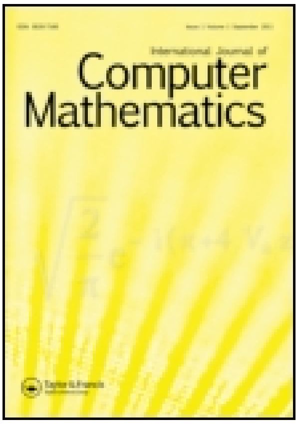

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

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Published online: 23 Dec 2010.

International Journal of Computer Mathematics
Vol. 88, No. 4, March 2011, 731–746

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(*Received 24 April 2009; revised version received 08 February 2010; accepted 15 March 2010*)

The star graph interconnection network has been recognized as an attractive alternative to the hypercube for its nice topological properties. Unlike previous research concerning the issue of embedding exactly one Hamiltonian cycle into an injured star network, this paper addresses the maximum number of fault-free mutually independent Hamiltonian cycles in the faulty star network. To be precise, let SG_n denote an n-dimensional star network in which $f \leq n-3$ edges may fail accidentally. We show that there exist $(n-2-f)$-mutually independent Hamiltonian cycles rooted at any vertex in SG_n if $n \in \{3, 4\}$, and there exist $(n-1-f)$-mutually independent Hamiltonian cycles rooted at any vertex in SG_n if $n \geq 5$.

Keywords: Hamiltonian; interconnection network; star graph; fault tolerance

2000 AMS Subject Classifications: 05C38; 05C45; 05C75; 05C90; 68M10

1. Introduction

The problem of finding Hamiltonian cycles in a graph is well known to be NP-complete and has been discussed in many areas. In 1969, Lovasz [27] asked whether every finite connected vertex-transitive graph has a Hamiltonian path, that is, a simple path that traverses every vertex exactly once.

DEFINITION 1 [4] *A graph is said to be vertex-transitive if for every pair u, v of vertices, there exists an automorphism of the graph that maps u into v. A graph is said to be edge-transitive if for any two edges a and b, there exists an automorphism of the graph that maps a into b.*

All known vertex-transitive graphs have a Hamiltonian path, but only four vertex-transitive graphs without any Hamiltonian cycle are known to exist. Since none of these four graphs

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ISSN 0020-7160 print/ISSN 1029-0265 online

DOI: 10.1080/00207161003786614
http://www.informaworld.com

is a Cayley graph, there is a folklore conjecture [6] that every Cayley graph with more than two vertices has a Hamiltonian cycle. In the last decades, this problem was extensively studied [2,3,5–7,10,11,17–19,28–30,35]. For those Cayley graphs for which the existence of Hamiltonian cycles has already been proved, more advanced properties, such as edge-Hamiltonicity, Hamiltonian connectivity, and Hamiltonian laceability, etc., are investigated [2,22]. In this paper, we address one of such properties, the concept of mutually independent Hamiltonian cycles [36,37], which is related to the number of Hamiltonian cycles in a given graph. Since its introduction, this topic has gained many researchers' attention [12,15,16,25,26,33]. In particular, Lin *et al.* [25] showed that the maximum number of mutually independent Hamiltonian cycles rooted at any vertex can be constructed recursively in the star graph interconnection network (for the detailed definitions, see Sections 2 and 3).

The interconnection network is of great interest in the area of parallel and distributed computer systems. Because it is usually multi-objected and complicated to design an interconnection network, its underlying topology can be modelled as a graph, whose vertices correspond to processors and whose edges correspond to connections/communication links. Hence, we use the terms graphs and networks interchangeably. Among various kinds of network topologies, the star graph is attractive for its high degree of symmetry. However, when some edges are removed at random from the star graph, the symmetry will be broken. Hence, we wonder, in a theoretical point of view, how many mutually independent Hamiltonian cycles can be formed in such an injured network. In this paper, the maximum number of fault-free mutually independent Hamiltonian cycles in the faulty star graph will be studied. To be precise, let SG_n denote an n-dimensional star graph with $f \leq n-3$ faulty edges. Then we aim at proving the following result: SG_n has $(n-2-f)$-mutually (respectively, $(n-1-f)$-mutually) independent Hamiltonian cycles rooted at any vertex if $n \in \{3,4\}$ (respectively, $n \geq 5$).

The rest of this paper is organized as follows. In Section 2, graph-theoretic notations and the definition of mutually independent Hamiltonian cycles are introduced. In Section 3, the star graph and its basic properties are presented. Section 4 consists of the proof of our main result. Finally, directions for future research are discussed in Section 5.

2. Preliminaries

Throughout this paper, graphs are simple, loopless, and undirected. For definitions and notations not defined here, see [4]. A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a non-empty set, and $E(G)$ is a subset of $\{\{u,v\} | \{u,v\}$ is a two-element subsets of $V(G)\}$. The set $V(G)$ is called the *vertex set* of G, and the set $E(G)$ is called the *edge set* of G. Two vertices u and v of G are *adjacent* if $\{u,v\} \in E(G)$. The *degree* of a vertex u in G is the number of edges incident to u. A graph G is *k-regular* if all its vertices have the same degree k. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint subsets, denoted by $V_0(G)$ and $V_1(G)$, such that every edge joins a vertex of $V_0(G)$ to a vertex of $V_1(G)$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a non-empty subset of $V(G)$. The subgraph of G *induced* by S is a subgraph of G with vertex set S, whose edge set consists of all the edges joining any two vertices in S. We use $G-S$ to denote the subgraph of G induced by $V(G)-S$. Let F be any subset of $E(G)$. Then we use $G-F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G)-F$. For any $S \subseteq V(G)$ and $F \subseteq E(G)$, graph $G-(S \cup F)$ is defined to be the graph $(G-F)-S$.

A *walk* of length $k \geq 1$ in a graph is a sequence of vertices, $W := v_1 v_2 \cdots v_{k+1}$, such that v_i and v_{i+1} are adjacent for $i = 1, 2, \ldots, k$. If $v_1 = x$ and $v_{k+1} = y$, we refer to W as an xy-walk. The notation xWy is also used simply to signify an xy-walk W. Moreover, we use W^{-1} to denote the reversed walk $v_{k+1} v_k \cdots v_1$. For any three vertices x, y, z in a graph, if xW_1y and yW_2z are walks, the sequence $x\ W_1\ y\ W_2\ z$, obtained by concatenating W_1 and W_2 at y, is a walk. A walk of length 0 consists of a single vertex. A *path* is a walk in which no vertex is repeated. For convenience, the ith vertex of a path P is denoted by $P(i)$. For any two vertices u and v in a graph G, the *distance* between u and v, denoted by $d_G(u, v)$, is the length of the shortest path between u and v. A *cycle* is a walk $v_1 v_2 \cdots v_{n+1}$ in which $n \geq 3$, $v_1 = v_{n+1}$, and the n vertices $v_1, v_2, \ldots, v_n$ are distinct.

A path (or cycle) in a graph G is a *Hamiltonian path* (or *Hamiltonian cycle*) of G if it spans G. A graph is *Hamiltonian* if it has a Hamiltonian cycle. A bipartite graph is *Hamiltonian laceable* [34] if there exists a Hamiltonian path between any two vertices that are in different partite sets. A Hamiltonian laceable graph G is said to be *hyper-Hamiltonian laceable* [22] if, for any $i \in \{0, 1\}$ and for any vertex $v \in V_i(G)$, there exists a Hamiltonian path in $G - \{v\}$ between any two vertices of $V_{1-i}(G)$.

Let G be a graph with N vertices. A *rooted* Hamiltonian cycle C in G can be described as $v_1\ v_2 \cdots v_N\ v_1$ to emphasize the order of vertices on C. Accordingly, v_1 is seen as the root vertex, and v_i is seen as the ith vertex on C. Two Hamiltonian cycles rooted at a given vertex s of G, namely $C_1 := v_1 v_2 \cdots v_N v_1$ and $C_2 := u_1 u_2 \cdots u_N u_1$ with $v_1 = u_1 = s$, are *independent* if $v_i \neq u_i$ for $2 \leq i \leq N$. A collection of m Hamiltonian cycles $C_1, \ldots, C_m$ in G, rooted at the same vertex, are said to be *m-mutually independent* if C_i and C_j are independent whenever $i \neq j$. Moreover, the *mutually independent Hamiltonicity* of G, denoted by $\mathcal{IHC}(G)$, is defined to be the maximum integer m such that for any vertex v of G, there exists a set of m-mutually independent Hamiltonian cycles rooted at v in G. The concept of mutually independent Hamiltonian cycles can be applied in many different areas [12,15,16,25,26,33].

3. The star graph

The hypercube has long been one of the most popular network topologies [21] because of its nice topological properties. The star graph, proposed by Akers and Krishnamurthy [1], is an attractive alternative to the hypercube for interconnecting processors in parallel computers. Since then, star networks have received many researchers' attention. For example, the diameter and fault diameter were computed in [1,20,32]. Moreover, Fragopoulou and Akl [8,9] studied how to embed directed edge-disjoint spanning trees into the star graph. The Hamiltonian properties of star graphs are addressed in [13,14,23,38]. In particular, because processors or links may fail accidentally to affect network performance, Tseng *et al.* [38] addressed fault-tolerant ring embedding in an injured star network if no more than $n - 3$ edge faults occur.

The definition of star graphs is described as follows. Let n be any positive integer. For convenience, we use $\mathbb{I}_n$ to denote the set $\{1, 2, \ldots, n\}$. A *permutation* $u_1 u_2 \cdots u_n$ on $\mathbb{I}_n$ is a sequence of all elements of $\mathbb{I}_n$. Every permutation can be written as a product of transpositions. An *even permutation* (respectively, *odd permutation*) is a permutation that can be written as a product of an even (respectively, odd) number of transpositions. The n-dimensional star graph SG_n is a graph whose vertex set is the set of all permutations on $\mathbb{I}_n$. Two vertices, $u_1 \cdots u_i \cdots u_n$ and $v_1 \cdots v_i \cdots v_n$, are adjacent through an edge of *dimension* i with $2 \leq i \leq n$ if $u_1 = v_i$, $v_1 = u_i$, and $u_j = v_j$ for $j \in \mathbb{I}_n - \{1, i\}$. Clearly, SG_n is $(n-1)$-regular with $n!$ vertices. Moreover, it is precisely a Cayley graph of the symmetric group with edge set consisting of all the transpositions of form $(1\ i)$, where $2 \leq i \leq n$. So it is vertex-transitive and edge-transitive [1]. The star graphs SG_2, SG_3, and SG_4 are illustrated in Figure 1.

For the sake of clarity, we use boldface letters to denote vertices of SG_n. Moreover, we use $\mathbf{e}$ to denote the vertex $12, \ldots, n$. It is known that SG_n is a bipartite graph with one partite set $V_0(\mathrm{SG}_n)$ consisting of all the even permutations and the other partite set $V_1(\mathrm{SG}_n)$ consisting of all the odd permutations. Let $\mathbf{u} = u_1 u_2 \cdots u_n$ be a vertex in SG_n. Then u_i is the ith *coordinate* of $\mathbf{u}$, denoted

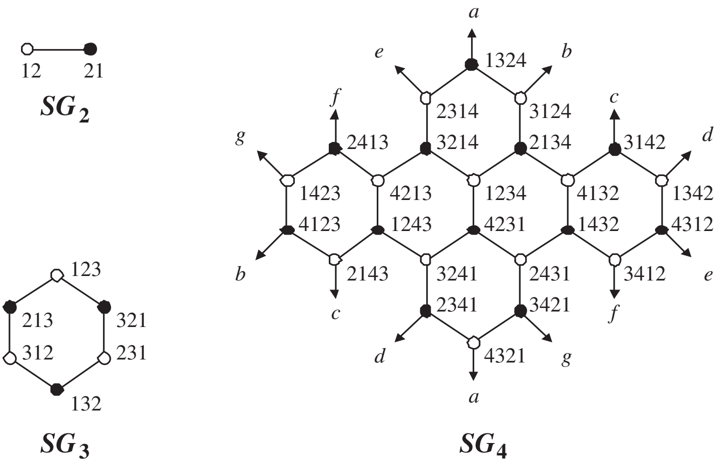

Figure 1. Illustrations for SG_2, SG_3, and SG_4.

by $(\mathbf{u})_i$, for $1 \le i \le n$. For any $2 \le i \le n$, the *i-neighbour* of vertex $\mathbf{u}$, denoted by $(\mathbf{u})^i$, is a vertex adjacent to $\mathbf{u}$ through an edge of dimension i. Obviously, $((\mathbf{u})^i)^i = \mathbf{u}$.

For any $1 \le i \le n$, let $SG_n^{\{i\}}$ denote the subgraph of SG_n induced by the set of vertices $\{\mathbf{u} \in V(SG_n) | (\mathbf{u})_n = i\}$. Then SG_n can be partitioned into n vertex-disjoint subgraphs $SG_n^{\{1\}}, aSG_n^{\{2\}}, \dots, SG_n^{\{n\}}$, and every of them is isomorphic to SG_{n-1}. Let $I \subseteq \mathbb{I}_n$. We use SG_n^I to denote the subgraph of SG_n induced by $\cup_{i\in I} V(SG_n^{\{i\}})$. For any pair i, j of distinct integers in $\mathbb{I}_n$, we use $E^{i,j}$ to denote the set of edges between $SG_n^{\{i\}}$ and $SG_n^{\{j\}}$.

In the rest of this section, we introduce some results to be used later.

THEOREM 1 [38] *Let $F \subset E(SG_n)$ with $|F| \le n-3$ for $n \ge 3$. Then $SG_n - F$ is Hamiltonian.*

Li *et al.* [23] introduced the *edge-fault-tolerant Hamiltonian laceability* of a bipartite graph G, which is the integer f such that for any $F \subseteq E(G)$ with $|F| \le f$, $G - F$ is still Hamiltonian laceable and there exists a subset F' of $E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not Hamiltonian laceable. Moreover, they also defined the *edge-fault-tolerant hyper-Hamiltonian laceability* of a graph G as the integer f such that for any $F \subseteq E(G)$ with $|F| \le f$, $G - F$ is hyper-Hamiltonian laceable and there exists a subset F' of $E(G)$ with $|F'| = f + 1$ such that $G - F'$ is no longer Hyper-Hamiltonian laceable.

THEOREM 2 [23] *The star graph SG_n is $(n-3)$-edge-fault-tolerant Hamiltonian laceable and $(n-4)$-edge-fault-tolerant hyper-Hamiltonian laceable for $n \ge 4$.*

LEMMA 1 [31] *Assume that $n \ge 3$. Then $|E^{i,j}| = (n-2)!$ for any $1 \le i \ne j \le n$. Moreover, there are $(n-2)!/2$ pairwise disjoint edges joining vertices of $V_t(SG_n^{\{i\}})$ to vertices of $V_{1-t}(SG_n^{\{j\}})$ for any $t \in \{0, 1\}$.*

LEMMA 2 *For $n \ge 3$, let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices of SG_n with $d_{SG_n}(\mathbf{u}, \mathbf{v}) \le 2$. Then $(\mathbf{u})_1 \ne (\mathbf{v})_1$.*

LEMMA 3 *Let $n \ge 5$ and $F \subset E(SG_n)$ with $|F| \le n-4$. Assume that $I = \{a_1, \dots, a_r\}$ is an r-element subset of $\mathbb{I}_n$ for any $r \in \mathbb{I}_n$. Suppose that $\mathbf{u} \in V_t(SG_n^{\{a_1\}})$ and $\mathbf{v} \in V_{1-t}(SG_n^{\{a_r\}})$ for any*

$t \in \{0, 1\}$. *Then there exists a Hamiltonian path* $H := \mathbf{x}_1 P_1 \mathbf{y}_1 \mathbf{x}_2 P_2 \mathbf{y}_2 \cdots \mathbf{x}_r P_r \mathbf{y}_r$ *in* $SG_n^I - F$ *such that* $\mathbf{x}_1 = \mathbf{u}, \mathbf{y}_r = \mathbf{v}$, *and* P_i *is a Hamiltonian path of* $SG_n^{\{a_i\}} - F$ *joining* $\mathbf{x}_i$ *to* $\mathbf{y}_i$ *for every* $1 \le i \le r$.

Proof Without loss of generality, we can assume that $t = 0$. Since $\mathrm{SG}_n^{\{a_1\}}$ is isomorphic to SG_{n-1}, this statement holds for $r = 1$ by Theorem 2. Thus, suppose that $r \ge 2$ and set $\mathbf{x}_1 = \mathbf{u}$ and $\mathbf{y}_r = \mathbf{v}$. By Lemma 1, there are $((n-2)!/2) > n - 4$ pairwise disjoint edges joining vertices of $V_1(\mathrm{SG}_n^{\{a_i\}})$ to vertices of $V_0(\mathrm{SG}_n^{\{a_{i+1}\}})$ for every $i \in \mathbb{I}_{r-1}$. Therefore, we can choose $\{\mathbf{y}_i, \mathbf{x}_{i+1}\} \in E^{a_i, a_{i+1}} - F$ with $\mathbf{y}_i \in V_1(\mathrm{SG}_n^{\{a_i\}})$ and $\mathbf{x}_{i+1} \in V_0(\mathrm{SG}_n^{\{a_{i+1}\}})$ for $i \in \mathbb{I}_{r-1}$. By Theorem 2, $\mathrm{SG}_n^{\{a_i\}} - F$ has a Hamiltonian path P_i joining $\mathbf{x}_i$ to $\mathbf{y}_i$ for every $i \in \mathbb{I}_r$. As a result, the sequence of vertices, $\mathbf{x}_1\ P_1\ \mathbf{y}_1\ \mathbf{x}_2\ P_2\ \mathbf{y}_2 \cdots \mathbf{x}_r\ P_r\ \mathbf{y}_r$, forms a desired Hamiltonian path of $\mathrm{SG}_n^I - F$ joining $\mathbf{u}$ to $\mathbf{v}$. ■

LEMMA 4 *Let* $n \ge 5$. *Assume that* $F \subset E(SG_n)$ *with* $|F| \le n - 4$, *and* $|F \cap SG_n^{\{i\}}| \le n - 5$ *for every* $i \in \mathbb{I}_n$. *Moreover, assume that* $I = \{a_1, \ldots, a_r\}$ *is an* r*-element subset of* $\mathbb{I}_n$ *for any* $2 \le r \le n$. *Suppose that* $\mathbf{u} \in V_t(SG_n^{\{a_1\}})$, $\mathbf{w} \in V_{1-t}(SG_n^{\{a_1\}})$, *and* $\mathbf{v} \in V_t(SG_n^{\{a_r\}})$ *for any* $t \in \{0, 1\}$. *Then there exists a Hamiltonian path* H *of* $(SG_n^I - F) - \{\mathbf{w}\}$ *joining* $\mathbf{u}$ *to* $\mathbf{v}$.

Proof Without loss of generality, we can assume that $t = 0$. By Lemma 1, there are $(n-2)!/2 > n - 3$ pairwise disjoint edges joining vertices of $V_0(\mathrm{SG}_n^{\{a_1\}})$ to vertices of $V_1(\mathrm{SG}_n^{\{a_2\}})$. Thus, we can choose a vertex $\mathbf{x}$ of $V_0(\mathrm{SG}_n^{\{a_1\}}) - \{\mathbf{u}\}$ with $(\mathbf{x})_1 = a_2$ and $\{\mathbf{x}, (\mathbf{x})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path P of $(\mathrm{SG}_n^{\{a_1\}} - F) - \{\mathbf{w}\}$ joining $\mathbf{u}$ to $\mathbf{x}$. By Lemma 3, there exists a Hamiltonian path Q of $\mathrm{SG}_n^{I-\{a_1\}} - F$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. As a result, the sequence of vertices, $\mathbf{u}\ P\ \mathbf{x}\ (\mathbf{x})^n\ Q\ \mathbf{v}$, forms a desired Hamiltonian path. ■

LEMMA 5 [24] *Let* $\mathbf{w}$ *and* $\mathbf{b}$ *denote two adjacent vertices of* SG_n *with* $n \ge 4$. *For any vertex* $\mathbf{u}$ *in* $V_t(SG_n) - \{\mathbf{w}, \mathbf{b}\}$, $t \in \{0, 1\}$, *and for any* $i \in \mathbb{I}_n$, *there exists a Hamiltonian path* P *of* $SG_n - \{\mathbf{w}, \mathbf{b}\}$ *joining* $\mathbf{u}$ *to some vertex* $\mathbf{v}$ *in* $V_{1-t}(SG_n) - \{\mathbf{w}, \mathbf{b}\}$ *with* $(\mathbf{v})_1 = i$.

LEMMA 6 *Let* $i \in \mathbb{I}_n$ *and* $F \subset E(SG_n)$ *with* $|F| \le n - 4$ *for* $n \ge 4$. *Suppose that* $\mathbf{w}$ *and* $\mathbf{b}$ *are two adjacent vertices of* SG_n, *and* $\mathbf{u} \in V_t(SG_n) - \{\mathbf{w}, \mathbf{b}\}$ *for any* $t \in \{0, 1\}$. *Then there exists a Hamiltonian path of* $(SG_n - F) - \{\mathbf{w}, \mathbf{b}\}$ *joining* $\mathbf{u}$ *to some vertex* $\mathbf{v}$ *of* $V_{1-t}(SG_n) - \{\mathbf{w}, \mathbf{b}\}$ *with* $(\mathbf{v})_1 = i$.

Proof Without loss of generality, we can assume that $t = 0$. Since SG_n is vertex-transitive, we can assume that $\mathbf{w} = \mathbf{e}$ and $\mathbf{b} = (\mathbf{e})^j$ with some $j \in \mathbb{I}_n - \{1\}$. We set $F_k = F \cap E(\mathrm{SG}_n^{\{k\}})$ for every $k \in \mathbb{I}_n$. The proof is done by induction on n. The induction basis, that is, the case $n = 4$, follows from Lemma 5. Suppose that this statement holds for SG_{n-1} with $n \ge 5$. We consider the dimensions of all edges in $F \cup \{\{\mathbf{e}, (\mathbf{e})^j\}\}$. If there is an edge in F whose dimension, say j', is different from j, then SG_n can be partitioned into n vertex-disjoint subgraphs with the j'th coordinate of each vertex (that is, the subgraph of SG_n induced by the vertices with the same j'th coordinate is SG_{n-1}). Otherwise, every edge of F has the same dimension j.

Case 1 The dimension j' exists. Without loss of generality, we can assume that $j' = n$. Thus, we have $\{\mathbf{e}, (\mathbf{e})^j\} \in E(\mathrm{SG}_n^{\{n\}})$ and $|F_k| \le n - 5$ for every $k \in \mathbb{I}_n$.

Subcase 1.1 Suppose that $\mathbf{u} \in V_0(\mathrm{SG}_n^{\{n\}})$. Since $|F| \le n - 4$, we can choose an integer $r \in \mathbb{I}_{n-1}$ such that $|F \cap E^{r,n}| = 0$. By the induction hypothesis, there exists a Hamiltonian path P of $(\mathrm{SG}_n^{\{n\}} - F_n) - \{\mathbf{e}, (\mathbf{e})^j\}$ joining $\mathbf{u}$ to a vertex $\mathbf{x} \in V_1(\mathrm{SG}_n^{\{n\}})$ with $(\mathbf{x})_1 = r$. We can choose a vertex $\mathbf{v}$ in $V_1(\mathrm{SG}_n^{\mathbb{I}_{n-1}-\{r\}})$ with $(\mathbf{v})_1 = i$. By Lemma 3, there exists a Hamiltonian path Q of $\mathrm{SG}_n^{\mathbb{I}_{n-1}} - F$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then the sequence of vertices, $\mathbf{u} P \mathbf{x} (\mathbf{x})^n Q \mathbf{v}$, is a desired path.

Subcase 1.2 Suppose that $\mathbf{u} \in V_0(\mathrm{SG}_n^{\{k\}})$ for some $k \in \mathbb{I}_{n-1}$. By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_1(\mathrm{SG}_n^{\{k\}})$ to vertices of $V_0(\mathrm{SG}_n^{\{n\}})$. We can pick out a vertex $\mathbf{y}$ of $V_1(\mathrm{SG}_n^{\{k\}})$ such that $(\mathbf{y})^n \in V_0(\mathrm{SG}_n^{\{n\}}) - \{\mathbf{e}\}$ and $\{\mathbf{y}, (\mathbf{y})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path H of $\mathrm{SG}_n^{\{k\}} - F_k$ joining $\mathbf{u}$ to $\mathbf{y}$. We can choose an integer r of $\mathbb{I}_{n-1} - \{k\}$ such that $|F \cap E^{r,n}| = 0$. By the induction hypothesis, there exists a Hamiltonian path P of $(\mathrm{SG}_n^{\{n\}} - F_n) - \{\mathbf{e}, (\mathbf{e})^j\}$ joining $(\mathbf{y})^n$ to a vertex $\mathbf{x}$ of $V_1(\mathrm{SG}_n^{\{n\}}) - \{(\mathbf{e})^j\}$ with $(\mathbf{x})_1 = r$. Besides, we choose a vertex $\mathbf{v}$ of $V_1(\mathrm{SG}_n^{\mathbb{I}_{n-1}-\{k,r\}})$ with $(\mathbf{v})_1 = i$. By Lemma 3, there exists a Hamiltonian path Q of $S_n^{\mathbb{I}_{n-1}-\{k\}} - F$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then the sequence of vertices, $\mathbf{u}$ H $\mathbf{y}$ $(\mathbf{y})^n$ P $\mathbf{x}$ $(\mathbf{x})^n$ Q $\mathbf{v}$, turns out to be a desired path.

Case 2 Every edge in F has the same dimension j. Without loss of generality, we may assume that $j = n$. Thus, we have $|F_t| = 0$ for every $t \in \mathbb{I}_n$.

Subcase 2.1 Suppose that $\mathbf{u} \in V_0(\mathrm{SG}_n^{\{k\}})$ for some $k \in \mathbb{I}_{n-1} - \{1\}$. By Lemma 1, there are $(n-2)!/2 > n-4$ pairwise disjoint edges joining vertices of $V_1(\mathrm{SG}_n^{\{k\}})$ to vertices of $V_0(\mathrm{SG}_n^{\{1\}})$. Thus, we can choose a vertex $\mathbf{x}$ of $V_1(\mathrm{SG}_n^{\{k\}})$ with $(\mathbf{x})_1 = 1$ and $\{\mathbf{x}, (\mathbf{x})^n\} \notin F$. By Theorem 2, there exits a Hamiltonian path H of $\mathrm{SG}_n^{\{k\}}$ joining $\mathbf{u}$ to $\mathbf{x}$. Similarly, we can choose a vertex $\mathbf{y}$ of $V_0(\mathrm{SG}_n^{\{1\}})$ with $(\mathbf{y})_1 = n$, $\{\mathbf{y}, (\mathbf{y})^n\} \notin F$, and $\mathbf{y} \neq (\mathbf{x})^n$. This can be done because there are $((n-2)!)/2 \geq n-2$ pairwise disjoint edges between the sets $V_0(\mathrm{SG}_n^{\{1\}})$ and $V_1(\mathrm{SG}_n^{\{k\}})$. By Theorem 2, $\mathrm{SG}_n^{\{1\}} - \{(\mathbf{e})^n\}$ has a Hamiltonian path P joining $(\mathbf{x})^n$ to $\mathbf{y}$. Let $\mathbf{v}$ be a vertex in $V_1(\mathrm{SG}_n^{\mathbb{I}_{n-1}-\{1,k\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4, there exists a Hamiltonian path Q of $(\mathrm{SG}_n^{\mathbb{I}_n-\{1,k\}} - F) - \{\mathbf{e}\}$ joining $(\mathbf{y})^n$ to $\mathbf{v}$. Then the sequence of vertices, $\mathbf{u}$ H $\mathbf{x}$ $(\mathbf{x})^n$ P $\mathbf{y}$ $(\mathbf{y})^n$ Q $\mathbf{v}$, turns out to be a desired path.

Subcase 2.2 Suppose that $\mathbf{u} \in V_0(\mathrm{SG}_n^{\{1\}})$. By Lemma 1, there are $(n-2)!/2 > n-4$ pairwise disjoint edges joining vertices of $V_0(\mathrm{SG}_n^{\{1\}})$ to vertices of $V_1(\mathrm{SG}_n^{\{n\}})$. Thus, we can choose a vertex $\mathbf{x}$ of $V_0(\mathrm{SG}_n^{\{1\}}) - \{\mathbf{u}\}$ with $(\mathbf{x})_1 = n$ and $\{\mathbf{x}, (\mathbf{x})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path H of $\mathrm{SG}_n^{\{1\}} - \{(\mathbf{e})^n\}$ joining $\mathbf{u}$ to $\mathbf{x}$. Furthermore, we choose a vertex $\mathbf{v}$ of $V_1(\mathrm{SG}_n^{\mathbb{I}_{n-1}-\{1\}})$ with $(\mathbf{v})_1 = i$. By Lemma 4, there exists a Hamiltonian path Q of $(\mathrm{SG}_n^{\mathbb{I}_n-\{1\}} - F) - \{\mathbf{e}\}$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Then the sequence of vertices, $\mathbf{u}$ H $\mathbf{x}$ $(\mathbf{x})^n$ Q $\mathbf{v}$, forms a desired path.

Subcase 2.3 Suppose that $\mathbf{u} \in V_0(\mathrm{SG}_n^{\{n\}})$. Since $|F| \leq n-4$, we can choose two integers k_1 and k_2 in $\mathbb{I}_{n-1} - \{1\}$ such that $\{(\mathbf{e})^{k_1}, ((\mathbf{e})^{k_1})^n\} \notin F$ and $\{(\mathbf{e})^{k_2}, ((\mathbf{e})^{k_2})^n\} \notin F$. Let $X = \{\{\mathbf{e}, (\mathbf{e})^t\} | t \in \mathbb{I}_{n-1} - \{1, k_1, k_2\}\}$. Obviously, $|X| = n-4$. Moreover, we can choose a vertex $\mathbf{x} \in V_1(\mathrm{SG}_n^{\{n\}})$ such that $(\mathbf{x})_1 \in \mathbb{I}_{n-1} - \{1, k_1, k_2\}$ and $\{\mathbf{x}, (\mathbf{x})^n\} \notin F$. Since $(\mathbf{x})_1 \neq k_1$ and $(\mathbf{x})_1 \neq k_2$, we have $\mathbf{x} \neq (\mathbf{e})^{k_1}$ and $\mathbf{x} \neq (\mathbf{e})^{k_2}$. By Theorem 2, there exists a Hamiltonian path H of $\mathrm{SG}_n^{\{n\}} - X$ joining $\mathbf{u}$ to $\mathbf{x}$. Because vertex $\mathbf{e}$ has precisely two neighbours, that is, $(\mathbf{e})^{k_1}$ and $(\mathbf{e})^{k_2}$, in $\mathrm{SG}_n^{\{n\}} - X$, the edges $\{\mathbf{e}, (\mathbf{e})^{k_1}\}$ and $\{\mathbf{e}, (\mathbf{e})^{k_2}\}$ must be consecutive on H. Thus, with no loss of generality, we can write $H = \mathbf{u}\ H_1 (\mathbf{e})^{k_1} \mathbf{e} (\mathbf{e})^{k_2} H_2 \mathbf{x}$. Let $\mathbf{y} = (\mathbf{e})^{k_2}$. Since $(\mathbf{y})_1 \neq (\mathbf{x})_1$, we have $i \neq (\mathbf{x})_1$ or $i \neq (\mathbf{y})_1$.

Subcase 2.3.1 Suppose that $i \neq (\mathbf{x})_1$. Let $k_3 = (\mathbf{x})_1$. We choose a vertex $\mathbf{v}$ of $V_1(\mathrm{SG}_n^{\{k_3\}})$ with $(\mathbf{v})_1 = i$. By Lemma 1, there are $(n-2)!/2 > n-4$ pairwise disjoint edges joining vertices of $V_1(\mathrm{SG}_n^{\{k_1\}})$ to vertices of $V_0(\mathrm{SG}_n^{\{1\}})$. Thus, we can choose a vertex $\mathbf{z}$ of $V_1(\mathrm{SG}_n^{\{k_1\}})$ with $(\mathbf{z})_1 = 1$ and $\{\mathbf{z}, (\mathbf{z})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path T of $\mathrm{SG}_n^{\{k_3\}}$ joining $(\mathbf{x})^n$ to $\mathbf{v}$. Similarly, there exist a Hamiltonian path P of $\mathrm{SG}_n^{\{k_1\}}$ joining $((\mathbf{e})^{k_1})^n$ to $\mathbf{z}$. By Lemma 4, there exists a Hamiltonian path Q of $(\mathrm{SG}_n^{\mathbb{I}_{n-1}-\{k_1,k_3\}} - F) - \{(\mathbf{e})^n\}$ joining $(\mathbf{z})^n$ to $(\mathbf{y})^n$. Then the sequence of vertices, $\mathbf{u}$ H_1 $(\mathbf{e})^{k_1}$ $((\mathbf{e})^{k_1})^n$ P $\mathbf{z}$ $(\mathbf{z})^n$ Q $(\mathbf{y})^n$ $\mathbf{y}$ H_2 $\mathbf{x}$ $(\mathbf{x})^n$ T $\mathbf{v}$, is a desired one.

Subcase 2.3.2 Suppose that $i \neq (\mathbf{y})_1$. Let $k_3 = (\mathbf{y})_1$. Then the proof of this case happens to be similar to that of Subcase 2.3.1. Thus, we omit the details. ■

LEMMA 7 *Let $\{a, b\} \subset \mathbb{I}_n$ with $a < b$, and let $F \subset E(SG_n)$ with $|F| \leq n - 4$ for $n \geq 4$. Suppose that $\mathbf{x} \in V_0(SG_n)$, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are two distinct neighbours of $\mathbf{x}$. Then there exists a Hamiltonian path of $(SG_n - F) - \{\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2\}$ between two vertices $\mathbf{u}$ and $\mathbf{v}$ in $V_0(SG_n) - \{\mathbf{x}\}$ such that $(\mathbf{u})_1 = a$ and $(\mathbf{v})_1 = b$.*

Proof Since SG_n is vertex-transitive, we can assume that $\mathbf{x} = \mathbf{e}$, $\mathbf{x}_1 = (\mathbf{e})^{i_1}$, and $\mathbf{x}_2 = (\mathbf{e})^{i_2}$ with some $\{i_1, i_2\} \subset \{2, 3, \ldots, n\}$. Then this lemma is proved by induction on n.

Suppose that $n = 4$. Thus, we have $|F| = 0$. Since SG_4 is edge-transitive, we can assume that $\mathbf{x}_1 = (\mathbf{e})^2 = 2134$ and $\mathbf{x}_2 = (\mathbf{e})^3 = 3214$. The required paths of $\mathrm{SG}_4 - \{1234, 2134, 3214\}$ are listed in Table 1.

Suppose that the statement holds for SG_{n-1} with $n \geq 5$. Let $F_k = F \cap E(\mathrm{SG}_n^{\{k\}})$ for every $k \in \mathbb{I}_n$. Without loss of generality, suppose that F contains at least one edge of dimension n. Thus, we have $|F_k| \leq n - 5$ for every $k \in \mathbb{I}_n$. Because $a < b$, we have $a \neq n$ and $b \neq 1$. Since $|F| \leq n - 4$, we can choose an integer c in $\mathbb{I}_{n-1} - \{1, a\}$ such that $|F \cap E^{c,n}| = 0$. Moreover, we can choose a vertex $\mathbf{v}$ of $V_0(\mathrm{SG}_n^{\{1\}})$ with $(\mathbf{v})_1 = b$.

Case 1 Suppose that $i_1 \neq n$ and $i_2 \neq n$. By the induction hypothesis, there exists a Hamiltonian path H of $(\mathrm{SG}_n^{\{n\}} - F_n) - \{\mathbf{e}, (\mathbf{e})^{i_1}, (\mathbf{e})^{i_2}\}$ joining a vertex $\mathbf{u}$ of $V_0(\mathrm{SG}_n^{\{n\}})$ with $(\mathbf{u})_1 = a$ to a vertex $\mathbf{y}$ of $V_0(\mathrm{SG}_n^{\{n\}})$ with $(\mathbf{y})_1 = c$. By Lemma 3, there exists a Hamiltonian path R of $\mathrm{SG}_n^{\mathbb{I}_{n-1}} - F$ joining $(\mathbf{y})^n$ to $\mathbf{v}$. As a result, the sequence of vertices, $\mathbf{u}\ H\ \mathbf{y}\ (\mathbf{y})^n\ R\ \mathbf{v}$, forms a desired path in $(\mathrm{SG}_n - F) - \{\mathbf{e}, (\mathbf{e})^{i_1}, (\mathbf{e})^{i_2}\}$.

Case 2 Either $i_1 = n$ or $i_2 = n$. Without loss of generality, we can assume that $i_2 = n$. We choose a vertex $\mathbf{u} \in V_0(\mathrm{SG}_n^{\{n\}})$ with $(\mathbf{u})_1 = a$. By Lemma 6, there exists a Hamiltonian path H of $(\mathrm{SG}_n^{\{n\}} - F_n) - \{\mathbf{e}, (\mathbf{e})^{i_1}\}$ joining a vertex $\mathbf{u}$ to some vertex $\mathbf{y}$ of $V_1(\mathrm{SG}_n^{\{n\}})$ with $(\mathbf{y})_1 = c$. By Lemma 4, there exists a Hamiltonian path Q of $(\mathrm{SG}_n^{\mathbb{I}_{n-1}} - F) - \{(\mathbf{e})^n\}$ joining $(\mathbf{y})^n$ to $\mathbf{v}$. As a result, the sequence of vertices, $\mathbf{u}\ H\ \mathbf{y}\ (\mathbf{y})^n\ Q\ \mathbf{v}$, is a desired path. ■

4. Mutually independent Hamiltonian cycles in faulty star graphs

Lin *et al.* [25] showed the next theorem.

THEOREM 3 [25] *$\mathcal{IHC}(SG_3) = 1$, $\mathcal{IHC}(SG_4) = 2$, and $\mathcal{IHC}(SG_n) = n - 1$ if $n \geq 5$.*

For the sake of clarity, our main result, Theorem 4, will be divided into three lemmas (Lemma 8–10).

LEMMA 8 *Let $f \in E(SG_4)$. Then $\mathcal{IHC}(SG_4 - \{f\}) = 1$.*

Proof Since SG_4 is edge-transitive, we can assume that $f = \{1234, 4231\}$. By Theorem 1, there exists a Hamiltonian cycle in $\mathrm{SG}_4 - \{f\}$. Thus, we have $\mathcal{IHC}(\mathrm{SG}_4 - \{f\}) \geq 1$. To show that $\mathcal{IHC}(\mathrm{SG}_4 - \{f\}) \leq 1$, it suffices to point out that there will be no two-mutually independent Hamiltonian cycles rooted at vertex 1234. In Table 2, we list all Hamiltonian cycles of $\mathrm{SG}_4 - \{f\}$ rooted at 1234. By brute force, we can check that there do not exist two-mutually independent Hamiltonian cycles. Hence, the proof is completed. ■

Table 1. The required Hamiltonian paths in $SG_4 - \{1234, 2134, 3214\}$.

$a = 1$ and $b = 2$	1324	3142	4132	1432	3412	4312	2314	1324	3124	4123	2143	1243	4213	2413	1423	3421	4321	2341	3241	4231	2431
$a = 1$ and $b = 3$	1423	2413	4213	1243	2143	4123	3124	1324	2314	4312	3412	1432	4132	3142	1342	2341	4321	3421	2431	4231	3241
$a = 1$ and $b = 4$	1324	3142	4132	1432	3412	4312	2314	1324	3124	4123	2143	1243	4213	2413	1423	3421	2431	4231	3241	2341	4321
$a = 2$ and $b = 3$	2314	1324	3124	4123	2143	1243	4213	2413	1423	3421	4321	2341	3241	4231	2431	1432	4132	3142	1342	4312	3412
$a = 2$ and $b = 4$	2314	1324	3124	4123	2143	1243	4213	2413	1423	3421	4321	2341	3241	4231	2431	1432	3412	4312	1342	3142	4132
$a = 3$ and $b = 4$	3124	1324	2314	4312	3412	1432	4132	3142	1342	2341	4321	3421	2431	4231	3241	1243	2143	4123	1423	2413	4213

Table 2. All Hamiltonian cycles rooted at 1234 in $SG_4 - \{\{1234, 4231\}\}$.

1234	2134	3124	1324	2314	4312	3412	1432	4132	3142	1342	2341	4321	3421	2431	4231	3241	1243	2143	4123	1423	2413	4213	3214	1234
1234	2134	3124	1324	4321	2341	3241	4231	2431	3421	1423	4123	2143	1243	4213	2413	3412	1432	4132	3142	1342	4312	2314	3214	1234
1234	2134	3124	4123	1423	2413	4213	1243	2143	3142	4132	1432	3412	4312	1342	2341	3241	4231	2431	3421	4321	1324	2314	3214	1234
1234	2134	4132	1432	2431	4231	3241	1243	2143	3142	1342	2341	4321	3421	1423	4123	3124	1324	2314	4312	3412	2413	4213	3214	1234
1234	2134	4132	3142	1342	4312	3412	1432	2431	4231	3241	2341	4321	3421	1423	2413	4213	1243	2143	4123	3124	1324	2314	3214	1234
1234	2134	4132	3142	2143	4123	3124	1324	2314	4312	1342	2341	4321	3421	1423	2413	3412	1432	2431	4231	3241	1243	4213	3214	1234
1234	3214	2314	1324	3124	4123	2143	1243	4213	2413	1423	3421	4321	2341	3241	4231	2431	1432	3412	4312	1342	3142	4132	2134	1234
1234	3214	2314	1324	4321	3421	2431	4231	3241	2341	1342	4312	3412	1432	4132	3142	2143	1243	4213	2413	1423	4123	3124	2134	1234
1234	3214	2314	4312	1342	3142	4132	1432	3412	2413	4213	1243	2143	4123	1423	3421	2431	4231	3241	2341	4321	1324	3124	2134	1234
1234	3214	4213	2413	1423	4123	2143	1243	3241	4231	2431	3421	4321	2341	1342	3142	4132	1432	3412	4312	2314	1324	3124	2134	1234
1234	3214	4213	2413	3412	4312	2314	1324	3124	4123	1423	3421	4321	2341	1342	3142	2143	1243	3241	4231	2431	1432	4132	2134	1234
1234	3214	4213	1243	3241	4231	2431	1432	3412	2413	1423	3421	4321	2341	1342	4312	2314	1324	3124	4123	2143	3142	4132	2134	1234

LEMMA 9 *Suppose that $n \geq 5$ and $F \subset E(SG_n)$ with $|F| = n - 3$. Let $\mathbf{u} \in V(SG_n)$. Then there exist two-mutually independent Hamiltonian cycles rooted at $\mathbf{u}$ in $SG_n - F$.*

Proof Because SG_n is edge-transitive, there exists an automorphism ϕ_1 of SG_n mapping any edge in F into an edge of dimension n. For convenience, let $\mathbf{w} = \phi_1(\mathbf{u})$. Moreover, let $\phi_2 : V(\mathrm{SG}_n) \to V(\mathrm{SG}_n)$ be a function defined as follows: $\phi_2(\mathbf{v}) = h((\mathbf{v})_1)h((\mathbf{v})_2), \ldots, h((\mathbf{v})_n)$ for any $\mathbf{v} \in V(\mathrm{SG}_n)$, where $h : \mathbb{I}_n \to \mathbb{I}_n$ is a function such that $h((\mathbf{w})_j) = j$ for each $j \in \mathbb{I}_n$. Clearly, ϕ_2 is also an automorphism of SG_n. It is easy to check that the composition of ϕ_1 and ϕ_2, namely $\phi_2 \circ \phi_1$, is an automorphism of SG_n such that $\phi_2 \circ \phi_1(\mathbf{u}) = \mathbf{e}$. For this reason, we can assume that $\mathbf{u} = \mathbf{e}$, and F contains at least one edge of dimension n. Let $F_k = F \cap E(\mathrm{SG}_n^{\{k\}})$ for every $k \in \mathbb{I}_n$. As a result, we have $|F_k| \leq n - 4$ for every $k \in \mathbb{I}_n$.

Case 1 Suppose that $\{\mathbf{e}, (\mathbf{e})^n\} \notin F$. Let $B = (b_{i,j})$ be a $2 \times n$ matrix with

$$b_{i,j} = \begin{cases} j & \text{if } i = 1, \\ n & \text{if } i = 2 \text{ and } j = 1, \\ j+1 & \text{if } i = 2 \text{ and } 2 \leq j \leq n-2, \\ 2 & \text{if } i = 2 \text{ and } j = n-1, \\ 1 & \text{if } i = 2 \text{ and } j = n. \end{cases}$$

By Lemma 3, there exists a Hamiltonian path P of $\mathrm{SG}_n^{\bigcup_{j=1}^{n}\{b_{1,j}\}} - F$ joining $(\mathbf{e})^n$ to $\mathbf{e}$. Similarly, there exists a Hamiltonian path H of $\mathrm{SG}_n^{\bigcup_{j=1}^{n}\{b_{2,j}\}} - F$ joining $\mathbf{e}$ to $(\mathbf{e})^n$. Then we set $C_1 := \mathbf{e}\ (\mathbf{e})^n\ P\ \mathbf{e}$ and $C_2 := \mathbf{e}\ H\ (\mathbf{e})^n\ \mathbf{e}$. Obviously, $\{C_1, C_2\}$ forms a set of two-mutually independent Hamiltonian cycles rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$ (see Figure 2(a) for illustration).

Case 2 Suppose that $\{\mathbf{e}, (\mathbf{e})^n\} \in F$ and $|F_n| = n - 4$. Obviously, we have $|F_k| = 0$ for every $k \in \mathbb{I}_{n-1}$. By Theorem 1, there exists a Hamiltonian cycle $H = \mathbf{e}\ R\ \mathbf{q}\ \mathbf{p}\ \mathbf{e}$ of

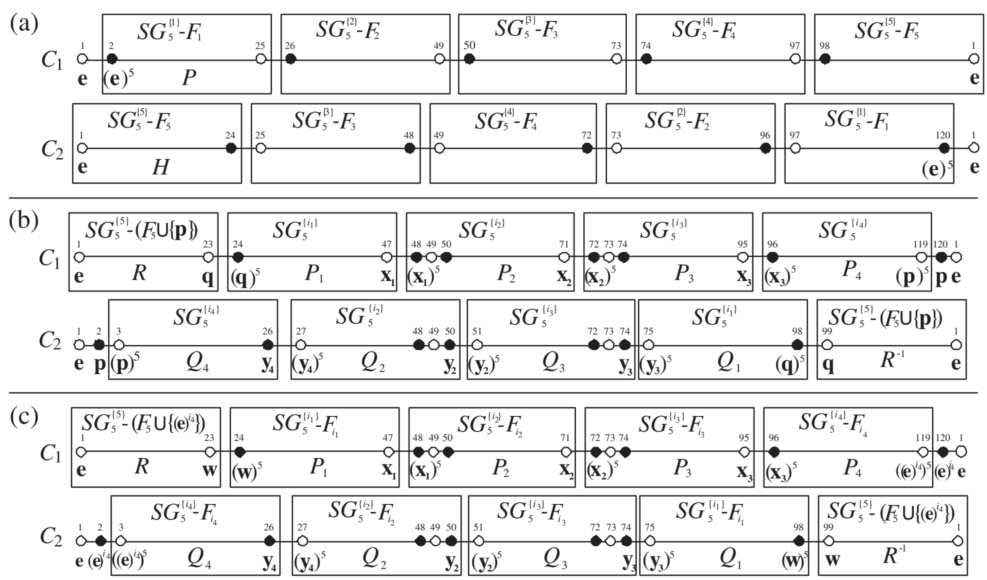

Figure 2. The two-mutually independent Hamiltonian cycles in $\mathrm{SG}_5 - F$ for Lemma 9.

$\mathrm{SG}_n^{\{n\}} - F_n$. Accordingly, we have that $\{\mathbf{p}, (\mathbf{p})^n\} \notin F$ and $\{\mathbf{q}, (\mathbf{q})^n\} \notin F$. By Lemma 2, $(\mathbf{p})_1 \neq (\mathbf{q})_1$. We set $(\mathbf{p})_1 = i_{n-1}$ and $(\mathbf{q})_1 = i_1$. Let $i_2 i_3 \cdots i_{n-2}$ be an arbitrary permutation of $\mathbb{I}_{n-1} - \{i_1, i_{n-1}\}$.

For $1 \leq k \leq n-2$, let $\mathbf{x}_k$ be a vertex of $V_0(\mathrm{SG}_n^{\{i_k\}})$ such that $(\mathbf{x}_k)_1 = i_{k+1}$ and $\{\mathbf{x}_k, (\mathbf{x}_k)^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path P_1 of $\mathrm{SG}_n^{\{i_1\}}$ joining $(\mathbf{q})^n$ to $\mathbf{x}_1$. Similarly, there is a Hamiltonian path P_k of $\mathrm{SG}_n^{\{i_k\}}$ joining $(\mathbf{x}_{k-1})^n$ to $\mathbf{x}_k$ for $2 \leq k \leq n-2$, and there is a Hamiltonian path P_{n-1} of $\mathrm{SG}_n^{\{i_{n-1}\}}$ joining $(\mathbf{x}_{n-2})^n$ to $(\mathbf{p})^n$. Then we set $C_1 := \mathbf{e}\ R\ \mathbf{q}\ (\mathbf{q})^n\ P_1\ \mathbf{x}_1\ (\mathbf{x}_1)^n\ P_2\ \mathbf{x}_2\ (\mathbf{x}_2)^n \cdots \mathbf{x}_{n-2}\ (\mathbf{x}_{n-2})^n\ P_{n-1}\ (\mathbf{p})^n\ \mathbf{p}\ \mathbf{e}$.

We can pick out a vertex $\mathbf{y}_{n-1}$ of $V_1(\mathrm{SG}_n^{\{i_{n-1}\}})$ such that $(\mathbf{y}_{n-1})_1 = i_2$ and $\{\mathbf{y}_{n-1}, (\mathbf{y}_{n-1})^n\} \notin F$. For $2 \leq k \leq n-3$, we have $|\{\mathbf{u} \in V_1(\mathrm{SG}_n^{\{i_k\}}) | (\mathbf{u})_1 = i_{k+1} \text{and}\ d_{\mathrm{SG}_n}(\mathbf{u}, (\mathbf{x}_{k-1})^n) = 2\}| = n-3 < (n-2)!/2$ if $n \geq 5$. Thus, we can choose a vertex $\mathbf{y}_k$ of $V_1(\mathrm{SG}_n^{\{i_k\}})$ such that $d_{\mathrm{SG}_n}(\mathbf{y}_k, (\mathbf{x}_{k-1})^n) > 2$, $(\mathbf{y}_k)_1 = i_{k+1}$, and $\{\mathbf{y}_k, (\mathbf{y}_k)^n\} \notin F$ for $2 \leq k \leq n-3$. Since $|\{\mathbf{u} \in V_1(\mathrm{SG}_n^{\{i_{n-2}\}}) | (\mathbf{u})_1 = i_1$ and $d_{\mathrm{SG}_n}(\mathbf{u}, (\mathbf{x}_{n-3})^n) = 2\}| = n-3 < (n-2)!/2$ if $n \geq 5$, we can choose a vertex $\mathbf{y}_{n-2}$ of $V_1(\mathrm{SG}_n^{\{i_{n-2}\}})$ such that $d_{\mathrm{SG}_n}(\mathbf{y}_{n-2}, (\mathbf{x}_{n-3})^n) > 2$, $(\mathbf{y}_{n-2})_1 = i_1$, and $\{\mathbf{y}_{n-2}, (\mathbf{y}_{n-2})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path Q_1 of $\mathrm{SG}_n^{\{i_1\}}$ joining $(\mathbf{y}_{n-2})^n$ to $(\mathbf{q})^n$. Again, there exists a Hamiltonian path Q_2 of $\mathrm{SG}_n^{\{i_2\}}$ joining $(\mathbf{y}_{n-1})^n$ to $\mathbf{y}_2$, there exists a Hamiltonian path Q_{n-1} of $\mathrm{SG}_n^{\{i_{n-1}\}}$ joining $(\mathbf{p})^n$ to $\mathbf{y}_{n-1}$, and there exists a Hamiltonian path Q_k of $\mathrm{SG}_n^{\{i_k\}}$ joining $(\mathbf{y}_{k-1})^n$ to $\mathbf{y}_k$ for each $3 \leq k \leq n-2$. Then we set $C_2 := \mathbf{e}\ \mathbf{p}\ (\mathbf{p})^n\ Q_{n-1}\ \mathbf{y}_{n-1}\ (\mathbf{y}_{n-1})^n\ Q_2\ \mathbf{y}_2\ (\mathbf{y}_2)^n\ Q_3\ \mathbf{y}_3\ (\mathbf{y}_3)^n \ldots (\mathbf{y}_{n-2})^n\ Q_1\ (\mathbf{q})^n\ \mathbf{q}\ R^{-1}\ \mathbf{e}$.

In summary, $\{C_1, C_2\}$ forms a set of 2-mutually independent Hamiltonian cycles rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$. Figure 2(b) illustrates C_1 and C_2 in S_5.

Case 3 Suppose that $\{\mathbf{e}, (\mathbf{e})^n\} \in F$ and $|F_n| \leq n-5$. Since $|F| = n-3$, there exists an integer of $\mathbb{I}_{n-1} - \{1\}$, say i_{n-1}, such that $|F \cap E^{i_{n-1},n}| = 0$. Assume that i_1 and i_2 are two integers of $\mathbb{I}_{n-1} - \{i_{n-1}\}$ such that $|F \cap E^{i_1,i_2}| = \max\{|F \cap E^{s,t}| | s, t \in \mathbb{I}_{n-1} - \{i_{n-1}\}\}$. Moreover, let $i_3 i_4 \cdots i_{n-2}$ be an arbitrary permutation of $\mathbb{I}_{n-1} - \{i_1, i_2, i_{n-1}\}$. Since $\{\mathbf{e}, (\mathbf{e})^n\} \in F$, we have $|F \cap E^{i_1,i_2}| \leq n-4$. Therefore, we have $|F \cap E^{i_{n-2},i_1}| \leq n-5$ and $|F \cap E^{i_k,i_{k+1}}| \leq n-5$ for $2 \leq k \leq n-3$.

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(\mathrm{SG}_n^{\{n\}})$ to vertices of $V_1(\mathrm{SG}_n^{\{i_1\}})$. Thus, we can choose a vertex $\mathbf{w} \in V_0(\mathrm{SG}_n^{\{n\}}) - \{\mathbf{e}\}$ such that $(\mathbf{w})_1 = i_1$ and $\{\mathbf{w}, (\mathbf{w})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path R of $(\mathrm{SG}_n^{\{n\}} - F_n) - \{(\mathbf{e})^{i_{n-1}}\}$ joining $\mathbf{e}$ to $\mathbf{w}$. For each $1 \leq k \leq n-2$, let $\mathbf{x}_k$ be a vertex of $V_0(\mathrm{SG}_n^{\{i_k\}})$ such that $(\mathbf{x}_k)_1 = i_{k+1}$ and $\{\mathbf{x}_k, (\mathbf{x}_k)^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path P_1 of $\mathrm{SG}_n^{\{i_1\}} - F_{i_1}$ joining $(\mathbf{w})^n$ to $\mathbf{x}_1$. Similarly, there exists a Hamiltonian path P_k of $\mathrm{SG}_n^{\{i_k\}} - F_{i_k}$ joining $(\mathbf{x}_{k-1})^n$ to $\mathbf{x}_k$ for each $2 \leq k \leq n-2$, and there exists a Hamiltonian path P_{n-1} of $\mathrm{SG}_n^{\{i_{n-1}\}} - F_{i_{n-1}}$ joining $(\mathbf{x}_{n-2})^n$ to $((\mathbf{e})^{i_{n-1}})^n$. Then we set $C_1 := \mathbf{e}\ R\ \mathbf{w}\ (\mathbf{w})^n\ P_1\ \mathbf{x}_1\ (\mathbf{x}_1)^n\ P_2\ \mathbf{x}_2\ (\mathbf{x}_2)^n \ldots (\mathbf{x}_{n-2})^n\ P_{n-1}\ ((\mathbf{e})^{i_{n-1}})^n\ (\mathbf{e})^{i_{n-1}}\ \mathbf{e}$.

Next, we can pick out a vertex $\mathbf{y}_{n-1}$ of $V_1(\mathrm{SG}_n^{\{i_{n-1}\}})$ such that $(\mathbf{y}_{n-1})_1 = i_2$ and $\{\mathbf{y}_{n-1}, (\mathbf{y}_{n-1})^n\} \notin F$. For any $2 \leq k \leq n-3$, we have $|\{\mathbf{u} \in V_1(\mathrm{SG}_n^{\{i_k\}}) | (\mathbf{u})_1 = i_{k+1}$ and $d_{\mathrm{SG}_n}(\mathbf{u}, (\mathbf{x}_{k-1})^n) = 2\}| = n-3$. By Lemma 1, there are $(n-2)!/2$ pairwise disjoint edges joining vertices of $V_1(\mathrm{SG}_n^{\{i_k\}})$ to vertices of $V_0(\mathrm{SG}_n^{\{i_{k+1}\}})$. It is noticed that $(n-2)!/2 > (n-3) + (n-5) = 2n-8$ if $n \geq 5$. Thus, we can choose a vertex $\mathbf{y}_k$ of $V_1(\mathrm{SG}_n^{\{i_k\}})$ such that $d_{\mathrm{SG}_n}(\mathbf{y}_k, (\mathbf{x}_{k-1})^n) > 2$, $(\mathbf{y}_k)_1 = i_{k+1}$, and $\{\mathbf{y}_k, (\mathbf{y}_k)^n\} \notin F$ for each $2 \leq k \leq n-3$. Since $(n-2)!/2 > |\{\mathbf{u} \in V_1(\mathrm{SG}_n^{\{i_{n-2}\}}) | (\mathbf{u})_1 = i_1$ and $d_{\mathrm{SG}_n}(\mathbf{u}, (\mathbf{x}_{n-3})^n) = 2\}| + (n-5) = (n-3) + (n-5) = 2n-8$ if $n \geq 5$, we can choose a vertex $\mathbf{y}_{n-2}$ of $V_1(\mathrm{SG}_n^{\{i_{n-2}\}})$ such that $d_{\mathrm{SG}_n}(\mathbf{y}_{n-2}, (\mathbf{x}_{n-3})^n) > 2$, $(\mathbf{y}_{n-2})_1 = i_1$, and $\{\mathbf{y}_{n-2}, (\mathbf{y}_{n-2})^n\} \notin F$. By Theorem 2, there exists a Hamiltonian path Q_1 of $\mathrm{SG}_n^{\{i_1\}} - F_{i_1}$ joining $(\mathbf{y}_{n-2})^n$ to $(\mathbf{w})^n$. Again, there exists a Hamiltonian path Q_2 of $\mathrm{SG}_n^{\{i_2\}} - F_{i_2}$ joining $(\mathbf{y}_{n-1})^n$ to $\mathbf{y}_2$, there exists a Hamiltonian path Q_{n-1} of $\mathrm{SG}_n^{\{i_{n-1}\}} - F_{i_{n-1}}$ joining $((\mathbf{e})^{i_{n-1}})^n$ to $\mathbf{y}_{n-1}$, and there exists a Hamiltonian path Q_k of $\mathrm{SG}_n^{\{i_k\}} - F_{i_k}$ joining $(\mathbf{y}_{k-1})^n$ to $\mathbf{y}_k$ for $3 \le k \le n-2$. We set $C_2 := \mathbf{e}(\mathbf{e})^{i_{n-1}}((\mathbf{e})^{i_{n-1}})^n Q_{n-1}\mathbf{y}_{n-1}(\mathbf{y}_{n-1})^n Q_2\mathbf{y}_2(\mathbf{y}_2)^n Q_3\mathbf{y}_3(\mathbf{y}_3)^n \cdots (\mathbf{y}_{n-2})^n Q_1(\mathbf{w})^n\mathbf{w}R^{-1}\mathbf{e}$.

As a result, $\{C_1, C_2\}$ turns out to be a set of two-mutually independent Hamiltonian cycles rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$. Figure 2(c) illustrates C_1 and C_2 in S_5. ■

LEMMA 10 *Let f be any integer of $\mathbb{I}_{n-4}$ for $n \ge 5$. Suppose that $F \subset E(SG_n)$ with $|F| = f$, and* $\mathbf{u}$ *is any vertex of SG_n. Then there exist $(n-1-f)$-mutually independent Hamiltonian cycles rooted at* $\mathbf{u}$ *in $SG_n - F$.*

Proof As explained in the proof of Lemma 9, there exists an automorphism of SG_n that can map any edge in F into an edge of dimension n and map $\mathbf{u}$ to $\mathbf{e}$ simultaneously. Hence, we can assume that $\mathbf{u} = \mathbf{e}$, and F contains at least one edge of dimension n. Let $F_k = F \cap E(\mathrm{SG}_n^{\{k\}})$ for every $k \in \mathbb{I}_n$. Thus, we have $|F_k| \le n-5$ for every $k \in \mathbb{I}_n$. Moreover, let $A_1 = E^{1,n} - \{\{\mathbf{e}, (\mathbf{e})^n\}\}$, and let $A_i = E^{i,n} \cup \{\{\mathbf{e}, (\mathbf{e})^i\}\}$ for $2 \le i \le n-1$.

Case 1 Suppose that $\{\mathbf{e}, (\mathbf{e})^n\} \in F$. It is noticed that there are at least $n-1-f$ elements of $|F \cap A_2|, |F \cap A_3|, \ldots, |F \cap A_{n-1}|$ equal to 0. Without loss of generality, we can assume that $|F \cap (\cup_{i=f+1}^{n-1} A_i)| = 0$. Thus, at least one of $|F \cap A_1|, \ldots, |F \cap A_f|$ equals to 0.

Subcase 1.1 Suppose that $|F \cap A_1| = 0$. Let $B = (b_{i,j})$ be an $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} f+i+j & \text{if } f+i+j \le n, \\ f+i+j-n & \text{otherwise.} \end{cases}$$

It is noticed that $b_{i,n-f-i} = n$ for every $1 \le i \le n-1-f$. Then we will construct a set of $(n-1-f)$-mutually independent Hamiltonian cycles $\{C_1, C_2, \ldots, C_{n-1-f}\}$ rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$.

Let $i \in \mathbb{I}_{n-2-f}$. We set $t_i = n-f-i$. By Lemma 7, there exists a Hamiltonian path Q_i of $(\mathrm{SG}_n^{\{b_{i,t_i}\}} - F_{b_{i,t_i}}) - \{\mathbf{e}, (\mathbf{e})^{b_{i,1}}, (\mathbf{e})^{b_{i,n}}\}$ joining two vertices $\mathbf{x}_i$ and $\mathbf{y}_i$ in $V_0(\mathrm{SG}_n^{\{b_{i,t_i}\}}) - \{\mathbf{e}\}$ such that $(\mathbf{x}_i)_1 = b_{i,t_i-1}$ and $(\mathbf{y}_i)_1 = b_{i,t_i+1}$. By Lemma 3, there exists a Hamiltonian path P_i of $\mathrm{SG}_n^{\cup_{j=1}^{t_i-1}\{b_{i,j}\}} - F$ joining $((\mathbf{e})^{b_{i,1}})^n$ to $(\mathbf{x}_i)^n$. Similarly, there exists a Hamiltonian path R_i of $\mathrm{SG}_n^{\cup_{j=t_i+1}^{n}\{b_{i,j}\}} - F$ joining $(\mathbf{y}_i)^n$ to $((\mathbf{e})^{b_{i,n}})^n$. Then we set $C_i := \mathbf{e}\ (\mathbf{e})^{b_{i,1}}\ ((\mathbf{e})^{b_{i,1}})^n\ P_i\ (\mathbf{x}_i)^n\ \mathbf{x}_i\ Q_i\ \mathbf{y}_i\ (\mathbf{y}_i)^n\ R_i\ ((\mathbf{e})^{b_{i,n}})^n\ (\mathbf{e})^{b_{i,n}}\ \mathbf{e}$.

By Lemma 6, $(\mathrm{SG}_n^{\{b_{n-1-f,1}\}} - F_{b_{n-1-f,n}}) - \{\mathbf{e}, (\mathbf{e})^{b_{n-1-f,n}}\}$ has a Hamiltonian path T joining $(\mathbf{e})^{b_{1,n}}$ to a vertex $\mathbf{z}$ of $V_0(\mathrm{SG}_n^{\{b_{n-1-f,1}\}}) - \{\mathbf{e}\}$ with $(\mathbf{z})_1 = b_{n-1-f,2}$. By Lemma 3, there exists a Hamiltonian path W of $\mathrm{SG}_n^{\cup_{j=2}^{n}\{b_{n-1-f,j}\}} - F$ joining $(\mathbf{z})^n$ to $((\mathbf{e})^{b_{n-1-f,n}})^n$. Then we set $C_{n-1-f} := \mathbf{e}\ (\mathbf{e})^{b_{1,n}}\ T\ \mathbf{z}\ (\mathbf{z})^n\ W\ ((\mathbf{e})^{b_{n-1-f,n}})^n\ (\mathbf{e})^{b_{n-1-f,n}}\ \mathbf{e}$.

As a result, $\{C_1, \ldots, C_{n-2-f}, C_{n-1-f}\}$ turns out be a set of $(n-1-f)$-mutually independent Hamiltonian cycles rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$. Figure 3 illustrates $\{C_1, C_2, C_3, C_4\}$ in $\mathrm{SG}_6 - F$ with $|F| = f = 1$.

Subcase 1.2 Suppose that $|F \cap A_1| > 0$. It is noticed that $f \ge 2$ in this subcase. Thus, at least one of $|F \cap A_2|, \ldots, |F \cap A_f|$ equals to 0. Without loss of generality, we can assume that

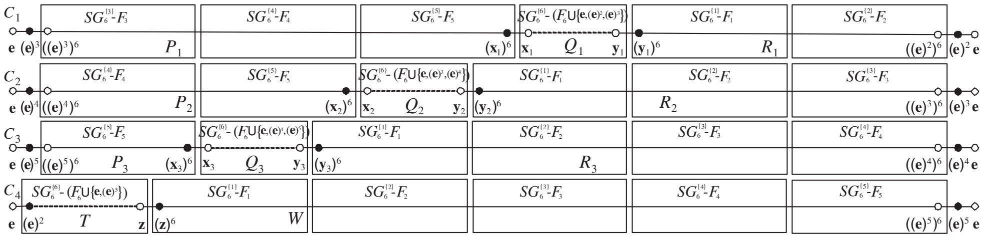

Figure 3. Mutually independent Hamiltonian cycles in $SG_6 - F$ with $|F| = 1$ for Subcase 1.1 of Lemma 10.

$|F \cap A_2| = 0$. Let $B = (b_{i,j})$ be an $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} f+i+j & \text{if } f+i+j \le n, \\ 2 & \text{if } f+i+j = n+1, \\ 1 & \text{if } f+i+j = n+2, \\ f+i+j-n & \text{otherwise.} \end{cases}$$

Using a similar manner to that of Subcase 1.1, we can construct a set of $(n-1-f)$-mutually independent Hamiltonian cycles $\{C_1, C_2, \ldots, C_{n-1-f}\}$ rooted at $\mathbf{e}$ in $SG_n - F$.

Case 2 Suppose that $\{\mathbf{e}, (\mathbf{e})^n\} \notin F$. It is noticed that there are at least $n-2-f$ elements of $|F \cap A_2|, |F \cap A_3| \ldots, |F \cap A_{n-1}|$ equal to 0. Without loss of generality, we can assume that $|F \cap (\cup_{i=f+2}^{n-1} A_i)| = 0$. Thus, at least one of $|F \cap A_1|, \ldots, |F \cap A_{f+1}|$ is 0.

Subcase 2.1 Suppose that $|F \cap A_1| = 0$. Let $B_n = (b_{i,j})$ be an $(n-1-f) \times n$ matrix with

$$B_5 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 4 & 2 & 3 & 1 \end{bmatrix},$$

and for $n \ge 6$

$$b_{i,j} = \begin{cases} j & \text{if } i = 1, \\ f+i+j & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j \le n, \\ f+i+j-n & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j > n, \\ n & \text{if } i = n-1-f \text{ and } j = 1, \\ 3 & \text{if } i = n-1-f \text{ and } j = 2, \\ 2 & \text{if } i = n-1-f \text{ and } j = 3, \\ n-1 & \text{if } i = n-1-f \text{ and } j = 4, \\ j-1 & \text{if } i = n-1-f \text{ and } 5 \le j \le n-1, \\ 1 & \text{if } i = n-1-f \text{ and } j = n. \end{cases}$$

Then we construct a set of $(n-1-f)$-mutually independent Hamiltonian cycles $\{C_1, C_2, \ldots, C_{n-1-f}\}$ rooted at $\mathbf{e}$ in $SG_n - F$ as follows.

We can pick out a vertex $\mathbf{v}$ of $V_1(SG_n^{\{b_{1,n}\}}) - \{(\mathbf{e})^{b_{n-2-f,1}}\}$ with $(\mathbf{v})_1 = b_{1,n-1}$. By Theorem 2, there exists a Hamiltonian path W of $(SG_n^{\{b_{1,n}\}} - F_{b_{1,n}}) - \{\mathbf{e}\}$ joining $\mathbf{v}$ to $(\mathbf{e})^{b_{n-2-f,1}}$. By Lemma 3, there exists a Hamiltonian path D of $SG_n^{\cup_{j=1}^{n-1}\{b_{1,j}\}} - F$ joining $(\mathbf{e})^n$ to $(\mathbf{v})^n$. We set $C_1 := \mathbf{e}\ (\mathbf{e})^n\ D\ (\mathbf{v})^n\ \mathbf{v}\ W\ (\mathbf{e})^{b_{n-2-f,1}}\ \mathbf{e}$.

Let $i \in \mathbb{I}_{n-2-f} - \{1\}$. We set $t_i = n - f - i$. By Lemma 7, there exists a Hamiltonian path Q_i of $(\mathrm{SG}_n^{\{b_{i,t_i}\}} - F_{b_{i,t_i}}) - \{\mathbf{e}, (\mathbf{e})^{b_{i,1}}, (\mathbf{e})^{b_{i,n}}\}$ joining two vertices $\mathbf{x}_i$ and $\mathbf{y}_i$ in $V_0(\mathrm{SG}_n^{\{b_{i,t_i}\}}) - \{\mathbf{e}\}$ such that $(\mathbf{x}_i)_1 = b_{i,t_i-1}$ and $(\mathbf{y}_i)_1 = b_{i,t_i+1}$. By Lemma 3, there exists a Hamiltonian path P_i of $\mathrm{SG}_n^{\cup_{j=1}^{t_i-1}\{b_{i,j}\}} - F$ joining $((\mathbf{e})^{b_{i,1}})^n$ to $(\mathbf{x}_i)^n$. Similarly, there exists a Hamiltonian path R_i of $S_n^{\cup_{j=t_i+1}^{n}\{b_{i,j}\}} - F$ joining $(\mathbf{y}_i)^n$ to $((\mathbf{e})^{b_{i,n}})^n$. Then we set $C_i := \mathbf{e}\ (\mathbf{e})^{b_{i,1}}\ ((\mathbf{e})^{b_{i,1}})^n\ P_i\ (\mathbf{x}_i)^n\ \mathbf{x}_i\ Q_i\ \mathbf{y}_i\ (\mathbf{y}_i)^n\ R_i\ ((\mathbf{e})^{b_{i,n}})^n\ (\mathbf{e})^{b_{i,n}}\ \mathbf{e}$.

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(\mathrm{SG}_n^{\{b_{n-1-f,k}\}})$ to vertices of $V_1(\mathrm{SG}_n^{\{b_{n-1-f,k-1}\}})$ for $3 \le k \le n-1$. Thus, we can choose a vertex $\mathbf{z}_k$ of $V_0(\mathrm{SG}_n^{\{b_{n-1-f,k}\}})$ such that $(\mathbf{z}_k)_1 = b_{n-1-f,k-1}$, $\{\mathbf{z}_k, (\mathbf{z}_k)^n\} \notin F$, and $\mathbf{z}_k \ne C_1((k-1)(n-1)!+1)$. By Lemma 4, there exists a Hamiltonian path T of $(\mathrm{SG}_n^{\cup_{j=1}^{2}\{b_{n-1-f,j}\}} - F) - \{\mathbf{e}\}$ joining $(\mathbf{e})^{b_{2,n}}$ to $(\mathbf{z}_3)^n$. By Theorem 2, there exists a Hamiltonian path H_k of $\mathrm{SG}_n^{\{b_{n-1-f,k}\}} - F_{b_{n-1-f,k}}$ joining $\mathbf{z}_k$ to $(\mathbf{z}_{k+1})^n$ for $3 \le k \le n-2$. By Lemma 3, there exists a Hamiltonian path H_{n-1} of $\mathrm{SG}_n^{\cup_{j=n-1}^{n}\{b_{n-1-f,j}\}} - F$ joining $\mathbf{z}_{n-1}$ to $(\mathbf{e})^n$. Then we set $C_{n-1-f} := \mathbf{e}(\mathbf{e})^{b_{2,n}}T(\mathbf{z}_3)^n\mathbf{z}_3H_3(\mathbf{z}_4)^n \cdots \mathbf{z}_{n-2}H_{n-2}(\mathbf{z}_{n-1})^n\mathbf{z}_{n-1}H_{n-1}(\mathbf{e})^n\mathbf{e}$.

Consequently, $\{C_1, C_2, \ldots, C_{n-2-f}, C_{n-1-f}\}$ is a set of $(n-1-f)$-mutually independent Hamiltonian cycles rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$. Figure 4(a) illustrates $\{C_1, C_2, C_3, C_4\}$ in $\mathrm{SG}_6 - F$ with $|F| = f = 1$.

Subcase 2.2 Suppose that $|F \cap A_1| > 0$. Thus, at least one of $|F \cap A_2|, \ldots, |F \cap A_{f+1}|$ equals to 0. Without loss of generality, we can assume that $|F \cap A_2| = 0$. Let $B_n = (b_{i,j})$ be an $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} n & \text{if } i = 1 \text{ and } j = 1, \\ j+1 & \text{if } i = 1 \text{ and } 2 \le j \le n-2, \\ 2 & \text{if } i = 1 \text{ and } j = n-1, \\ 1 & \text{if } i = 1 \text{ and } j = n, \\ f+i+j & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j \le n, \\ 2 & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j = n+1, \\ 1 & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j = n+2, \\ f+i+j-n & \text{if } 2 \le i \le n-2-f \text{ and } f+i+j \ge n+3, \\ j & \text{if } i = n-1-f. \end{cases}$$

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(\mathrm{SG}_n^{\{b_{1,2}\}})$ to vertices of $V_1(\mathrm{SG}_n^{\{b_{1,1}\}})$. Thus, we can choose a vertex $\mathbf{z}$ of $V_0(\mathrm{SG}_n^{\{b_{1,2}\}})$ such that $(\mathbf{z})_1 = b_{1,1}$, $\{\mathbf{z}, (\mathbf{z})^n\} \notin F$, and $(\mathbf{z})^n \ne (\mathbf{e})^{b_{2,n}}$. By Theorem 2, there exists a Hamiltonian path T of $(\mathrm{SG}_n^{\{b_{1,1}\}} - F_{b_{1,1}}) - \{\mathbf{e}\}$ joining $(\mathbf{e})^{b_{2,n}}$ to $(\mathbf{z})^n$. By Lemma 3, there exists a Hamiltonian path H of $\mathrm{SG}_n^{\cup_{j=2}^{n}\{b_{1,j}\}} - F$ joining $\mathbf{z}$ to $(\mathbf{e})^n$. Then we set $C_1 := \mathbf{e}\ (\mathbf{e})^{b_{2,n}}\ T\ (\mathbf{z})^n\ \mathbf{z}\ H\ (\mathbf{e})^n\ \mathbf{e}$.

Let $i \in \mathbb{I}_{n-2-f} - \{1\}$. We set $t_i = n - f - i$. By Lemma 7, there exists a Hamiltonian path Q_i of $(\mathrm{SG}_n^{\{b_{i,t_i}\}} - F_{b_{i,t_i}}) - \{\mathbf{e}, (\mathbf{e})^{b_{i,1}}, (\mathbf{e})^{b_{i,n}}\}$ joining two vertices $\mathbf{x}_i$ and $\mathbf{y}_i$ in $V_0(\mathrm{SG}_n^{\{b_{i,t_i}\}}) - \{\mathbf{e}\}$ such that $(\mathbf{x}_i)_1 = b_{i,t_i-1}$ and $(\mathbf{y}_i)_1 = b_{i,t_i+1}$. By Lemma 3, there exists a Hamiltonian path P_i of $\mathrm{SG}_n^{\cup_{j=1}^{t_i-1}\{b_{i,j}\}} - F$ joining $((\mathbf{e})^{b_{i,1}})^n$ to $(\mathbf{x}_i)^n$. Similarly, there exists a Hamiltonian path R_i of $\mathrm{SG}_n^{\cup_{j=t_i+1}^{n}\{b_{i,j}\}} - F$ joining $(\mathbf{y}_i)^n$ to $((\mathbf{e})^{b_{i,n}})^n$. Then we set $C_i := \mathbf{e}\ (\mathbf{e})^{b_{i,1}}\ ((\mathbf{e})^{b_{i,1}})^n\ P_i\ (\mathbf{x}_i)^n\ \mathbf{x}_i\ Q_i\ \mathbf{y}_i\ (\mathbf{y}_i)^n\ R_i\ ((\mathbf{e})^{b_{i,n}})^n\ (\mathbf{e})^{b_{i,n}}\ \mathbf{e}$.

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(\mathrm{SG}_n^{\{b_{n-1-f,2}\}})$ to vertices of $V_1(\mathrm{SG}_n^{\{b_{n-1-f,3}\}})$. Thus, we can pick out a vertex $\mathbf{w}$ of $V_0(\mathrm{SG}_n^{\{b_{n-1-f,2}\}})$

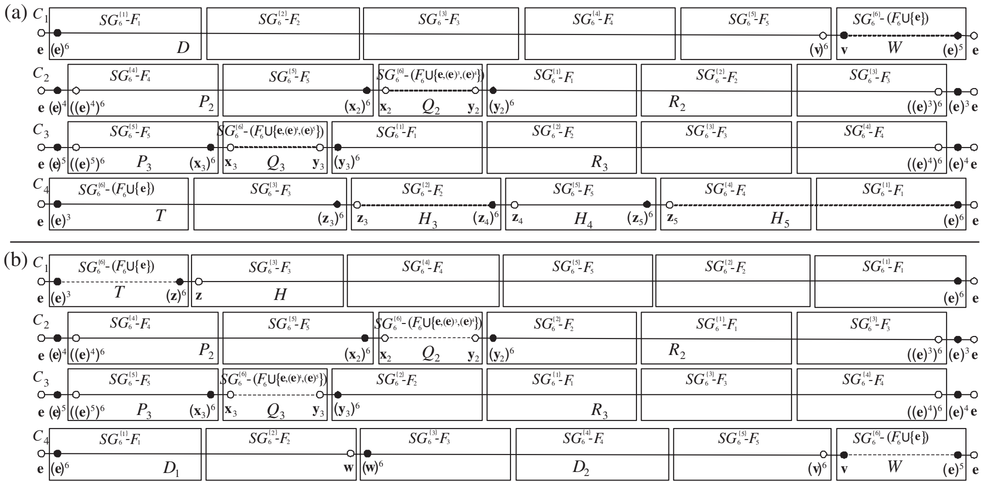

Figure 4. Mutually independent Hamiltonian cycles in $\mathrm{SG}_6 - F$ with $|F| = 1$ for Case 2 of Lemma 10.

such that $(\mathbf{w})_1 = b_{n-1-f,3}$, $\{\mathbf{w}, (\mathbf{w})^n\} \notin F$, and $d_{\mathrm{SG}_n}(\mathbf{w}, (\mathbf{y}_{n-2-f})^n) > 1$. Moreover, we choose a vertex $\mathbf{v}$ of $V_1(\mathrm{SG}_n^{\{b_{n-1-f,n}\}})$ such that $(\mathbf{v})_1 = b_{n-1-f,n-1}$ and $\{\mathbf{v}, (\mathbf{v})^n\} \notin F$. By Lemma 3, there exists a Hamiltonian path D_1 of $\mathrm{SG}_n^{\cup_{j=1}^{2}\{b_{n-1-f,j}\}} - F$ joining $(\mathbf{e})^n$ to $\mathbf{w}$. Similarly, there exists a Hamiltonian path D_2 of $\mathrm{SG}_n^{\cup_{j=3}^{n-1}\{b_{n-1-f,j}\}} - F$ joining $(\mathbf{w})^n$ to $(\mathbf{v})^n$. By Theorem 2, there exists a Hamiltonian path W of $(\mathrm{SG}_n^{\{b_{n-1-f,n}\}} - F_{b_{n-1-f,n}}) - \{\mathbf{e}\}$ joining $\mathbf{v}$ to $(\mathbf{e})^{b_{n-2-f,1}}$. Then we set $C_{n-1-f} := \mathbf{e}\ (\mathbf{e})^n\ D_1\ \mathbf{w}\ (\mathbf{w})^n\ D_2\ (\mathbf{v})^n\ \mathbf{v}\ W\ (\mathbf{e})^{b_{n-2-f,1}}\ \mathbf{e}$.

Hence, $\{C_1, C_2, \ldots, C_{n-2-f}, C_{n-1-f}\}$ forms a set of $(n-1-f)$-mutually independent Hamiltonian cycles rooted at $\mathbf{e}$ in $\mathrm{SG}_n - F$. Figure 4(b) illustrates $\{C_1, C_2, C_3, C_4\}$ in $\mathrm{SG}_6 - F$ with $|F| = f = 1$. ■

Combining Theorem 3 and Lemmas 8–10, we summarize those results as follows.

THEOREM 4 *Let $F \subset E(SG_n)$ with $|F| \leq n-3$ for $n \geq 3$, and let $\mathbf{u} \in V(SG_n)$. Then there exist $(n-2-|F|)$-mutually independent Hamiltonian cycles rooted at $\mathbf{u}$ in $SG_n - F$ if $n \in \{3,4\}$, and there exist $(n-1-|F|)$-mutually independent Hamiltonian cycles rooted at $\mathbf{u}$ in $SG_n - F$ if $n \geq 5$.*

5. Conclusion

In this paper, we study the problem of finding mutually independent Hamiltonian cycles in a faulty star graph. That is, given a set of faulty edges $F \subset E(\mathrm{SG}_n)$ with $|F| \leq n-3$, we show that $\mathrm{SG}_n - F$ has a set of $(n-2-|F|)$-mutually (respectively, $(n-1-|F|)$-mutually) independent Hamiltonian cycles rooted at any vertex if $n \in \{3,4\}$ (respectively, $n \geq 5$). We believe that a similar result could be obtained for the graph generated by any transposition tree of order n [1] when at most $n-3$ edges fail. On the other hand, we also believe that our current result can be further refined; to be precise, we would like to show that $\mathcal{IHC}(\mathrm{SG}_n - F) = \delta(\mathrm{SG}_n - F)$, where $\delta(\mathrm{SG}_n - F)$ denotes the minimum degree of graph $\mathrm{SG}_n - F$.

The edge faults considered in this work are random and independent. To guarantee that such a faulty star graph remains Hamiltonian in this situation, the maximum number of faulty edges cannot exceed $n - 3$. For this reason, we can make a more general condition on the nature of faulty edges such that every vertex still has at least two neighbours in a faulty star graph. This kind of sets of faulty edges is called conditionally faulty. Let $F \subset E(\mathrm{SG}_n)$ be *conditionally faulty*. Then we believe that mutually independent Hamiltonian cycles can be constructed in SG_n if $|F| \leq 3n - 10$. These results convince us that the star graph is really robust enough to interconnect computing units in parallel and distributed systems.

Acknowledgements

The authors express the most immense gratitude to Editor-in-Chief, Editor, and the anonymous referees for their careful reading and constructive comments. They greatly improve the quality of the paper. This work was supported in part by the National Science Council of the Republic of China under Contract NSC 98-2218-E-468-001-MY3. J.J.M. Tan was supported in part by the National Science Council of the Republic of China under Contract NSC 96-2221-E-009-134-MY3 and in part by the Aiming for the Top University and Elite Research Center Development Plan.

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