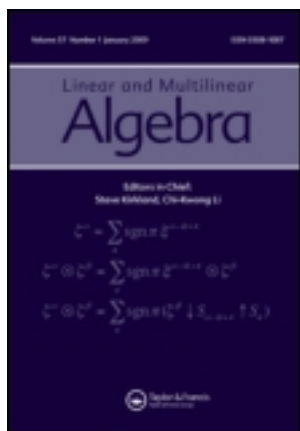


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Operator norms and lower bounds of generalized Hausdorff matrices

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Operator norms and lower bounds of generalized Hausdorff matrices

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Let $A = (a_{n,k})_{n,k \geq 0}$ be a non-negative matrix. Denote by $L_{p,q}(A)$ the supremum of those L satisfying the following inequality:

$$\left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{n,k} x_k \right)^q \right)^{1/q} \geq L \left(\sum_{k=0}^{\infty} x_k^p \right)^{1/p} \quad (X \in \ell_p, X \geq 0).$$

The purpose of this article is to establish a Bennett-type formula for $\|H_{\mu}^0\|_{p,p}^{\downarrow}$ and a Hardy-type formula for $L_{p,p}^{\downarrow}(H_{\mu}^{\alpha})$ and $L_{p,p}(H_{\mu}^{\alpha})$, where H_{μ}^{α} is a generalized Hausdorff matrix and $0 < p \leq 1$. Similar results are also established for $L_{p,q}(H_{\mu}^{\alpha})$ and $L_{p,q}((H_{\mu}^{\alpha})^t)$ for other ranges of p and q . Our results extend [Chen and Wang, *Lower bounds of Copson type for Hausdorff matrices*, Linear Algebra Appl. 422 (2007), pp. 208–217] and [Chen and Wang, *Lower bounds of Copson type for Hausdorff matrices: II*, Linear Algebra Appl. 422 (2007) pp. 563–573] from H_{μ}^0 to H_{μ}^{α} with $\alpha \geq 0$ and completely solve the value problem of $\|H_{\mu}^0\|_{p,p}^{\downarrow}$, $L_{p,p}^{\downarrow}(H_{\mu}^{\alpha})$, $L_{p,p}(H_{\mu}^{\alpha})$, and $L_{p,p}((H_{\mu}^{\alpha})^t)$ for $\alpha \in \mathbb{N} \cup \{0\}$.

Keywords: operator norms; lower bound; generalized Hausdorff matrices

AMS Subject Classifications: Primary 26D15; 40G05; 47A30; 47B37

1. Introduction

Let ℓ_p denote the space of all complex sequences $X = \{x_n\}_{n=0}^{\infty}$ such that $\|X\|_p := (\sum_{n=0}^{\infty} |x_n|^p)^{1/p} < \infty$. We have $\|X\|_p \geq \|X\|_q$ for $0 < p \leq q \leq \infty$. Write $X \geq 0$ if $x_n \geq 0$ for all n . We also write $X \downarrow$ for the case $x_0 \geq x_1 \geq \dots$. For $p, q \in \mathbb{R} \setminus \{0\}$ and $A = (a_{n,k})_{n,k \geq 0} \geq 0$, denote by $L_{p,q}(A)$ the supremum of those L obeying the following inequality:

$$\left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{n,k} x_k \right)^q \right)^{1/q} \geq L \left(\sum_{k=0}^{\infty} x_k^p \right)^{1/p} \quad (X \in \ell_p, X \geq 0).$$

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The study of $L_{p,q}(A)$ goes back to Copson [9] (see also [4,11,12]). It is clear that $L_{p,q}(A) \leq L_{p,q}^\downarrow(A) \leq \|A\|_{p,q}^\downarrow \leq \|A\|_{p,q}$, where

$$L_{p,q}^\downarrow(A) = \inf_{\|X\|_p=1, X \geq 0, X \downarrow} \|AX\|_q, \quad \|A\|_{p,q}^\downarrow = \sup_{\|X\|_p=1, X \geq 0, X \downarrow} \|AX\|_q,$$

and

$$\|A\|_{p,q} = \sup_{\|X\|_p=1} \|AX\|_q.$$

Moreover, by (1.3), given below, we infer that for any $m \geq 0$ and $1 \leq p < \infty$,

$$|A|_{p,p} \geq \left(\sum_{k=0}^m \sum_{n=0}^{\infty} |a_{n,k}|^p \right)^{1/p} \geq (m+1)^{1/p} L_{p,p}(A),$$

where

$$|A|_{p,p} = \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_{n,k}|^p \right)^{1/p}.$$

This gives

$$L_{p,p}(A) \leq \frac{1}{(m+1)^{1/p}} |A|_{p,p}.$$

Whenever $|A|_{p,p} < \infty$, the rights side of the last inequality tends to 0 as $m \rightarrow \infty$. In this case, $L_{p,p}(A) = 0$. Thus, for $1 \leq p < \infty$, $L_{p,p}(A) > 0 \Rightarrow |A|_{p,p} = \infty$. As indicated in [8, Section 3], $0 < L_{p,p}(A_W^{NM}) < \infty \Rightarrow \|A_W^{NM}\|_{p,p} < \infty$, where $1 \leq p < \infty$ and A_W^{NM} is any Nörlund matrix. In [7, Corollaries 3.3 and 3.5], we further indicated that there are many cases for which $0 < L_{p,q}(H_\mu^0) < \infty$, but $\|H_\mu^0\|_{p,q} = \infty$, where $0 < q \leq p \leq 1$, μ is a Borel probability measure on $[0, 1]$ and $H_\mu^\alpha = (h_{n,k}^\alpha)_{n,k \geq 0}$ is the generalized Hausdorff matrix associated with μ , defined by

$$h_{n,k}^\alpha = \begin{cases} \binom{n+\alpha}{n-k} \int_0^1 \theta^{k+\alpha} (1-\theta)^{n-k} d\mu(\theta) & (n \geq k), \\ 0 & (n < k). \end{cases}$$

As indicated in Theorem 3.1 and the remark before Lemma 2.4, $\|H_\mu^0\|_{p,q}^\downarrow$ may be finite, but $\|H_\mu^0\|_{p,q} = \infty$. These display the significance of studying $L_{p,q}(A)$, $L_{p,q}^\downarrow(A)$ and $\|A\|_{p,q}^\downarrow$.

In [5, Theorem 4] and [1, Theorems 2 and 3], Bennett and Grosse-Erdmann established the following general results:

$$\|A\|_{p,q}^\downarrow = \sup_{r \geq 0} (1+r)^{-1/p} \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^r a_{n,k} \right)^q \right)^{1/q} \quad (0 < p \leq 1; p \leq q < \infty), \tag{1.1}$$

$$L_{p,q}^\downarrow(A) = \inf_{r \geq 0} (1+r)^{-1/p} \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^r a_{n,k} \right)^q \right)^{1/q} \quad (1 \leq p \leq \infty; 0 < q \leq p), \tag{1.2}$$

$$L_{p,q}(A) = \inf_{k \geq 0} \left(\sum_{n=0}^{\infty} a_{n,k}^q \right)^{1/q} \quad (1 \leq p \leq \infty; 0 < q \leq p). \tag{1.3}$$

As indicated in [1,2,6,8,14,15], some further investigations for the extreme values appeared in (1.1)–(1.3) are necessary even for ‘nice’ matrices A , such as $A = H_\mu^\alpha$ or $(H_\mu^\alpha)^t$, where $(\cdot)^t$ denotes the transpose of (\cdot) . The known results for H_μ^0 and $(H_\mu^0)^t$ are listed below. In [2, Theorem H], Bennett proved that

$$\|H_\mu^0\|_{p,p}^\downarrow = \int_0^1 \theta^{-1/p} d\mu(\theta) \quad (1 \leq p < \infty). \tag{1.4}$$

In [2, Theorem 1] and [3, Theorem 7.18], he also showed that

$$L_{p,p}^\downarrow(H_\mu^0) = \left(\sum_{n=0}^\infty \left(\int_0^1 (1-\theta)^n d\mu(\theta) \right)^p \right)^{1/p} \quad (1 \leq p < \infty) \tag{1.5}$$

and

$$L_{p,p}((H_\mu^0)^t) = \int_0^1 \theta^{-1/p^*} d\mu(\theta) \quad (0 < p \leq 1), \tag{1.6}$$

where $1/p + 1/p^* = 1$. Recently, we found several new results on $L_{p,q}(H_\mu^0)$ or on $L_{p,q}((H_\mu^0)^t)$ with p different from the one given in (1.6). For instance, in [6], we proved that $L_{p,q}(H_\mu^0) = \mu(\{1\})$ and $L_{p,q}((H_\mu^0)^t) = ((\mu(\{0\}))^q + (\mu(\{1\}))^q)^{1/q}$, where $1 < q \leq p \leq \infty$. The case $0 < q \leq 1 \leq p \leq \infty$ was also examined there. In [7, Theorem 2.3], we also claimed that if

$$\int_{(0,1]} \theta^{-(1/p+\epsilon)} d\mu(\theta) < \infty \quad \text{for some } \epsilon > 0, \tag{1.7}$$

then the following formula holds:

$$L_{p,p}(H_\mu^0) = \int_{(0,1]} \theta^{-1/p} d\mu(\theta) \quad (0 < p \leq 1). \tag{1.8}$$

In the same article, we pointed out that such a result is false for $\epsilon = 0$.

From the last paragraph, we see that the values of $\|H_\mu^0\|_{p,p}^\downarrow$ and $L_{p,p}^\downarrow(H_\mu^0)$ for $0 < p < 1$ are unclear. Moreover, no significant formulae are found for these values whenever H_μ^0 is replaced by H_μ^α . The purpose of this article is to solve these problems. In particular, we establish a Bennett-type formula for $\|H_\mu^0\|_{p,q}^\downarrow$ with $0 < p \leq q \leq 1$ (cf Theorem 3.1), and we also prove in Theorem 4.3 that under (1.7) and $\alpha > 0$, the values of $L_{p,p}^\downarrow(H_\mu^\alpha)$ and $L_{p,p}(H_\mu^\alpha)$ are determined by the integral given in (1.8). The details will be given in Sections 3–6. Our results extend [6] and [7] from H_μ^0 to H_μ^α with $\alpha \geq 0$.

2. Preliminaries

We list several results which will be used in the following sections. The first one indicates that (1.7) is stronger than (2.1), where

$$\sum_{n=0}^\infty \left(\int_{(0,1]} (1-\theta)^n d\mu(\theta) \right)^p < \infty. \tag{2.1}$$

LEMMA 2.1 *Let $0 < p \leq 1$. Then (1.7) \Rightarrow (2.1).*

Proof By [16, Chapter III, Equation (1.15)], there exists a suitable constant C such that

$$\left(\int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right)^p \leq C \left(\int_{(0,1]} A_n^{1/p+\epsilon-1} (1 - \theta)^n d\mu(\theta) \right)^p (n + 1)^{-\epsilon p - 1 + p}, \quad (2.2)$$

where $A_n^\alpha = \binom{n+\alpha}{n}$ and $\epsilon > 0$ is given in (1.7). Summing up both sides of (2.2) from $n = 0$ to ∞ first and then applying the Hölder inequality, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right)^p \\ & \leq C \left(\sum_{n=0}^{\infty} \int_{(0,1]} A_n^{1/p+\epsilon-1} (1 - \theta)^n d\mu(\theta) \right)^p \left(\sum_{n=0}^{\infty} (n + 1)^{\frac{-\epsilon p}{1-p} - 1} \right)^{1-p} \\ & \leq C \left(\frac{1-p}{\epsilon p} \right)^{1-p} \left(\int_{(0,1]} \sum_{n=0}^{\infty} A_n^{1/p+\epsilon-1} (1 - \theta)^n d\mu(\theta) \right)^p. \end{aligned} \quad (2.3)$$

For $\theta \in (0, 1]$, we have $\sum_{n=0}^{\infty} A_n^{1/p+\epsilon-1} (1 - \theta)^n = \theta^{-(1/p+\epsilon)}$ (see [16, Chapter III, Equation (1.9)]). Plugging this into the right side of (2.3), we obtain

$$\sum_{n=0}^{\infty} \left(\int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right)^p \leq C \left(\frac{1-p}{\epsilon p} \right)^{1-p} \left(\int_{(0,1]} \theta^{-(1/p+\epsilon)} d\mu(\theta) \right)^p.$$

Hence, (1.7) \Rightarrow (2.1). ■

In general, (1.7) does not imply (2.4), where

$$\sum_{n=0}^{\infty} \left(\int_0^1 (1 - \theta)^n d\mu(\theta) \right)^p < \infty. \quad (2.4)$$

The measure $\mu = (1/2)\delta_0 + (1/2)\delta_1$ gives a counterexample, where δ_θ denotes the Dirac measure at θ , defined by

$$\delta_\theta(E) = \begin{cases} 1 & \text{if } \theta \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

We also point out that (2.1) does not imply (1.7). Consider the measure $\mu = \sum_{k=0}^{\infty} \gamma^{-1} m_k^{-1/p} (\log m_k)^{-1} \delta_{m_k^{-1}}$, where $m_k \uparrow \infty$, $\log m_k \geq (k + 1)^{2/p}$ and $\gamma = \sum_{k=0}^{\infty} m_k^{-1/p} (\log m_k)^{-1}$. For such μ and m_k , we have

$$\frac{m_k^{-1}}{1 - (1 - 1/m_k)^p} \rightarrow 1/p \quad \text{as } k \rightarrow \infty,$$

which implies

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right)^p & \leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (1 - 1/m_k)^{np} \gamma^{-p} m_k^{-1} (\log m_k)^{-p} \\ & = \gamma^{-p} \sum_{k=0}^{\infty} \frac{m_k^{-1}}{1 - (1 - 1/m_k)^p} (\log m_k)^{-p} \\ & \leq \gamma^{-p} \sum_{k=0}^{\infty} \frac{m_k^{-1}}{1 - (1 - 1/m_k)^p} (k + 1)^{-2} < \infty. \end{aligned}$$

This shows that (2.1) holds. However, (1.7) fails because

$$\int_{(0,1]} \theta^{-(1/p+\epsilon)} d\mu(\theta) = \gamma^{-1} \sum_{k=0}^{\infty} m_k^\epsilon (\log m_k)^{-1} = \infty \quad \text{for all } \epsilon > 0.$$

In [2, Theorem 1], Bennett proved that for $1 \leq q < \infty$,

$$\frac{1}{r+1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^r h_{n,k}^0 \right)^q \tag{2.6}$$

is increasing in r . In the following, we claim that (2.6) is a decreasing function in r for $0 < q \leq 1$. This result will be used to establish Theorem 3.1.

LEMMA 2.2 *Let $0 < q \leq 1$. The following assertions are true.*

- (i) *For any fixed $r \in \mathbb{N} \cup \{0\}$, the sequence $\{\sum_{k=0}^r h_{n,k}^0\}_{n=0}^{\infty}$ is decreasing.*
- (ii) *The sequence defined by (2.6) decreases with r ($r = 0, 1, \dots$).*

Proof Consider (i). We have $\sum_{k=0}^r h_{n,k}^0 = \int_0^1 (\sum_{k=0}^r e_{n,k}^0(\theta)) d\mu(\theta)$, where

$$e_{n,k}^\alpha(\theta) = \begin{cases} \binom{n+\alpha}{n-k} \theta^{k+\alpha} (1-\theta)^{n-k} & (n \geq k), \\ 0 & (n < k). \end{cases} \tag{2.7}$$

It suffices to show that $\{\sum_{k=0}^r e_{n,k}^0(\theta)\}_{n=0}^{\infty}$ is decreasing for all $\theta \in [0, 1]$. By definition, $\sum_{k=0}^r e_{n,k}^0(\theta) = 1$ for $n = 0, 1, \dots, r$, so it suffices to show that $\{\sum_{k=0}^r e_{n,k}^0(\theta)\}_{n=r}^{\infty}$ is decreasing. Without loss of generality, we only discuss those $n \geq r + 1$. Clearly, $\sum_{k=0}^r e_{n,k}^0(\theta) = \sum_{k=0}^n e_{n,k}^0(\theta) - \sum_{k=r+1}^n e_{n,k}^0(\theta) = 1 - \sum_{k=r+1}^n e_{n,k}^0(\theta)$. On the other hand,

$$\sum_{k=r+1}^n e_{n,k}^0(\theta) = \sum_{k=r+1}^n \binom{n}{k} \theta^k (1-\theta)^{n-k} = \sum_{j=0}^m \binom{m+r+1}{m-j} \theta^{j+r+1} (1-\theta)^{m-j},$$

where $m = n - (r + 1)$. By [13, Lemma 2], $\{\sum_{k=r+1}^n e_{n,k}^0(\theta)\}_{n=r+1}^{\infty}$ is increasing. Hence, $\{\sum_{k=0}^r e_{n,k}^0(\theta)\}_{n=r}^{\infty}$ is decreasing. This verifies (i). Next, we prove (ii). Fix $r \in \mathbb{N} \cup \{0\}$. Let $X = \{x_k\}_{k=0}^{\infty}$ and $Y = \{y_k\}_{k=0}^{\infty}$ be defined by

$$x_{n(r+2)+j} = \sum_{k=0}^r h_{n,k}^0 \quad (n \geq 0; 0 \leq j \leq r+1)$$

and

$$y_{n(r+1)+j} = \sum_{k=0}^{r+1} h_{n,k}^0 \quad (n \geq 0; 0 \leq j \leq r).$$

The assertion (i) ensures that $\{x_k\}_{k=0}^{\infty}$ is decreasing. By [2, Equation (27)], we have $\sum_{k=0}^N x_k \leq \sum_{k=0}^N y_k$ for all N . On the other hand,

$$\sum_{k=0}^{\infty} x_k = (r+2) \sum_{k=0}^r \sum_{n=0}^{\infty} h_{n,k}^0 = (r+2)(r+1) \int_0^1 \theta^{-1} d\mu(\theta) \tag{2.8}$$

and

$$\sum_{k=0}^{\infty} y_k = (r+1) \sum_{k=0}^{r+1} \sum_{n=0}^{\infty} h_{n,k}^0 = (r+2)(r+1) \int_0^1 \theta^{-1} d\mu(\theta). \tag{2.9}$$

So $\sum_{k=N}^{\infty} x_k \geq \sum_{k=N}^{\infty} y_k$ for all N . By [3, Lemma 20.21], we get $\sum_{k=0}^{\infty} x_k^q \geq \sum_{k=0}^{\infty} y_k^q$. To replace x_k in (2.8) and y_k in (2.9) by x_k^q and y_k^q , respectively, we see that $\frac{1}{r+1} \sum_{n=0}^{\infty} (\sum_{k=0}^r h_{n,k}^0)^q \geq \frac{1}{r+2} \sum_{n=0}^{\infty} (\sum_{k=0}^{r+1} h_{n,k}^0)^q$. This completes the proof. ■

Lemma 2.2(i) may be false if $h_{n,k}^0$ is replaced by $h_{n,k}^\alpha$ with $\alpha > 0$. Consider $\mu = \delta_\theta$, where δ_θ is defined by (2.5). Fix n and r . We have

$$\lim_{\theta \rightarrow 0^+} \frac{\sum_{k=0}^r h_{n+1,k}^\alpha}{\sum_{k=0}^r h_{n,k}^\alpha} = \lim_{\theta \rightarrow 0^+} \frac{\binom{n+1+\alpha}{n+1} (1-\theta)^{n+1} + \theta \{\dots\}}{\binom{n+\alpha}{n} (1-\theta)^n + \theta \{\dots\}} = 1 + \frac{\alpha}{n+1} > 1.$$

Hence, there exists some $\theta \in (0, 1)$, depending on n and r , such that $\sum_{k=0}^r h_{n+1,k}^\alpha > \sum_{k=0}^r h_{n,k}^\alpha$. This shows that $\{\sum_{k=0}^r h_{n,k}^\alpha\}_{n=0}^\infty$ may not be decreasing. For $q = 1$, we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^r h_{n,k}^\alpha \right)^q = \sum_{k=0}^r \sum_{n=0}^{\infty} h_{n,k}^\alpha = (r+1) \int_0^1 \theta^{-1} d\mu(\theta) \quad (r = 0, 1, \dots),$$

and so $\frac{1}{r+1} \sum_{n=0}^{\infty} (\sum_{k=0}^r h_{n,k}^\alpha)^q$ is decreasing in r for all $\alpha \geq 0$. We are unable to determine whether it is true for $\alpha > 0$ and $0 < q < 1$.

To prove Theorem 4.2, we need the following lemma, which extends [3, Proposition 19.2] and [6, Lemma 2.1] from $e_{n,k}^0(\theta)$ to $e_{n,k}^\alpha(\theta)$, where $\alpha > 0$.

LEMMA 2.3 *Let $\alpha > 0$. The following assertions hold.*

- (i) For $\Omega \subseteq [0, 1]$, $\rho_k := \|\{\int_\Omega e_{n,k}^\alpha(\theta) d\mu(\theta)\}_{n=k}^\infty\|_p$ decreases for $1 \leq p \leq \infty$, and increases for $0 < p \leq 1$.
- (ii) Moreover, if $\Omega \subseteq [\delta, 1 - \delta]$ for some $0 < \delta < 1/2$, then $\rho_k \downarrow 0$ for $1 < p \leq \infty$, $\uparrow \infty$ for $0 < p < 1$, and is equal to $\int_\Omega \theta^{-1} d\mu(\theta)$ for $p = 1$.

Proof Consider (i). We have

$$\int_\Omega e_{n,k}^\alpha(\theta) d\mu(\theta) = \mu(\Omega) \times \int_0^1 e_{n,k}^\alpha(\theta) d\mu^*(\theta),$$

where $d\mu^* = \frac{\chi_\Omega}{\mu(\Omega)} d\mu$ is a probability measure on $[0, 1]$ and χ_Ω denotes the characteristic function of the set Ω . Hence, we only need to prove the case $\Omega = [0, 1]$. Without loss of generality, we assume $\mu((0, 1]) \neq 0$. Let $dv(\theta) = \theta^\alpha d\mu(\theta) / \int_0^1 \theta^\alpha d\mu(\theta)$ and $v_k = \int_0^1 \theta^k dv(\theta)$. Define the matrix $S = (s_{i,j})_{i,j \geq 0}$ by

$$s_{i,j} = \begin{cases} \frac{\Delta^i v_{k+1}}{\Delta^j v_k} & (i \geq j), \\ 0 & (i < j), \end{cases}$$

where $\Delta v_k = v_k - v_{k+1}$. Following the proof of [3, Proposition 19.2], we can prove $\sum_{i=0}^\infty s_{i,j} = 1$ for all j and $\sup_{i \geq 0} \sum_{j=0}^\infty s_{i,j} \leq 1$. From [3, Propositions 7.1 and 7.4] and $\|S\|_{\infty, \infty} = \sup_{i \geq 0} \sum_{j=0}^\infty s_{i,j}$, we get

$$\|S\|_{p,p} \leq 1 \quad (1 \leq p \leq \infty) \quad \text{and} \quad L_{p,p}(S) \geq 1 \quad (0 < p \leq 1). \tag{2.10}$$

On the other hand,

$$\begin{aligned} \sum_{j=0}^i s_{i,j} \binom{k+\alpha+j}{j} \Delta^j v_k &= \Delta^i v_{k+1} \sum_{j=0}^i \binom{k+\alpha+j}{j} \\ &= \binom{k+1+\alpha+i}{i} \Delta^i v_{k+1}. \end{aligned}$$

Moreover, $(\int_0^1 \theta^\alpha d\mu(\theta)) \Delta^i v_k = \Delta^i \mu_k$ for all k , where

$$\mu_k = \int_0^1 \theta^{k+\alpha} d\mu(\theta) \quad (k = 0, 1, \dots). \tag{2.11}$$

By (2.10),

$$\begin{aligned} \left\| \left\{ \binom{k+1+\alpha+i}{i} \Delta^i v_{k+1} \right\}_{i=0}^\infty \right\|_p &\leq \|S\|_{p,p} \left\| \left\{ \binom{k+\alpha+j}{j} \Delta^j v_k \right\}_{j=0}^\infty \right\|_p \\ &\leq \left\| \left\{ \binom{k+\alpha+j}{j} \Delta^j v_k \right\}_{j=0}^\infty \right\|_p \quad (1 \leq p \leq \infty). \end{aligned}$$

This implies that $\rho_k = \| \{ \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k \}_{n=k}^\infty \|_p$ decreases with k for $1 \leq p \leq \infty$. Analogously, it follows from $L_{p,p}(S) \geq 1$ that ρ_k increases with k for $0 < p \leq 1$. Next, let $\Omega \subseteq [\delta, 1 - \delta]$, where $0 < \delta < 1/2$. For $1 < p < \infty$, by Minkowski's inequality, we get

$$\begin{aligned} \rho_k &\leq \int_\Omega \left(\sum_{n=k}^\infty \left(e_{n,k}^\alpha(\theta) \right)^p \right)^{1/p} d\mu(\theta) \\ &\leq \int_\Omega \left(\sum_{n=k}^\infty e_{n,k}^\alpha(\theta) \right)^{1/p} \left(\sup_{n \geq k} e_{n,k}^\alpha(\theta) \right)^{1-1/p} d\mu(\theta) \\ &= \int_\Omega \theta^{-1/p} \left(\sup_{n \geq k} e_{n,k}^\alpha(\theta) \right)^{1-1/p} d\mu(\theta). \end{aligned} \tag{2.12}$$

By [3, Equation (19.13)], for $\theta \in \Omega$,

$$\begin{aligned} \sup_{m \geq k} e_{m,k}^\alpha(\theta) &\leq \theta^{\alpha - ([\alpha] + 1)} \left(\sup_{m \geq k} e_{m+[\alpha]+1, k+[\alpha]+1}^0(\theta) \right) \\ &\rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \tag{2.13}$$

Applying Fatou's Lemma to (2.12) first and then using (2.13), we infer that

$$\limsup_{k \rightarrow \infty} \rho_k \leq \int_\Omega \theta^{-1/p} \left(\limsup_{k \rightarrow \infty} \left(\sup_{n \geq k} e_{n,k}^\alpha(\theta) \right)^{1-1/p} \right) d\mu(\theta) = 0.$$

Combining this with (i), we conclude that $\rho_k \downarrow 0$ for $1 < p \leq \infty$. We leave the proof of the cases $p = \infty$, $0 < p < 1$ and $p = 1$ to the readers. ■

From Lemma 2.3(ii) and the definition, we see that for $\alpha > 0$, if $\mu((0, 1)) \neq 0$, then $\|H_\mu^\alpha\|_{p,q} = \infty$ for all $p > 0$ and $0 < q < 1$. By the definition, we can also prove that this result still holds for $\alpha = 0$.

The following two lemmas will be used in the proof of Theorem 4.3.

LEMMA 2.4 Let $p, q > 0$. The following assertions hold.

- (i) $h_{n,k}^{\alpha+s} = h_{n+s,k+s}^\alpha$ for all $n, k \geq 0$, where $\alpha \geq 0$ and $s \in \mathbb{N} \cup \{0\}$.
- (ii) The function $\alpha \mapsto L_{p,q}(H_\mu^\alpha)$ is increasing on $\mathbb{N} \cup \{0\}$.

Proof The assertion (i) follows from the definition. Let $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$ and $s := \alpha_1 - \alpha_2 \geq 0$. By (i), $\|H_\mu^{\alpha_1} X\|_q = \|H_\mu^{\alpha_2} X'\|_q$ for all $X = \{x_k\}_{k=0}^\infty \geq 0$, where $X' = \{x'_k\}_{k=0}^\infty$ is defined by

$$x'_k = \begin{cases} 0 & (0 \leq k < s), \\ x_{k-s} & (k \geq s). \end{cases}$$

This leads us to (ii) and the proof is complete. ■

Lemma 2.4(ii) may be false if we replace $L_{p,q}(H_\mu^\alpha)$ by $L_{p,q}^\downarrow(H_\mu^\alpha)$. A counterexample is given by $\mu = (1/2)\delta_0 + (1/2)\delta_1$. For such measure, by definition, $L_{p,p}^\downarrow(H_\mu^0) = \infty > 1/2 = L_{p,p}^\downarrow(H_\mu^1)$ for $p > 0$.

LEMMA 2.5 Let $\alpha \geq 0$ and $\rho > 1/p > 0$. Set $X_\alpha^\rho = \{x_k^\rho\}_{k=0}^\infty$ with

$$x_k^\rho = \binom{k + \alpha}{k} / \binom{k + \alpha + \rho}{k}. \tag{2.14}$$

Then $X_\alpha^\rho \geq 0$, $X_\alpha^\rho \downarrow$, and $X_\alpha^\rho \in \ell_p$. Moreover, if $\int_0^1 \theta^{-\rho} d\mu(\theta) < \infty$, then

$$\|H_\mu^\alpha X_\alpha^\rho\|_p \leq \|X_\alpha^\rho\|_p \int_0^1 \theta^{-\rho} d\mu(\theta). \tag{2.15}$$

Proof Clearly, $X_\alpha^\rho \geq 0$ and $X_\alpha^\rho \downarrow$. By [16, Chapter III, Equation (1.15)], we have $x_k^\rho \simeq \frac{\Gamma(1+\alpha+\rho)}{\Gamma(1+\alpha)} k^{-\rho}$, and so $X_\alpha^\rho \in \ell_p$. It is easy to see that

$$\binom{n + \alpha}{n - k} \binom{k + \alpha}{k} / \binom{k + \alpha + \rho}{k} = \binom{n + \alpha + \rho}{n - k} \binom{n + \alpha}{n} / \binom{n + \alpha + \rho}{n}.$$

This implies

$$\begin{aligned} \|H_\mu^\alpha X_\alpha^\rho\|_p &= \left(\sum_{n=0}^\infty \left(\sum_{k=0}^n \binom{n + \alpha}{n - k} \Delta^{n-k} \mu_k x_k^\rho \right)^p \right)^{1/p} \\ &= \left(\sum_{n=0}^\infty \left(x_n^\rho \sum_{k=0}^n \binom{n + \alpha + \rho}{n - k} \Delta^{n-k} \mu_k \right)^p \right)^{1/p}, \end{aligned} \tag{2.16}$$

where μ_k is defined by (2.11). Since $d\mu$ is a Borel probability measure and $\int_0^1 \theta^{-\rho} d\mu(\theta) < \infty$, by [13, Theorem 3], we get

$$\int_0^1 \theta^{-\rho} d\mu(\theta) = \sup_{n \geq 0} \sum_{k=0}^n \binom{n + \alpha + \rho}{n - k} \Delta^{n-k} \mu_k. \tag{2.17}$$

Inserting (2.17) into (2.16), the desired result follows. ■

Lemma 2.5 provides an upper bound estimate for the values of $L_{p,p}^\downarrow(H_\mu^\alpha)$ and $L_{p,p}(H_\mu^\alpha)$. We shall see this point later.

3. Investigation of the values of $\|H_\mu^0\|_{p,q}^\downarrow$ for the case $0 < p \leq q \leq 1$

Equation (1.1) is the special case $b_n = 1$ of [5, Theorem 4]. It allows us to derive the following result. Our result compliments the case $0 < p \leq q \leq 1$, which (1.4) does not deal with.

THEOREM 3.1 *Let $0 < p \leq q \leq 1$. Then*

$$\|H_\mu^0\|_{p,q}^\downarrow = \left(\sum_{n=0}^\infty \left(\int_0^1 (1-\theta)^n d\mu(\theta) \right)^q \right)^{1/q}. \tag{3.1}$$

Proof From (1.1) and Lemma 2.2(ii), we obtain

$$\|H_\mu^0\|_{q,q}^\downarrow = \left(\sum_{n=0}^\infty (h_{n,0}^0)^q \right)^{1/q} = \left(\sum_{n=0}^\infty \left(\int_0^1 (1-\theta)^n d\mu(\theta) \right)^q \right)^{1/q}. \tag{3.2}$$

On the other hand, the definition of $\|H_\mu^0\|_{p,q}^\downarrow$ implies

$$\left(\sum_{n=0}^\infty (h_{n,0}^0)^q \right)^{1/q} \leq \|H_\mu^0\|_{p,q}^\downarrow \leq \|H_\mu^0\|_{q,q}^\downarrow. \tag{3.3}$$

Putting (3.2)–(3.3) together yields (3.1). ■

We are unable to determine whether (3.1) still holds for $\|H_\mu^\alpha\|_{p,q}^\downarrow$ with $\alpha > 0$.

4. Investigation of the values of $L_{p,q}^\downarrow(H_\mu^\alpha)$ and $L_{p,q}(H_\mu^\alpha)$ for the case $0 < q \leq p \leq 1$

Now, we come to the evaluation of $L_{p,q}^\downarrow(H_\mu^\alpha)$ and $L_{p,q}(H_\mu^\alpha)$. We state the results here and give their proofs in the following section. The following result extends [7, Theorem 2.3, Equation (2.1)] from H_μ^0 to H_μ^α , where $\alpha \geq 0$.

THEOREM 4.1 *Let $\alpha \geq 0$ and $0 < q \leq p \leq 1$. Then*

$$L_{p,q}^\downarrow(H_\mu^\alpha) \geq L_{p,q}(H_\mu^\alpha) \geq \int_{(0,1]} \theta^{-1/q} d\mu(\theta). \tag{4.1}$$

Clearly, both inequalities in (4.1) are equalities whenever the integral at the right side of (4.1) is ∞ . In the following, we investigate the equality problem for other cases.

THEOREM 4.2 *Let $0 < q \leq p \leq 1$. The following assertions are true.*

- (i) *Both inequalities in (4.1) are equalities for $\alpha > 0$ (respectively $\alpha = 0$) if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ (respectively $\mu(\{1\}) = 1$) or the integral in (4.1) is ∞ .*

- (ii) For $\alpha \geq 0$, the second inequality in (4.1) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the integral in (4.1) is ∞ .

Theorem 4.2(ii) extends [7, Theorem 2.3(i)] from $\alpha = 0$ to $\alpha \geq 0$. For $\alpha = 0$ and $\mu(\{0\}) + \mu(\{1\}) = 1$, Theorem 4.2(ii) ensures that the second inequality in (4.1) becomes an equality for the case $0 < q < p \leq 1$. However, the choice $\mu = (1/2)\delta_0 + (1/2)\delta_1$ indicates that the same phenomenon may not happen to the first inequality. The same measure also shows that the condition $\mu(\{1\}) = 1$ in Theorem 4.2(i) cannot be replaced by $\mu(\{0\}) + \mu(\{1\}) = 1$.

For $0 < q = p \leq 1$, we have the following result.

THEOREM 4.3 *Let $0 < p \leq 1$. The following assertions hold.*

- (i) For $\alpha > 0$, if (1.7) or $p = 1$ holds, then

$$L_{p,p}^\downarrow(H_\mu^\alpha) = L_{p,p}(H_\mu^\alpha) = \int_{(0,1)} \theta^{-1/p} d\mu(\theta). \tag{4.2}$$

Equation (4.2) still holds for the case that $\alpha = 0$, $p = 1$, and $\mu(\{0\}) = 0$. If we only assume $\alpha = 0$ and $p = 1$, then the second equality in (4.2) remains valid.

- (ii) For $\alpha \in \mathbb{N} \cup \{0\}$, $L_{p,p}^\downarrow(H_\mu^\alpha) < \infty$ if and only if (2.1) (respectively (2.4)) is satisfied for $\alpha \in \mathbb{N}$ (respectively $\alpha = 0$). Moreover, in this case, (4.2) holds.
- (iii) For $\alpha \in \mathbb{N} \cup \{0\}$, $L_{p,p}(H_\mu^\alpha) < \infty \iff$ (2.1) is satisfied. Moreover, the second equality in (4.2) holds.

Theorem 4.3(i) extends [7, Theorem 2.3(ii) and (iii)] from $\alpha = 0$ to $\alpha \geq 0$. We remark that Equation (4.2) may be false for $\alpha = 0$. A counterexample is given by the measure $\mu = (1/2)\delta_0 + (1/2)\delta_1$. For this measure,

$$L_{p,p}^\downarrow(H_\mu^0) = \infty > 1/2 = \int_{(0,1)} \theta^{-1/p} d\mu(\theta) \quad (0 < p \leq 1).$$

The same measure also shows that the condition $\mu(\{0\}) = 0$ required in Theorem 4.3(i) cannot be removed. Moreover, condition (2.4) in Theorem 4.3(ii) cannot be replaced by (1.7). By the way, we are unable to determine whether the results of Theorem 4.3(ii) and (iii) remain valid for all $\alpha > 0$.

By [3, Proposition 7.9], $L_{p,q}((H_\mu^\alpha)^\ell) = L_{q^*,p^*}(H_\mu^\alpha)$, where $-\infty < p \leq q < 0$, $1/p + 1/p^* = 1$ and $1/q + 1/q^* = 1$. We have $0 < p^* \leq q^* < 1$. This enables us to restate the statements of Theorems 4.1–4.3 in terms of $L_{p,q}((H_\mu^\alpha)^\ell)$. The following is one of them.

THEOREM 4.4 *Let $-\infty < p \leq q < 0$. Then*

$$L_{p,q}((H_\mu^\alpha)^\ell) \geq \int_{(0,1)} \theta^{-1/p^*} d\mu(\theta) \quad (\alpha \geq 0). \tag{4.3}$$

Moreover, the equality holds for $p = q$ if (4.4) is satisfied:

$$\int_{(0,1)} \theta^{-(1/p^* + \epsilon)} d\mu(\theta) < \infty \quad \text{for some } \epsilon > 0. \tag{4.4}$$

Theorem 4.4 fills up the gap $-\infty < p \leq q < 0$, which [3, Theorem 7.18] (see (1.6)) does not deal with.

5. Proofs of Theorems 4.1–4.3

Proof of Theorem 4.1 It suffices to show that $L_{p,q}(H_\mu^\alpha) \geq \int_{(0,1]} \theta^{-1/q} d\mu(\theta)$. Let $E^\alpha(\theta) = (e_{n,k}^\alpha(\theta))_{n,k \geq 0}$ denote the generalized Euler matrix defined by (2.7). For $0 < \theta \leq 1$, the column sums of $E^\alpha(\theta)$ are all equal to θ^{-1} . According to [13, Lemma 2] and [10, Equation (4.2.4)], $\sup_{n \geq 0} \sum_{k=0}^\infty e_{n,k}^\alpha(\theta) = 1$. By [3, Proposition 7.4], we see that $L_{q,q}(E^\alpha(\theta)) \geq \theta^{-1/q}$. Applying Minkowski's inequality, we get

$$\begin{aligned} \|H_\mu^\alpha X\|_q &= \left\| \int_0^1 E^\alpha(\theta) X d\mu(\theta) \right\|_q \geq \int_{(0,1]} \|E^\alpha(\theta) X\|_q d\mu(\theta) \\ &\geq \left(\int_{(0,1]} \theta^{-1/q} d\mu(\theta) \right) \|X\|_q \geq \left(\int_{(0,1]} \theta^{-1/q} d\mu(\theta) \right) \|X\|_p, \end{aligned}$$

where $X \geq 0$. This leads us to (4.1). ■

Proof of Theorem 4.2 First, consider (ii). Obviously, the second inequality in (4.1) is an equality if the integral in (4.1) is ∞ . For the case that $\mu(\{0\}) + \mu(\{1\}) = 1$, we have $\mu((0, 1)) = 0$, and so for $\alpha > 0$,

$$\begin{aligned} \|H_\mu^\alpha e_0\|_q &= \left(\sum_{n=0}^\infty (h_{n,0}^\alpha)^q \right)^{1/q} = \left(\sum_{n=0}^\infty \left(\binom{n+\alpha}{n} \int_0^1 \theta^\alpha (1-\theta)^n d\mu(\theta) \right)^q \right)^{1/q} \\ &= \mu(\{1\}) = \int_{(0,1]} \theta^{-1/q} d\mu(\theta), \end{aligned} \tag{5.1}$$

where $e_0 = (1, 0, \dots)$. This shows that $L_{p,q}(H_\mu^\alpha) \leq \int_{(0,1]} \theta^{-1/q} d\mu(\theta)$. From [7, Equation (2.3)], we find that the same inequality still holds for the case $\alpha = 0$. Combining these with (4.1), we conclude that the second inequality in (4.1) is an equality.

Conversely, assume that $\mu(\{0\}) + \mu(\{1\}) \neq 1$ and $\int_{(0,1]} \theta^{-1/q} d\mu(\theta) < \infty$. We shall prove that the second inequality in (4.1) is not an equality. Let $e_{n,k}^\alpha(\theta)$ be defined by (2.7). Then

$$\int_{(0,1]} \theta^{-1/q} d\mu(\theta) = \int_{(0,1]} \left(\sum_{n=0}^\infty e_{n,0}^\alpha(\theta) \right)^{1/q} d\mu(\theta).$$

By assumption, $0 < q < 1$. This implies $\sum_{n=0}^\infty e_{n,0}^\alpha(\theta) < \sum_{n=0}^\infty (e_{n,0}^\alpha(\theta))^q$ for all $\theta \in (0, 1)$. Since $\mu((0, 1)) \neq 0$, it follows from Minkowski's inequality that

$$\int_{(0,1]} \theta^{-1/q} d\mu(\theta) < \int_{(0,1]} \left(\sum_{n=0}^\infty (e_{n,0}^\alpha(\theta))^q \right)^{1/q} d\mu(\theta) \leq \left\| \left\{ h_{n,0}^\alpha \right\}_{n=0}^\infty \right\|_q. \tag{5.2}$$

This enables us to find $0 < \beta < 1$ such that

$$\int_{(0,1]} \theta^{-1/q} d\mu(\theta) < \beta \left\| \left\{ h_{n,0}^\alpha \right\}_{n=0}^\infty \right\|_q. \tag{5.3}$$

We claim that

$$L_{p,q}(H_\mu^\alpha) \geq \min \left(\beta^{\frac{q-p}{q}} \int_{(0,1]} \theta^{-1/q} d\mu(\theta), \beta \left\| \left\{ h_{n,0}^\alpha \right\}_{n=0}^\infty \right\|_q \right). \tag{5.4}$$

Let $X \geq 0$ with $\|X\|_p = 1$. Then $x_{k_0} \geq \beta$ for some k_0 or $x_k < \beta$ for all k . For the first case, by Lemma 2.3, we get

$$\|H_\mu^\alpha X\|_q \geq x_{k_0} \left(\sum_{n=0}^\infty (h_{n,k_0}^\alpha)^q \right)^{1/q} \geq \beta \left\| \left\{ h_{n,0}^\alpha \right\}_{n=0}^\infty \right\|_q \quad (\alpha > 0).$$

As for the second case, we have $\sum_{k=0}^\infty x_k^q \geq \beta^{q-p} \sum_{k=0}^\infty x_k^p = \beta^{q-p}$. Applying (4.1), we infer that

$$\|H_\mu^\alpha X\|_q \geq \left(\int_{(0,1]} \theta^{-1/q} d\mu(\theta) \right) \|X\|_q \geq \beta^{\frac{q-p}{q}} \int_{(0,1]} \theta^{-1/q} d\mu(\theta).$$

Hence, no matter which case occurs, $\|H_\mu^\alpha X\|_q$ is always greater than or equal to the minimum given in (5.4), and consequently, (5.4) follows. We have $\beta^{(q-p)/q} > 1$. By (5.3)–(5.4), the second inequality in (4.1) is not an equality. This completes the proof of (ii).

Next, we prove (i). Obviously, (5.1) is true for the case $\alpha = 0$ if $\mu(\{1\}) = 1$. Putting (4.1) and (5.1) together, the ‘if’ part of (i) follows. For the ‘only if’ part, we assume that both inequalities in (4.1) are equalities. By (ii), $\mu(\{0\}) + \mu(\{1\}) = 1$ or the integral in (4.1) is ∞ . This finishes the proof of the case $\alpha > 0$. As for $\alpha = 0$, we shall further conclude $\mu(\{1\}) = 1$. Under the condition $\mu(\{0\}) + \mu(\{1\}) = 1$, we have

$$H_\mu^0 = \begin{pmatrix} 1 & 0 & \dots & \dots \\ \mu(\{0\}) & \mu(\{1\}) & 0 & \dots \\ \mu(\{0\}) & 0 & \mu(\{1\}) & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

If $\mu(\{0\}) \neq 0$, then by definition, $L_{p,q}^\downarrow(H_\mu^0) = \infty$. But we have assumed that both inequalities in (4.1) are equalities. This leads us to $L_{p,q}^\downarrow(H_\mu^0) = \int_{(0,1]} \theta^{-1/q} d\mu(\theta) = \mu(\{1\}) < \infty$, which is a contradiction. Hence, $\mu(\{0\}) = 0$ and the desired result follows. ■

Proof of Theorem 4.3 Consider (i). For $\alpha > 0$, we have $e_{n,k}^\alpha(0) = 0$ for all n, k , so

$$\|H_\mu^\alpha e_0\|_1 = \sum_{n=0}^\infty h_{n,0}^\alpha = \int_{(0,1]} \sum_{n=0}^\infty e_{n,0}^\alpha(\theta) d\mu(\theta) = \int_{(0,1]} \theta^{-1} d\mu(\theta), \quad (5.5)$$

where $e_0 = (1, 0, \dots)$. Thus, $L_{1,1}^\downarrow(H_\mu^\alpha) \leq \int_{(0,1]} \theta^{-1} d\mu(\theta)$. Combining this with (4.1), we get (4.2) for the case $p = 1$. Obviously, the above argument also applies to the case that $\alpha = 0, p = 1$, and $\mu(\{0\}) = 0$. To replace e_0 in (5.5) by $e_1 = (0, 1, 0, \dots)$, the same argument also proves the second equality in (4.2) for the case that $\alpha = 0$ and $p = 1$. Next, consider (1.7). Without loss of generality, we assume $\mu((0, 1]) \neq 0$, and consequently, the probability measure $\mu_{(0)}$ exists, where $d\mu_{(\tau)} = \frac{\chi_{(\tau,1]}}{\mu((\tau, 1])} d\mu$. By (1.7), $\int_0^1 \theta^{-\rho} d\mu_{(0)}(\theta) < \infty$ for $1/p < \rho < 1/p + \epsilon$. For such ρ , Lemma 2.5 ensures that $X_\alpha^\rho \geq 0$, $X_\alpha^\rho \downarrow$, and

$$\|H_{\mu_{(0)}}^\alpha X_\alpha^\rho\|_p \leq \|X_\alpha^\rho\|_p \int_0^1 \theta^{-\rho} d\mu_{(0)}(\theta), \quad (5.6)$$

where X_α^ρ is defined by (2.14). We have assumed $\alpha > 0$, so

$$h_{n,k}^\alpha = \binom{n+\alpha}{n-k} \int_{(0,1]} \theta^{k+\alpha} (1-\theta)^{n-k} d\mu(\theta) \quad (n \geq k). \tag{5.7}$$

Inserting (5.7) into (5.6), we obtain

$$\|H_\mu^\alpha X_\alpha^\rho\|_p \leq \|X_\alpha^\rho\|_p \int_{(0,1]} \theta^{-\rho} d\mu(\theta). \tag{5.8}$$

We have $\theta^{-\rho} \leq \theta^{-(1/p+\epsilon)} \in L^1((0, 1])$. By (1.7) and Lebesgue's dominated convergence theorem,

$$\lim_{\rho \rightarrow 1/p} \int_{(0,1]} \theta^{-\rho} d\mu(\theta) = \int_{(0,1]} \theta^{-1/p} d\mu(\theta).$$

Putting this with (4.1) and (5.8) together yields

$$\begin{aligned} \int_{(0,1]} \theta^{-1/p} d\mu(\theta) &\leq L_{p,p}(H_\mu^\alpha) \leq L_{p,p}^\downarrow(H_\mu^\alpha) \\ &\leq \lim_{\rho \rightarrow 1/p} \frac{\|H_\mu^\alpha X_\alpha^\rho\|_p}{\|X_\alpha^\rho\|_p} \leq \int_{(0,1]} \theta^{-1/p} d\mu(\theta), \end{aligned} \tag{5.9}$$

which gives (4.2) for the case (1.7).

Next, consider (iii). Let $\alpha \in \mathbb{N} \cup \{0\}$. Obviously, (iii) holds if $\mu((0, 1]) = 0$. Hence, we assume that $\mu((0, 1]) \neq 0$. Suppose that $L_{p,p}(H_\mu^\alpha) < \infty$. Then for $X = (\dots, x_{k_0}, \dots) \geq 0$, we have $\frac{\|H_\mu^\alpha X\|_p}{\|X\|_p} \geq \frac{x_{k_0} \|\{h_{n,k_0}^\alpha\}_{n=0}^\infty\|_p}{\|X\|_p}$, so $\|\{h_{n,k}^\alpha\}_{n=0}^\infty\|_p < \infty$ for some k . By [3, Proposition 19.2],

$$\begin{aligned} I &:= \sum_{n=0}^\infty \left(\int_{(0,1]} (1-\theta)^n d\mu(\theta) \right)^p \\ &\leq (\mu((0, 1]))^p \sum_{n=\alpha+k}^\infty \left(\binom{n}{\alpha+k} \int_0^1 \theta^{k+\alpha} (1-\theta)^{n-\alpha-k} d\mu_{(0)}(\theta) \right)^p \\ &= \sum_{n=k}^\infty \left(\binom{n+\alpha}{n-k} \int_{(0,1]} \theta^{k+\alpha} (1-\theta)^{n-k} d\mu(\theta) \right)^p \leq \|\{h_{n,k}^\alpha\}_{n=0}^\infty\|_p^p \quad \text{for all } k. \end{aligned}$$

Thus, $I < \infty$. Conversely, assume $I < \infty$. We shall show that $L_{p,p}(H_\mu^\alpha) < \infty$. From Lemma 2.4(ii), we know that $L_{p,p}(H_\mu^\alpha)$ is increasing in α . Without loss of generality, we assume $\alpha \in \mathbb{N}$. We have $h_{n,k}^\alpha = h_{n+\alpha,k+\alpha}^0$ and

$$h_{n,k}^0 = \begin{cases} \binom{n}{n-k} \int_{(0,1]} \theta^k (1-\theta)^{n-k} d\mu(\theta) & (n \geq k \geq 1), \\ 0 & (n < k). \end{cases} \tag{5.10}$$

Applying (1.1) twice and Theorem 3.1 once, we get

$$\begin{aligned} L_{p,p}(H_\mu^\alpha) &\leq L_{p,p}^\downarrow(H_\mu^\alpha) \leq \|H_\mu^\alpha\|_{p,p}^\downarrow = \sup_{r \geq 0} \left(\frac{1}{r+1} \sum_{n=0}^\infty \left(\sum_{k=0}^r h_{n,k}^\alpha \right)^p \right)^{1/p} \\ &\leq \sup_{r \geq 0} \left(\frac{r+\alpha+1}{r+1} \frac{1}{r+\alpha+1} \sum_{n=0}^\infty \left(\sum_{k=1}^{r+\alpha} h_{n,k}^0 \right)^p \right)^{1/p} \\ &\leq (1+\alpha)^{1/p} \mu((0, 1]) \|H_{\mu_{(0)}}^0\|_{p,p}^\downarrow = (1+\alpha)^{1/p} I^{1/p} < \infty. \end{aligned} \tag{5.11}$$

Hence, $L_{p,p}(H_\mu^\alpha) < \infty$ if and only if (2.1) holds. For (iii), it remains to prove the equality. By (4.1) and the monotonicity of $\alpha \mapsto L_{p,p}(H_\mu^\alpha)$, it suffices to show that $L_{p,p}(H_\mu^\alpha) \leq \int_{(0,1]} \theta^{-1/p} d\mu(\theta)$ for $\alpha \geq 1$. We have assumed $\mu((0, 1]) \neq 0$ at the beginning of the proof of (iii), so $\mu((\tau_0, 1]) > 0$ for some $\tau_0 > 0$. Let $0 < \tau < \tau_0$ and $\rho > 1/p$. We have $\int_0^1 \theta^{-\rho} d\mu_{(\tau)}(\theta) < \infty$. To modify the argument from (5.6) to (5.9), we can find $\rho_\tau > 1/p$ such that

$$\|H_{\mu_{(\tau)}}^\alpha X_\alpha^{\rho_\tau}\|_p^p \leq \left\{ \left(\int_{(0,1]} \theta^{-1/p} d\mu_{(\tau)}(\theta) \right)^p + \tau \right\} \|X_\alpha^{\rho_\tau}\|_p^p.$$

It is clear that $X_\alpha^{\rho_\tau} \downarrow$ and

$$\begin{aligned} \|H_\mu^\alpha X_\alpha^{\rho_\tau}\|_p^p &\leq (\mu((\tau, 1]))^p \|H_{\mu_{(\tau)}}^\alpha X_\alpha^{\rho_\tau}\|_p^p + \sum_{n=0}^\infty \left(\sum_{k=0}^\infty |h_{n,k}^\alpha - \mu((\tau, 1]) \hat{h}_{n,k}| X_k^{\rho_\tau} \right)^p \\ &\leq \left\{ \left(\int_{(\tau,1]} \theta^{-1/p} d\mu(\theta) \right)^p + \tau + (\|A_\tau\|_{p,p}^\downarrow)^p \right\} \|X_\alpha^{\rho_\tau}\|_p^p, \end{aligned} \tag{5.12}$$

where $H_{\mu_{(\tau)}}^\alpha = (\hat{h}_{n,k})_{n,k \geq 0}$ and $A_\tau = (a_{n,k}^\tau)_{n,k \geq 0}$, defined by

$$\hat{h}_{n,k} = \begin{cases} \binom{n+\alpha}{n-k} \int_0^1 \theta^{k+\alpha} (1-\theta)^{n-k} d\mu_{(\tau)}(\theta) & (n \geq k), \\ 0 & (n < k), \end{cases}$$

and

$$a_{n,k}^\tau = |h_{n,k}^\alpha - \mu((\tau, 1]) \hat{h}_{n,k}| \quad (n, k \geq 0).$$

We claim that $\|A_\tau\|_{p,p}^\downarrow \rightarrow 0$ as $\tau \downarrow 0$. If so, the limiting case of (5.12) gives $L_{p,p}(H_\mu^\alpha) \leq L_{p,p}^\downarrow(H_\mu^\alpha) \leq \int_{(0,1]} \theta^{-1/p} d\mu(\theta)$. This is what we want. From the definitions of $h_{n,k}^\alpha$ and $\hat{h}_{n,k}$, we obtain

$$a_{n,k}^\tau = \begin{cases} \binom{n+\alpha}{n-k} \int_{[0,\tau]} \theta^{k+\alpha} (1-\theta)^{n-k} d\mu(\theta) & (n \geq k), \\ 0 & (n < k). \end{cases}$$

Without loss of generality, we only consider those τ with $\mu([0, \tau]) \neq 0$. Using an argument similar to (5.11), we can prove

$$\begin{aligned} \|A_\tau\|_{p,p}^\downarrow &\leq \mu([0, \tau])^{1/p} (1 + \alpha)^{1/p} \left(\sum_{n=0}^\infty \left(\int_{(0,\tau]} (1-\theta)^n \frac{d\mu(\theta)}{\mu([0, \tau])} \right)^p \right)^{1/p} \\ &= (1 + \alpha)^{1/p} \left(\sum_{n=0}^\infty \left(\int_{(0,\tau]} (1-\theta)^n d\mu(\theta) \right)^p \right)^{1/p} \rightarrow 0 \quad \text{as } \tau \downarrow 0. \end{aligned}$$

The last fact is based on (2.1) and Lebesgue's dominated convergence theorem. This finishes the proof of (iii).

Finally, we prove (ii). We have $X_\alpha^\rho \downarrow$, so the proof of (iii) can apply to the corresponding parts of (ii). In particular, the case $\alpha \in \mathbb{N}$ of (ii) holds. It remains to prove the case $\alpha = 0$. For $X \in \ell_p$ and $X \downarrow 0$, we have

$$\|H_\mu^0 X\|_p \geq x_0 \| \{h_{n,0}^0\}_{n=0}^\infty \|_p = x_0 \left(\sum_{n=0}^\infty \left(\int_0^1 (1-\theta)^n d\mu(\theta) \right)^p \right)^{1/p}.$$

Hence, (2.4) holds if $L_{p,p}^\downarrow(H_\mu^0) < \infty$. Conversely, assume that (2.4) holds. We have $L_{p,p}^\downarrow(H_\mu^0) \leq \|H_\mu^0\|_{p,p}^\downarrow$. By (2.4) and Theorem 3.1, we obtain $L_{p,p}^\downarrow(H_\mu^0) < \infty$. Hence, $L_{p,p}^\downarrow(H_\mu^0) < \infty$ if and only if (2.4) holds. From

$$\infty > \sum_{n=0}^\infty \left(\int_0^1 (1-\theta)^n d\mu(\theta) \right)^p \geq \sum_{n=0}^\infty (\mu(\{0\}))^p,$$

we see $\mu(\{0\})=0$. This implies (5.10), even if $n \geq k \geq 1$ is replaced by $n \geq k$. Moreover, $\mu((0, 1]) \neq 0$, and so $\mu((\tau_0, 1]) > 0$ for some $\tau_0 > 0$. Following the preceding proof of (iii), we can easily derive (4.2) for the case $\alpha=0$ of (ii). We leave it to the readers. ■

6. Investigation of the values of $L_{p,q}(H_\mu^\alpha)$ and $L_{p,q}((H_\mu^\alpha)^t)$ with $1 \leq p \leq \infty$ and $0 < q \leq p$

In [6, Theorems 2.2 and 3.1], we found suitable formulae for $L_{p,q}(H_\mu^0)$ and $L_{p,q}((H_\mu^0)^t)$, where $1 \leq p \leq \infty$ and $0 < q \leq p$. In the following, we extend them to the cases $L_{p,q}(H_\mu^\alpha)$ and $L_{p,q}((H_\mu^\alpha)^t)$ with $\alpha > 0$.

THEOREM 6.1 *Let $1 \leq p \leq \infty$ and $0 < q \leq p$. Then the following two assertions hold.*

(i) *For $0 < q \leq 1$,*

$$L_{p,q}(H_\mu^\alpha) = \begin{cases} \left\{ \sum_{n=0}^\infty \left(\binom{n+\alpha}{n} \int_0^1 \theta^\alpha (1-\theta)^n d\mu(\theta) \right)^q \right\}^{1/q} & \text{if } \alpha > 0 \text{ or } \alpha = 0, \mu(\{0\}) = 0, \\ \left\{ \sum_{n=1}^\infty \left(n \int_0^1 \theta (1-\theta)^{n-1} d\mu(\theta) \right)^q \right\}^{1/q} & \text{if } \alpha = 0 \text{ and } \mu(\{0\}) > 0. \end{cases}$$

(ii) *If $1 < q \leq \infty$ and $(\sum_{n=0}^\infty |h_{n,k_0}^\alpha|^q)^{1/q} < \infty$ for some $k_0 \geq 0$, then*

$$L_{p,q}(H_\mu^\alpha) = \lim_{n \rightarrow \infty} \mu_n = \mu(\{1\}),$$

where μ_n is defined by (2.11). We also have

$$L_{p,q}(H_\mu^\alpha) = \infty \quad \text{if} \quad \left(\sum_{n=0}^\infty |h_{n,k}^\alpha|^q \right)^{1/q} = \infty \quad \text{for all } k \geq 0.$$

Proof The case $\alpha=0$ has been proven in [6, Theorem 2.2], so we assume $\alpha > 0$. Applying (1.3), we get $L_{p,q}(H_\mu^\alpha) = \inf_{k \geq 0} (\sum_{n=0}^\infty (h_{n,k}^\alpha)^q)^{1/q}$. Hence, (i) follows from Lemma 2.3. For (ii), we have $e_{n,k}^\alpha(0) = 0$ for all $n, k \geq 0$, $e_{k,k}^\alpha(1) = 1$, and $e_{n,k}^\alpha(1) = 0$ for all $n > k \geq 0$. To replace $e_{n,k}(\theta)$ by $e_{n,k}^\alpha(\theta)$, the same proof as [6, Theorem 2.2(ii)] leads us to the desired result of (ii). ■

THEOREM 6.2 *Let $1 \leq p \leq \infty$ and $0 < q \leq p$. Then the following two assertions hold.*

- (i) $L_{p,q}((H_\mu^\alpha)^t) = \int_0^1 \theta^\alpha d\mu(\theta)$ if $0 < q \leq 1$.
- (ii) $L_{p,q}((H_\mu^\alpha)^t) = ((\mu(\{0\}))^q + (\mu(\{1\}))^q)^{1/q}$ if $1 < q \leq \infty$.
- (iii) $L_{p,q}((H_\mu^\alpha)^t) = \mu(\{1\})$ if $\alpha > 0$ and $1 < q \leq \infty$.

Proof By (1.3), $L_{p,q}((H_\mu^\alpha)') = \inf_{n \geq 0} (\sum_{k=0}^n (h_{n,k}^\alpha)^q)^{1/q}$. Consider the case $0 < q \leq 1$. Then $\| \{h_{n,k}^\alpha\}_{k=0}^\infty \|_1 \leq \| \{h_{n,k}^\alpha\}_{k=0}^\infty \|_q$. Moreover, by [13, Lemma 2], we know that $\sum_{k=0}^n h_{n,k}^\alpha$ is increasing in n . Putting these together yields

$$\int_0^1 \theta^\alpha d\mu(\theta) = h_{0,0}^\alpha \leq \sum_{k=0}^n h_{n,k}^\alpha \leq \left(\sum_{k=0}^n (h_{n,k}^\alpha)^q \right)^{1/q} \quad \text{for } n \geq 1.$$

Hence, $L_{p,q}((H_\mu^\alpha)') = h_{0,0}^\alpha$ and (i) follows. The case (ii) has been proved in [6, Theorem 3.1], so we consider (iii). For $m \geq 2$, we have

$$\mu(\{1\}) \leq \inf_{n \geq 0} h_{n,n}^\alpha \leq L_{p,q}((H_\mu^\alpha)') \leq h_{m,0}^\alpha + h_{m,m}^\alpha + \left(\sum_{k=1}^{m-1} (h_{m,k}^\alpha)^q \right)^{1/q}. \tag{6.1}$$

Since $\alpha > 0$, it follows from Lebesgue’s dominated convergence theorem that

$$h_{m,0}^\alpha = \int_{(0,1]} \binom{m+\alpha}{m} \theta^\alpha (1-\theta)^m d\mu(\theta) \rightarrow 0 \quad \text{as } m \rightarrow \infty \tag{6.2}$$

and

$$h_{m,m}^\alpha = \int_{(0,1]} \theta^{m+\alpha} d\mu(\theta) \rightarrow \mu(\{1\}) \quad \text{as } m \rightarrow \infty. \tag{6.3}$$

Like [6, Equation (3.2)], applying [10, Equation (4.2.4)], we can prove

$$\begin{aligned} \left(\sum_{k=1}^{m-1} (h_{m,k}^\alpha)^q \right)^{1/q} &\leq \int_{(0,1)} \left(\sum_{k=1}^{m-1} (e_{m,k}^\alpha(\theta))^q \right)^{1/q} d\mu(\theta) \\ &\leq \left(\sup_{0 < \theta < 1} \sum_{k=1}^{m-1} e_{m,k}^\alpha(\theta) \right)^{1/q} \int_{(0,1)} \left(\sup_{1 \leq k \leq m-1} e_{m,k}^\alpha(\theta) \right)^{1-1/q} d\mu(\theta) \\ &\leq \int_{(0,1)} \left(\sup_{1 \leq k \leq m-1} e_{m,k}^\alpha(\theta) \right)^{1-1/q} d\mu(\theta), \end{aligned} \tag{6.4}$$

where $e_{m,k}^\alpha(\theta)$ is defined by (2.7). For $\theta \in (0, 1)$, we have

$$\sup_{1 \leq k \leq m-1} e_{m,k}^\alpha(\theta) \leq \sup_{k > k_0} \left(\sup_{n \geq k} e_{n,k}^\alpha(\theta) \right) + \sup_{1 \leq k \leq k_0} e_{m,k}^\alpha(\theta) \quad (m > k_0 \geq 1).$$

As proved in (2.13), $\sup_{n \geq k} e_{n,k}^\alpha(\theta) \rightarrow 0$ as $k \rightarrow \infty$. On the other hand, for each k with $1 \leq k \leq k_0$, $e_{m,k}^\alpha(\theta) \rightarrow 0$ as $m \rightarrow \infty$. Hence, $\sup_{1 \leq k \leq m-1} e_{m,k}^\alpha(\theta) \rightarrow 0$ as $m \rightarrow \infty$. Putting this with (6.4) together and applying Lebesgue’s dominated convergence theorem, we obtain

$$\left(\sum_{k=1}^{m-1} (h_{m,k}^\alpha)^q \right)^{1/q} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \tag{6.5}$$

Hence, (iii) follows from (6.1)–(6.3) and (6.5). We complete the proof. ■

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