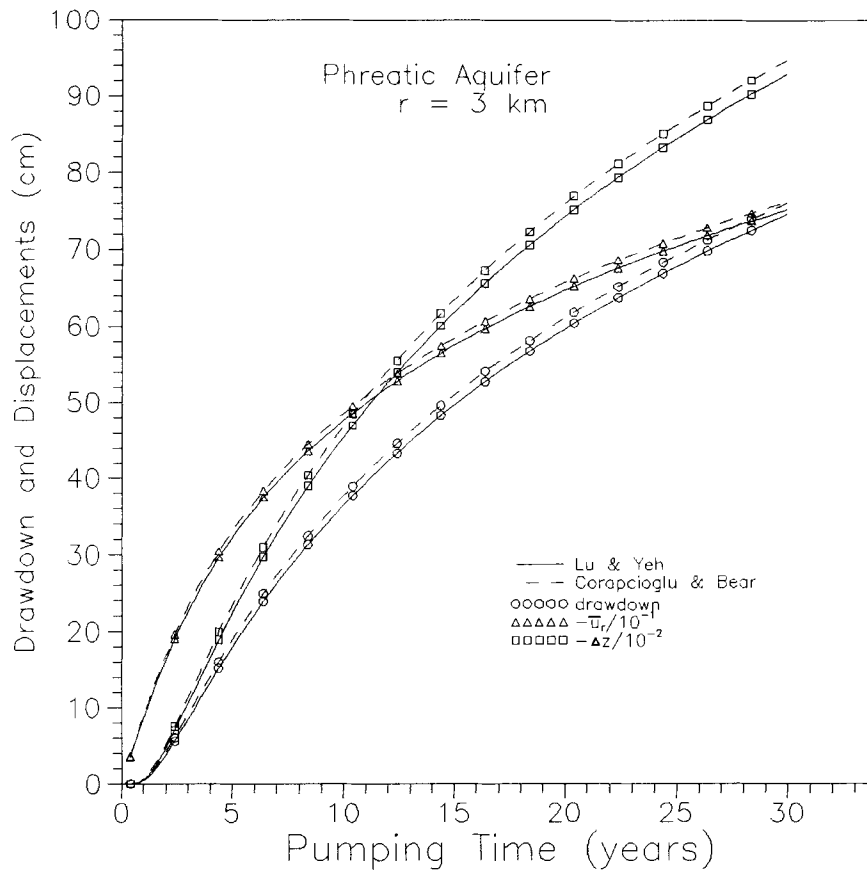


Figure 4. Temporal distribution of the drawdown, horizontal displacement  $\bar{u}_r$  and vertical displacement  $\Delta z$  at a distance of 3 km from the pumping well in a phreatic aquifer

Solving equation (39) with the boundary condition of equation (29b) and letting  $u_w \mathbf{W}(u_w) \approx 0$  and  $e^{-u_w} \approx 1$ , where  $u_w = r_w^2/4t(S_y/\bar{T} + 1/C_v^*)^{-1}$ , one obtains the following solution for the horizontal displacement:

$$\bar{u}_r(r, t) = -\frac{\rho g t Q_w}{4\pi \bar{T}(S_y/\bar{T} + 1/C_v^*)r} \left[ \frac{1}{\lambda + G} - \frac{\lambda S_y}{2G(\lambda + G)} \right] \left\{ u \mathbf{W}(u) - e^{-u} + 1 \right\} \quad (40)$$

The first term in the brackets in equations (38) and (40) is notably introduced by the occurrence of the drawdown. The second term, which includes  $S_y$ , is introduced by the change in body force. This term produced by the change of the body force will vanish, so that  $C_v^* = C_v$ , if the body force is omitted. The displacement behaviour will then become similar to that of the confined aquifer, which is affected only by the pressure decline.

The drawdown, horizontal displacement and vertical displacement, i.e. equations (35), (40) and (38), respectively, are presented here, again utilizing data from the paper of Corapcioglu and Bear.<sup>2</sup> Consider a phreatic aquifer with  $\bar{K} = 6.69 \times 10^{-3}$  cm/s,  $B^0 = D^0 = 142$  m,  $Q_w = 50$  l/s,  $S_y = 0.4$ ,  $C_v^{**} = [S_y/\bar{T} + 1/C_v^*]^{-1} = 2.27 \times 10^2$  cm<sup>2</sup>/s,  $\gamma_w = \rho g = 9.806 \times 10^{-3}$  N/cm<sup>3</sup>,  $\lambda = 4.475 \times 10^3$  N/cm<sup>2</sup> and  $G = 4.475 \times 10^3$  N/cm<sup>2</sup>. The water compressibility  $\beta$  is neglected. The

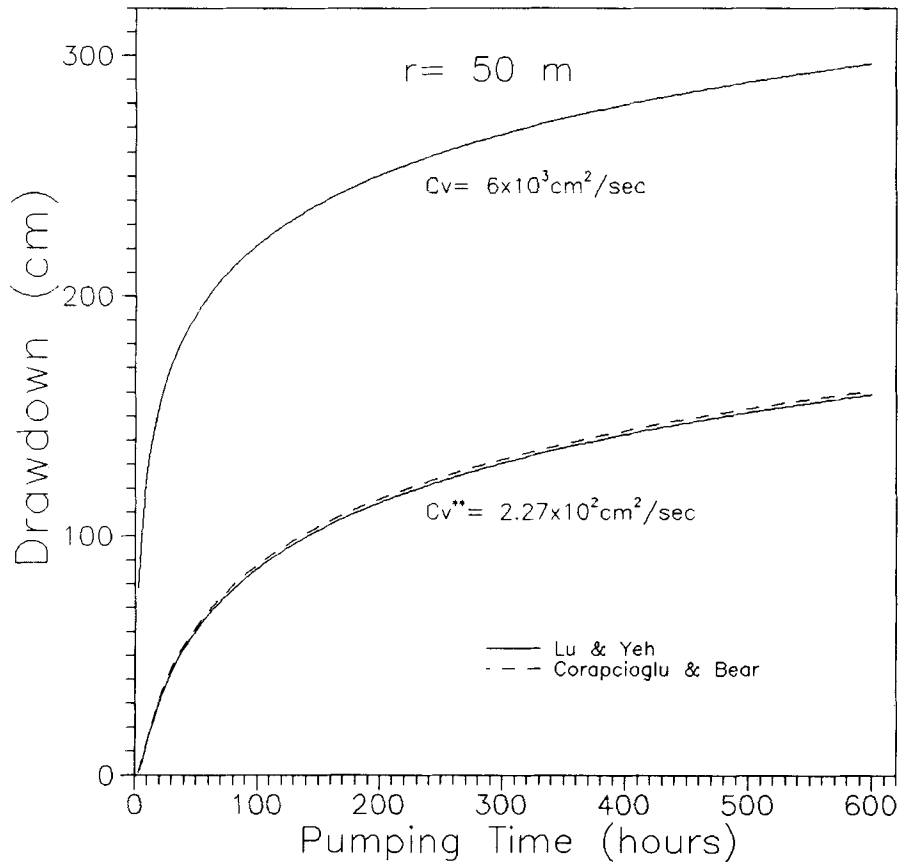


Figure 5. Comparison of the drawdowns for pumping in confined and phreatic aquifers at a distance of 50 m from the pumping well

results are drawn in Figures 3 and 4. The relationship of the drawdown, horizontal displacement and vertical displacement versus the distance from the well after continuous pumping for 3 yrs is shown in Figure 3. The relationship of the drawdown, horizontal displacement and vertical displacement versus time at a distance of 3 km from the pumping well is shown in Figure 4. The horizontal displacement can be observed from Figure 3 to have a maximum value of 2.16 cm at a distance of approximately 1.7 km from the pumping well. The magnitude of the horizontal displacement is of one order of magnitude or more greater than that of the vertical displacement beyond a distance of 1.9 km from the well. The influence range of the vertical displacement in a phreatic aquifer is smaller than that in a confined aquifer. Both vertical and horizontal displacements can be observed from Figure 4 to increase with time if the well is pumped continuously. Comparisons of the drawdowns, vertical displacements and horizontal displacements of confined and phreatic aquifers at a distance of 50 m from the pumping well are drawn in Figures 5, 6 and 7, respectively. Both horizontal and vertical displacements in the phreatic case can be seen to be smaller than those occurring in the confined case. Results by Corapcioglu and Bear<sup>2</sup> are also drawn in Figures 3–7. Their results, indicated by dashed lines in Figures 3–7, are slightly larger than our results in the case of the phreatic aquifer. These discrepancies are the result of errors made in their equations (74), (74'), (82) and (86).

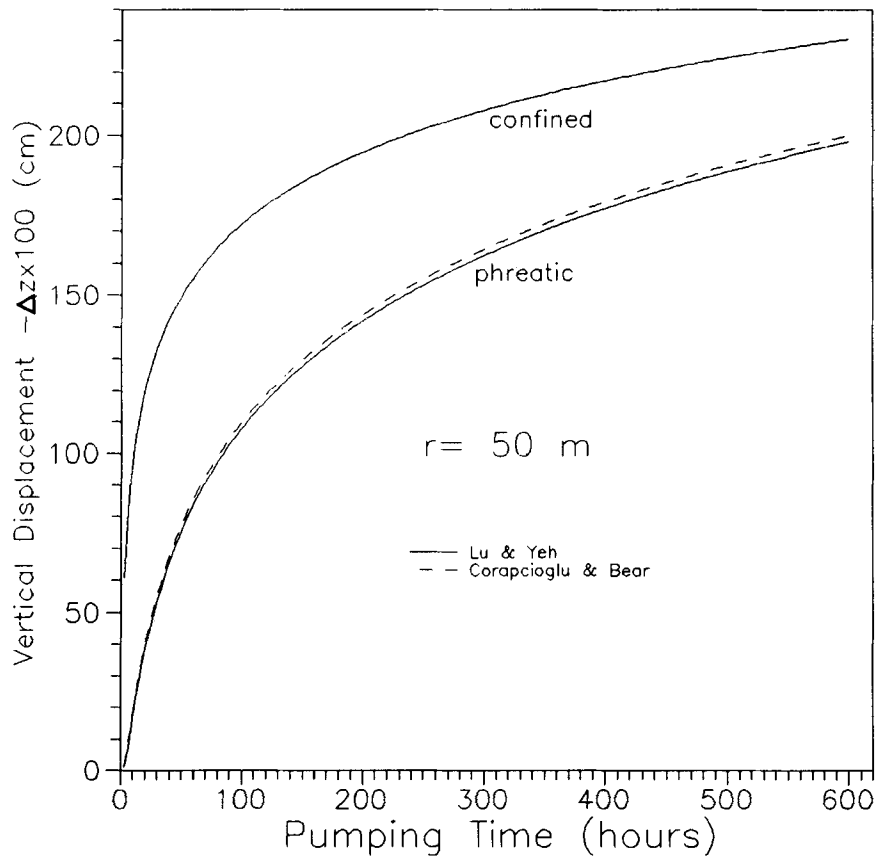


Figure 6. Comparison of the vertical displacements for pumping in confined and phreatic aquifers at a distance of 50 m from the pumping well

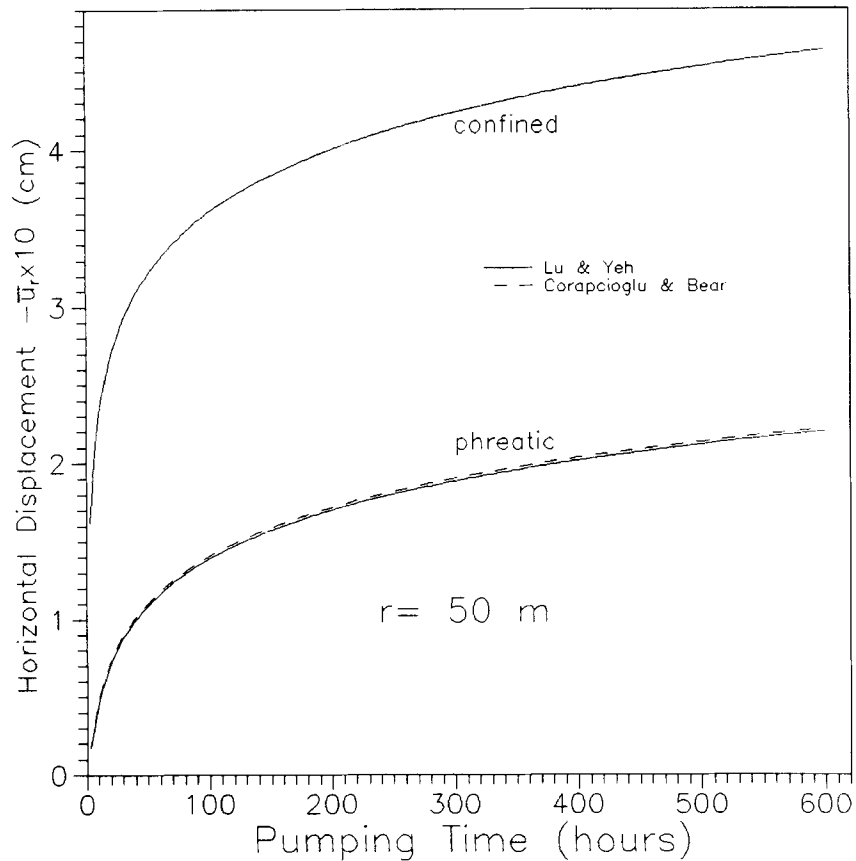


Figure 7. Comparison of the horizontal displacements for pumping in confined and phreatic aquifers at a distance of 50 m from the pumping well

### SUMMARY AND CONCLUSIONS

The coupled equations presented in this investigation take into consideration the effect of the body force in the equilibrium equation of stress field. An elastic constitutive relationship between the effective stress and the displacement was introduced in order to couple the equilibrium and flow equations. These equations were integrated over the vertical thickness of the aquifers. Assumptions made in the derivation include (1) no external force acting on the top and bottom bounding boundaries of the compressible layer exists and (2) the horizontal displacements have no variations along the vertical direction. The concept of plane total stress was also employed in this study. Two regional equations coupling the averaged dilation and pressure were then obtained. Since the integrated effect of the body force was considered in this approach, fewer assumptions were necessary and the procedures were less complicated than those of Bear and Corapcioglu,<sup>1,2</sup> especially in the case of phreatic aquifers. The pumping data sets provided in the papers of Bear and Corapcioglu<sup>1,2</sup> in 1981 and 1983, respectively, were analysed as simple case studies. The drawdown and displacements in a confined aquifer were demonstrated to be the same as the results presented by Bear and Corapcioglu.<sup>1</sup> In a phreatic aquifer, however, the results estimated by the proposed approach were slightly different from those of Corapcioglu and Bear.<sup>2</sup> Some equations in Corapcioglu and Bear<sup>2</sup> could possibly have been incorrect.

The horizontal displacement and vertical displacement in a phreatic aquifer can be observed from Figures 1–7 to be smaller than those in a confined aquifer. The influence range of the vertical displacement in a phreatic aquifer was also smaller than that in a confined aquifer. The horizontal displacement, which has been neglected by some researchers, was found to be quite significant and should not be neglected. For both confined and phreatic aquifers, the magnitude of the horizontal displacement was generally found to be almost one order of magnitude or more greater than that of the vertical displacement at distances beyond several kilometres from the pumping well. It is interesting to note that the mechanisms of displacements in the confined and phreatic aquifers are not the same. Displacements occurring in confined aquifers are induced by the pressure decline only. On the other hand, displacements occurring in phreatic aquifers are caused by the drop of the water table and the change in body force. Displacement behaviour in a phreatic aquifer, however, will appear similar to that in a confined aquifer if the change of the body force is neglected. Therefore, it is suggested that the change of the body force should be considered when phreatic aquifers are pumped.

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