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## The theory of wavelet transform method on chaotic synchronization of coupled map lattices

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The wavelet transform method originated by Wei *et al.* [Phys. Rev. Lett. **89**, 284103.4 (2002)] was proved [Juang and Li, J. Math. Phys. **47**, 072704.16 (2006); Juang *et al.*, J. Math. Phys. **47**, 122702.11 (2006); Shieh *et al.*, J. Math. Phys. **47**, 082701.10 (2006)] to be an effective tool to reduce the order of coupling strength for coupled chaotic systems to acquire the synchrony regardless the size of oscillators. In Juang *et al.*, [IEEE Trans. Circuits Syst., I: Regul. Pap. **56**, 840 (2009)] such method was applied to coupled map lattices (CMLs). It was demonstrated that by adjusting the wavelet constant of the method can greatly increase the applicable range of coupling strengths, the parameters, range of the individual oscillator, and the number of nodes for local synchronization of CMLs. No analytical proof is given there. In this paper, the optimal or near optimal wavelet constant can be explicitly identified. As a result, the above described scenario can be rigorously verified. © 2011 American Institute of Physics. [doi:10.1063/1.3525802]

### I. INTRODUCTION

Simulation of natural phenomena is one of the most important research fields, and coupled map lattices (CMLs) are a paradigm for studying fundamental questions in spatially extended dynamical systems. This is because of their wide range of applications such as in turbulence, pattern formation in natural systems, and solitons. They also exhibit a very rich phenomenology, including a wide variety of both spatial and temporal periodic structures, intermittence, chaos, domain walls, kink dynamics, etc. As a matter of fact, one of the most interesting aspects of CMLs is the presence of attracting manifolds. Such attracting manifolds lead to notions such as partial synchronization,<sup>8</sup> weak and strong synchronization,<sup>16,19</sup> and (complete) synchronization.<sup>13,14,17,21–23</sup>

As to the study of local synchronization in CMLs, the notion of master stability functions (MSFs) that allows one to isolate the contribution of the network structure in terms of the eigenvalues of the coupling matrix was introduced in Refs. 1, 4, 9, 15, and 25 to determine the possible range of coupling strength. This function then defines a region of stably synchronous state in terms of the coupling strength and the eigenvalues of the coupling matrix. Most of the work done in finding such a region of stability of the synchronous state is numerical. In a few certain cases, such as when the coupling matrix is symmetric, the MSFs can be further reduced to a number of inequalities<sup>2,5–7</sup>

$$L_{\max} + \ln |1 + d\lambda_i| < 0, \quad i = 2, \dots, m.$$

Here  $L_{\max}$  is the largest Lyapunov exponent of the individual map,  $\lambda_i$  are the nonzero eigenvalues of the  $m \times m$  coupling matrix, and  $d$  is the coupling strength. The Gershgorin disk theory is then applied to obtain some sufficient conditions<sup>2</sup> on the coupling strength for local synchronization.

In Ref. 12, the optimal synchronization interval for coupling strengths of CMLs with symmetric coupling was analytically obtained. In particular, they also identified the best choice of coupling

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strength in the sense that such a coupling strength gives the fastest convergence rate of initial values toward the synchronous manifold. Furthermore, it was shown that such a coupling strength is independent of the choice of the individual chaotic map. Those results described above were also generalized<sup>13</sup> to the case that the coupling topology is allowed to be nonsymmetric. Due to the nonlinear coupling between oscillators for CMLs, both the second largest and the smallest eigenvalues of the coupling matrix play a role in determining the synchronization interval as opposed to linear coupling between those of coupled chaotic systems, which results in only the second largest eigenvalue being relevant in determining its synchronization interval. Such nonlinear coupling of CMLs also produces size instability. For instance, assume the dynamics of the individual oscillator is governed by the quadratic equations  $f_\mu(x) = \mu x(1-x)$ . Fixed  $\mu = 3.5708 > \mu_\infty = 3.5699456$ , the corresponding CMLs loses its synchronization provided that the number of oscillators grows larger than 20.<sup>12</sup> To improve on such size instability phenomenon, the wavelet transform method initiated in Ref. 20 was applied. The associated system remains synchronized when the number of oscillators reaches is doubled.<sup>12</sup> It was also demonstrated there that by adjusting the wavelet constant of the method can greatly increase the applicable range of coupling strengths, the parameters, and the number of nodes for local synchronization of CMLs.

In this paper, the optimal or near optimal wavelet constant can be explicitly identified. As a result, the above described scenario can be rigorously verified. We shall begin with describing the model of CMLs and briefly mention the wavelet transform method. The dynamic of CMLs with a symmetric coupling network can be described in the following vector form:<sup>17,21</sup>

$$X(n+1) = (I + \epsilon A)F(X(n)), \quad (1.1)$$

where  $X(n) = (x_1(n), x_2(n), \dots, x_N(n))^T$ ;  $I$  is the identity matrix;  $\epsilon$  is the coupling strength;  $A$  is a symmetric coupling matrix having zero row sums with zero being a simple eigenvalue; and  $F(x_1, x_2, \dots, x_N) = (f(x_1), \dots, f(x_N))^T$ . Here,  $f(x)$  describes the chaotic dynamics of an individual oscillator.

The wavelet transform method is a way<sup>24</sup> of reconstructing the network topology so as to affect the stability of synchronous manifold of (1.1). For more details, see Refs. 12 and 18. Here, we only described the needed formulation for our purpose. Let  $N = 2n$ . Write  $A$  as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}_{n \times n},$$

where the dimension of each block matrix  $A_{kl}$  is  $2 \times 2$ . By an  $i$ -scale wavelet operator  $W$ ,<sup>3,20</sup> the matrix  $A$  is transformed into  $W(A)$  of the form

$$W(A) = \begin{pmatrix} \tilde{A}_{11} & \cdots & \tilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{n1} & \cdots & \tilde{A}_{nn} \end{pmatrix}_{n \times n},$$

where each entry of  $\tilde{A}_{kl}$  is the average of entries of  $A_{kl}$ ,  $1 \leq k, l \leq n$ . After reconstruction,<sup>18</sup> the coupling matrix  $A$  becomes  $A + \alpha W(A)$ . Here  $\alpha$  is a wavelet constant.

In this paper, we consider the nearest neighbor coupling with mixed boundary conditions, which is given as follows:

$$A = A(\beta) = \begin{pmatrix} -1 - \beta & 1 & 0 & \cdots & 0 & \beta \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ \beta & 0 & \cdots & 0 & 1 & -1 - \beta \end{pmatrix}_{N \times N}.$$

Note that  $\beta = 1$  corresponds to periodic boundary conditions, while  $\beta = 0$  is associated with Neumann boundary conditions. The newly constructed coupling matrix  $A(\beta) + \alpha W(A(\beta)) =: C(\alpha, \beta)$  is then of the following form.

$$C(\alpha, \beta) = A(\beta) + \alpha \begin{pmatrix} \tilde{A}_1(\beta) & \tilde{A}_2(1) & \mathbf{0} & \cdots & \mathbf{0} & \tilde{A}_2(\beta) \\ \tilde{A}_2(1) & \tilde{A}_1(1) & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \tilde{A}_1(1) & \tilde{A}_2(1) \\ \tilde{A}_2(\beta) & \mathbf{0} & \cdots & \mathbf{0} & \tilde{A}_2(1) & \tilde{A}_1(\beta) \end{pmatrix}_{n \times n}, \tag{1.2a}$$

where

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{A}_1(\beta) = \begin{pmatrix} \frac{-1-\beta}{4} & \frac{-1-\beta}{4} \\ \frac{-1-\beta}{4} & \frac{-1-\beta}{4} \end{pmatrix}, \text{ and } \tilde{A}_2(\beta) = \begin{pmatrix} \frac{\beta}{4} & \frac{\beta}{4} \\ \frac{\beta}{4} & \frac{\beta}{4} \end{pmatrix}.$$

The eigenvalues of  $C(\alpha, \beta)$  are denoted by  $\lambda_i(\alpha, \beta)$  with

$$0 = \lambda_1(\alpha, \beta) \geq \lambda_2(\alpha, \beta) \geq \cdots \geq \lambda_N(\alpha, \beta). \tag{1.2b}$$

For fixed  $\beta$ , the graphs of  $\lambda_i(\alpha, \beta)$  are called the eigencurves of  $C(\alpha, \beta)$ .

We conclude this introductory section by recording some of needed theorems derived in Refs. 10 and 12. Let  $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$  be the eigenvalues of the coupling matrix  $A$ . It was shown, e.g., Ref. 17, that if

$$L_{\max} + \ln |1 + \epsilon \lambda_i| < 0 \quad i = 2, \dots, N. \tag{1.3}$$

for all the nonzero eigenvalues  $\lambda_i$ . Note that the second largest eigenvalue alone is not enough to ensure that all other eigenvalues satisfied (1.3). To achieve synchronization of CMLs, we need to find  $\epsilon$  so that the maximum of  $|1 + \epsilon \lambda_i|, i = 2, \dots, N$ , is a minimum. Specifically, we need to solve a min-max problem of the form

$$\min_{\epsilon \in \mathbb{R}} \max_{2 \leq i \leq N} |1 + \epsilon \lambda_i|. \tag{1.4}$$

The min-max problem (1.3) was solved in Ref. 12. For ease of reference, we record their result in the following.

**Theorem 1.1:** (Theorem 1 of Ref. 12) The min-max problem (1.3) can be achieved when  $\epsilon = \epsilon_{2,N} = \frac{-2}{\lambda_2 + \lambda_N}$ . Let

$$t_{i,j} := \left| \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right|. \tag{1.5}$$

Then

$$\min_{\epsilon \in \mathbb{R}} \max_{2 \leq i \leq N} |1 + \epsilon \lambda_i| = t_{2,N}. \tag{1.6}$$

Consequently, system (1.1) is (locally) synchronized if and only if

$$L_{\max} + \ln |t_{2,N}| =: \delta_{N,f} < 0. \tag{1.7}$$

If (1.6) holds, then there exists an optimal neighborhood  $N_{N,f}$ , the synchronization interval, of  $\epsilon_{2,N}$  so that (1.1) is (locally) synchronized whenever  $\epsilon \in N_{N,f}$ . Here

$$N_{N,f} = \left( \frac{1 - e^{-L_{\max}}}{-\lambda_2}, \frac{1 + e^{-L_{\max}}}{-\lambda_N} \right). \tag{1.8}$$

The interval  $N_{N,f}$  (if it exists) is optimal in the sense that if  $\epsilon$  is not in  $N_{N,f}$ , then system (1.1) will not acquire (local) synchronization. Moreover,  $\epsilon_{2,N}$ , which is independent of the choice of the individual chaotic map, is the best choice of coupling strength for local synchronization of 1.1 in the sense that such a coupling strength gives the fastest convergence rate of initial values toward the synchronous manifold.

Clearly, Theorem 1.1 is still valid for newly constructed coupling matrix  $C(\alpha, \beta)$ . Note that the corresponding  $\delta_{N,f}$ ,  $\epsilon_{2,N}$ ,  $N_{N,f}$ , and  $t_{2,N}$  now depend on the wavelet constant  $\alpha$  and the boundary constant  $\beta$  as well. To emphasize such dependence, we may write  $\delta_{N,f}$ ,  $\epsilon_{2,N}$ ,  $N_{N,f}$ , and  $t_{2,N}$  as  $\delta_{N,f}(\alpha, \beta)$ ,  $\epsilon_{2,N}(\alpha, \beta)$ ,  $N_{N,f}(\alpha, \beta)$ , and  $t_{2,N}(\alpha, \beta)$ , respectively.

**Theorem 1.2:** (Theorem 2.1 of Ref. 10) Let  $N \times N$ ,  $N = 2n$ ,  $n \in \mathbb{N}$ , be the dimension of the matrix  $C(\alpha, 1)$ . Let dimension of each block matrix in  $C(\alpha, 1)$  be  $2 \times 2$ . Then the eigenvalues  $\lambda_m^\pm(\alpha, 1)$  of  $C(\alpha, 1)$  are of the following form:

$$\lambda_m^\pm(\alpha, 1) = \frac{1}{2} \left( \alpha \cos \frac{2m\pi}{n} - \alpha - 4 \right) \pm \frac{1}{2} \left[ \left( \alpha \cos \frac{2m\pi}{n} - \alpha - 4 \right)^2 + 4 \left( \alpha \cos^2 \frac{2m\pi}{n} + 2(\alpha + 1) \cos \frac{2m\pi}{n} - 2 - 3\alpha \right) \right]^{1/2}, \quad m = 0, 1, 2, \dots, n-1.$$

Likewise, treating  $\alpha$  as a parameter, the graphs of  $\lambda_m^\pm(\alpha, 1)$  are to be termed eigencurves of  $C(\alpha, 1)$ .

**Proposition 1.1:** (Proposition 2.4 of Ref. 10) If  $\frac{n}{2}$  is not a positive integer, then the eigencurves  $\lambda_m^\pm(\alpha, 1)$ ,  $m = 1, 2, \dots, n-1$ , are strictly decreasing in  $\alpha \in (0, \infty)$ . If  $\frac{n}{2} (> 1)$  is a positive integer, then  $\lambda_m^\pm(\alpha, 1)$ ,  $m = 1, 2, \frac{n}{2} - 1, \frac{n}{2} + 1, \dots, n-1$ , and  $\lambda_{\frac{n}{2}}^-(\alpha, 1)$  are strictly decreasing in  $\alpha \in (0, \infty)$ .

**Theorem 1.3:** (Theorem 2.2 of Ref. 10) Let  $N$  be any positive even integer. The dimension of each block matrix in  $C(\alpha, 1)$  is  $2 \times 2$ . Then

- (i) suppose  $N$  is a multiple of four and  $N > 4$ . For each  $\alpha > 0$ , let  $\lambda_2(\alpha, 1)$  be the second largest eigenvalue of  $C(\alpha, 1)$ . Then  $\lambda_2(\alpha, 1) = \lambda_1^+(\alpha, 1)$ , for  $0 \leq \alpha \leq \frac{1}{\sin^2(\frac{\pi}{n})} =: \alpha_1$ ; and  $\lambda_2(\alpha, 1) = \lambda_{\frac{n}{2}}^+(\alpha, 1) = -2$  for all  $\alpha \in [\alpha_1, \infty)$ .
- (ii) Suppose  $N$  is not a multiple of four. Then there exists a  $\tilde{\alpha}_c$  such that  $\lambda_2(\alpha, 1) = \lambda_{[\frac{n}{2}]}^+(\alpha, 1)$  for all  $\alpha \geq \tilde{\alpha}_c$ . Here  $[\frac{n}{2}]$  = the largest positive integer that is less than or equal to  $\frac{n}{2}$ . Moreover,  $\lambda_2(\alpha, 1) < -2$  whenever  $\alpha > \alpha_1$ .

**Proposition 1.2:** (Proposition 2.5 of Ref. 10) (i) In the  $\alpha - \lambda$  plane,  $\lambda_t^+(\alpha, 1)$  intersect with  $\lambda = -2 + k$  at  $\alpha_{t,k}$ , where

$$\alpha_{t,k} = \frac{2(1+t) - k^2}{(1-t)(1+t+k)}.$$

- (ii) For  $-1 \leq t < 1$ ,  $\lim_{\alpha \rightarrow \infty} \lambda_t^+(\alpha, 1) = -(t+3)$ .

**Theorem 1.4:** (Theorem 3.1 of Ref. 10) Let  $N$  be any positive even integer. The dimension of each block matrix in  $C(\alpha, 0)$  is  $2 \times 2$ . Let  $\lambda_m^\pm(\alpha, 0)$  be defined as follows:

$$\lambda_m^\pm(\alpha, 0) = \frac{1}{2} \left( \alpha \cos \frac{m\pi}{n} - \alpha - 4 \right) \pm \frac{1}{2} \left[ \left( \alpha \cos \frac{m\pi}{n} - \alpha - 4 \right)^2 + 4 \left( \alpha \cos^2 \frac{m\pi}{n} + 2(\alpha + 1) \cos \frac{m\pi}{n} - 2 - 3\alpha \right) \right]^{1/2}$$

Then  $\lambda_m^\pm(\alpha, 0)$ ,  $m = 1, 2, \dots, n-1$ ,  $\lambda_0^+(\alpha, 0) = 0$  and  $\lambda_n^+(\alpha, 0) = -2$  are eigenvalues of  $C(\alpha, 0)$  for each  $\alpha > 0$ .

**Theorem 1.5:** (Theorem 3.2 of Ref. 10) For each  $\alpha$ , let  $\lambda_2(\alpha, 0)$  be the second largest eigenvalue of  $C(\alpha, 0)$ . Then  $\lambda_2(\alpha, 0) = \lambda_1^+(\alpha, 0)$ , for  $0 \leq \alpha \leq \frac{1}{\sin^2(\frac{\pi}{2n})} =: \bar{\alpha}_1$ ; and  $\lambda_2(\alpha, 0) = \lambda_n^+(\alpha, 0) = -2$  for all  $\alpha \in [\bar{\alpha}_1, \infty)$ .

For the wavelet transform method applying on the coupled chaotic systems, the theoretical verification of its effectiveness was provided in Refs. 10, 11, and 18. It should be noted that in determining the synchronization interval of coupled chaotic systems only the second largest eigenvalue of the coupling matrix is involved.

**II. MAIN RESULTS FOR  $C(\alpha, 1)$**

In this section, we shall verify the effectiveness of the wavelet transform method applying on (1.1) for  $A$  being the nearest neighbor coupling with periodic boundary conditions. To this end, in addition to study the least eigencurve  $\lambda_N(\alpha, \beta)$  of  $C(\alpha, 1)$ , we need to further explore some additional properties of its second eigencurve  $\lambda_2(\alpha, \beta)$ , which were not studied in Ref. 10. We begin with letting  $t = \cos(\frac{2m\pi}{n})$ . Then we have

$$\begin{aligned} \lambda_m^\pm(\alpha, 1) &= \frac{1}{2} \left\{ \alpha(t - 1) - 4 \pm [(t - 1)^2\alpha^2 + 4(t^2 - 1)\alpha + 8(1 + t)]^{1/2} \right\} \\ &=: \lambda_t^\pm(\alpha, 1). \end{aligned}$$

It should be noted that  $\lambda_i^+(\alpha, \beta)$  (respectively,  $\lambda_i^-(\alpha, \beta)$ ) may intersect with  $\lambda_j^+(\alpha, \beta)$  (respectively,  $\lambda_j^-(\alpha, \beta)$ ) for  $i \neq j$ . Hence,  $\lambda_i(\alpha, \beta)$  may consist of pieces from various  $\lambda_m^+(\alpha, \beta)$  or  $\lambda_m^-(\alpha, \beta)$ ,  $m = 0, 1, 2, \dots, n - 1$ . See Theorem 1.3, for example. This disorder structure of  $\lambda_m^\pm(\alpha, \beta)$ ,  $m = 0, 1, 2, \dots, n - 1$ , makes the analytical identification of  $\lambda_i(\alpha, \beta)$  a nontrivial matter. The proof of the following proposition is elementary and thus skipped.

*Proposition 2.1:* Let  $\alpha > 0$  and  $\lambda_N(\alpha, 1), \lambda_2(\alpha, 1) \leq 0$ , then the following statements are equivalent.

- (1)  $\frac{\lambda_N(\alpha, 1) - \lambda_2(\alpha, 1)}{\lambda_N(\alpha, 1) + \lambda_2(\alpha, 1)}$  attains its minimum at  $\alpha_{\min}$ .
- (2)  $\frac{\lambda_N(\alpha, 1)}{\lambda_2(\alpha, 1)}$  attains its minimum at  $\alpha_{\min}$ .

Some new information concerning the eigencurve  $\lambda_2(\alpha, 1)$  for  $N$  is not a multiple of four are obtained in the following theorem.

**Theorem 2.1:** Let  $N$  be any positive even integer and  $N$  is not a multiple of four. Then there exists a  $\tilde{\alpha}_c$  such that

$$\lambda_2(\alpha, 1) = \begin{cases} \lambda_1^+(\alpha, 1), & \text{for } 0 \leq \alpha \leq \alpha_1 := \frac{1}{\sin^2(\frac{\pi}{n})}; \\ \lambda_{[\frac{n}{2}]}^+(\alpha, 1), & \text{for } \alpha \geq \tilde{\alpha}_c (\geq \alpha_1). \end{cases}$$

Here  $[\frac{n}{2}] =$  the largest positive integer that is less than or equal to  $\frac{n}{2}$ . Moreover,  $\lambda_2(\alpha, 1) < -2$  whenever  $\alpha > \alpha_1$ .

*Proof:* For  $\alpha_{t,k}$  to be positive in Proposition 1.2, we must have

$$2(1 + t) > k^2.$$

Now,

$$\begin{aligned} (1 - t)^2(1 + t + k)^2 \frac{d\alpha_{t,k}}{dt} &= 2(t + 1)^2 - k^3 + 4k - 2tk^2 > (1 + t)k^2 - k^3 + 4k - 2tk^2 \\ &= -k(k^2 + (t - 1)k - 4) = -k(k - t_+)(k - t_-), \end{aligned}$$

where  $t_{\pm} = (1 - t \pm \sqrt{16 + (1 - t)^2})/2$ . Note that we have used  $2(1 + t) > k^2$  to justify the above inequality. Moreover  $t_- < 0$  and  $t_+ \geq 2$ . Thus,  $\frac{d\alpha_{t,k}}{dt} > 0$  whenever  $0 \leq k < 2$ , and  $\lambda = \lambda_t^+(\alpha, 1)$  have the intersections intersect at the positive  $\alpha_{t,k}$ . Upon using Proposition 1.1, we conclude that for  $0 \leq m \leq n - 1$ , the portion of the graphs of  $\lambda_m^+(\alpha, 1)$  lying above the line  $\lambda = -2$  do not intersect each other. For  $N$  is not a multiple of four, we have  $n$  is an odd number. Since  $\alpha_1 \cdot \left(\cos \frac{2\pi}{n} - 1\right) = -2$ , we have that

$$\begin{aligned} \lambda_1^+(\alpha_1, 1) &= \frac{1}{2} \left\{ \alpha_1 \cdot \left(\cos \frac{2\pi}{n} - 1\right) - 4 + \left[ \alpha_1^2 \cdot \left(\cos \frac{2\pi}{n} - 1\right)^2 \right. \right. \\ &\quad \left. \left. + 4\alpha_1 \cdot \left(\cos \frac{2\pi}{n} - 1\right) \left(\cos \frac{2\pi}{n} + 1\right) + 8 \left(1 + \cos \frac{2\pi}{n}\right) \right]^{1/2} \right\} \\ &= \frac{1}{2} \left\{ (-2) - 4 + \left[ (-2)^2 - 8 \cdot \left(\cos \frac{2\pi}{n} + 1\right) + 8 \left(1 + \cos \frac{2\pi}{n}\right) \right]^{1/2} \right\} \\ &= -2. \end{aligned}$$

Hence  $\lambda_2(\alpha, 1)$  equal to  $\lambda_1^+(\alpha, 1)$  for  $0 \leq \alpha \leq \alpha_1 := \frac{1}{\sin^2(\frac{\pi}{n})}$  as asserted.

By using Proposition 1.2(ii), there exists a  $\tilde{\alpha}_c$  such that  $\lambda_2(\alpha, 1) = \lambda_{[\frac{n}{2}]_1}^+(\alpha, 1)$  for all  $\alpha_1 \leq \tilde{\alpha}_c \leq \alpha$ . And, we obtain the following inequality:

$$\alpha_1 \cdot \left(\cos \frac{2[\frac{n}{2}]\pi}{n} - 1\right) = -\frac{1}{2 \sin^2 \frac{\pi}{2n}} \leq -2,$$

for  $n \geq 3$  is an odd number. Consequently,

$$\begin{aligned} \lambda_{[\frac{n}{2}]_1}^+(\alpha_1, 1) &\leq \frac{1}{2} \left\{ (-2) - 4 + \left[ (-2)^2 - 8 \cdot \left(\cos \frac{2[\frac{n}{2}]\pi}{n} + 1\right) + 8 \left(1 + \cos \frac{2[\frac{n}{2}]\pi}{n}\right) \right]^{1/2} \right\} \\ &= -2 = \lambda_1^+(\alpha_1, 1). \end{aligned}$$

Hence,  $\lambda_2(\alpha, 1) < -2$  whenever  $\alpha > \alpha_1$ . □

We next study some properties of the least eigencurve  $\lambda_N(\alpha, 1)$  of  $C(\alpha, 1)$ . Some direct calculations would yield the following proposition.

*Proposition 2.2:* (i) In the  $\alpha - \lambda$  plane,  $\lambda_t^-(\alpha, 1)$  intersect with  $\lambda = -4 + k$  at  $\alpha_{t,k}^*$ , where

$$\alpha_{t,k}^* = \frac{2 - 2t + k^2 - 4k}{(1 - t)(1 - t - k)} > 0 \text{ for } k < 0.$$

(ii) For  $-1 \leq t < 1$ ,  $(m = 1, 2, \dots, n - 1)$ .

$$\lim_{\alpha \rightarrow \infty} \lambda_t^-(\alpha, 1) = \lim_{\alpha \rightarrow \infty} \lambda_m^-(\alpha, 1) = -\infty.$$

**Theorem 2.2:** Let  $N$  be any positive even integer. Then

$$\lambda_N(\alpha, 1) = \begin{cases} \lambda_0^-(\alpha, 1) = -4, & \text{for } 0 \leq \alpha \leq \frac{2}{1 - \cos(\frac{2}{n}[\frac{n}{2}]\pi)} := \alpha_1^*; \\ \lambda_{[\frac{n}{2}]}^-(\alpha, 1), & \text{for } \alpha \geq \alpha_1^*. \end{cases}$$

In particular, if  $N$  is a multiple of four,  $\alpha_1^* = 1$  and  $\lambda_{[\frac{n}{2}]}^-(\alpha, 1) = -2\alpha - 2$ .

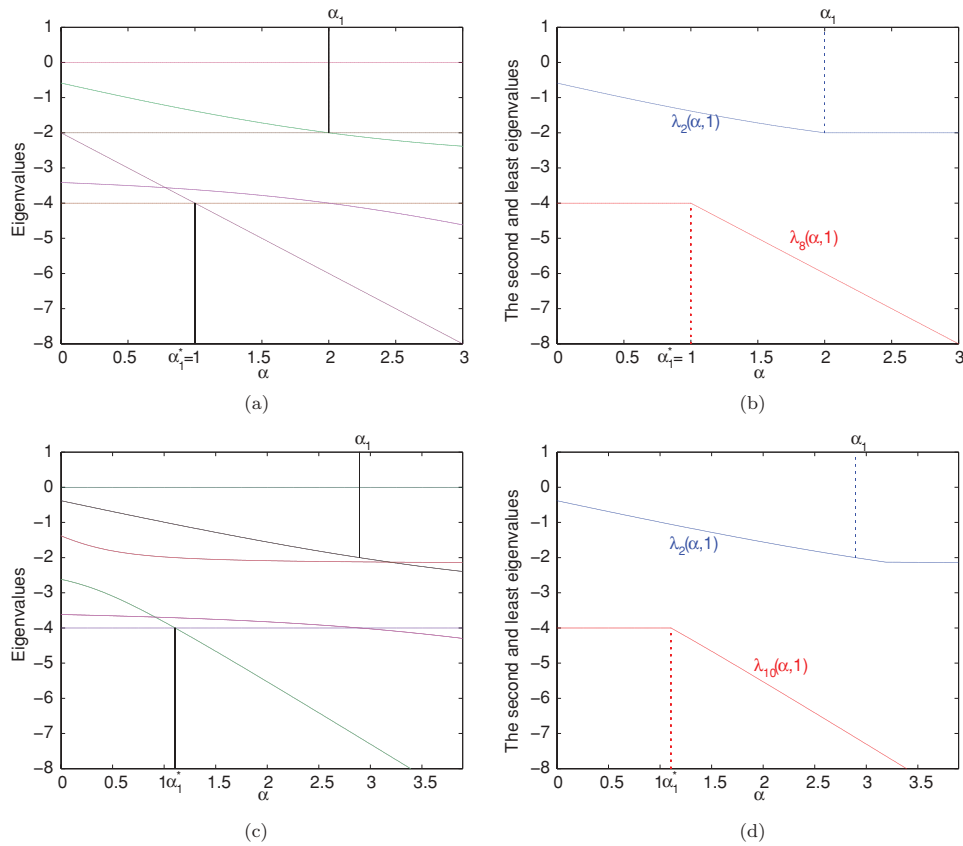


FIG. 1. (Color online) (a) All eigenvalues of  $C(\alpha, 1)$  for  $N = 8$  which is multiple of four. (b) The second and least eigenvalues for  $N = 8$ . (c) All eigenvalues of  $C(\alpha, 1)$  for  $N = 10$  which is not multiple of four. (d) The second and least eigenvalues for  $N = 10$ .

*Proof:* First, we note that  $\lambda_0^-(\alpha, 1) = -4$  for all  $\alpha$  by Theorem 1.2. Furthermore, let  $k < 0$ , we consider the following inequality:

$$\begin{aligned} (1-t)^2(1-t-k)^2 \frac{d\alpha_{t,k}^*}{dt} &= 2(1-t)^2 - 8k(1-t) + 6k^2 - 2tk^2 - k^3 \\ &> 2(1-t)^2 - 8k(1-t) + 2k^2(1-t) - k^3 > 0. \end{aligned}$$

Thus,  $\frac{d\alpha_{t,k}^*}{dt} > 0$ , whenever  $\lambda = -4 + k, k < 0$ . Upon using the Proposition 2.2, we conclude that for  $0 < m < n - 1$ , the portion of the graphs of  $\lambda_m^-(\alpha, 1)$  lying below the line  $\lambda = -4$  does not intersect each other. Moreover, let  $k = 0, t = \cos(\frac{2}{n}[\frac{n}{2}]\pi)$  in  $\alpha_{t,k}^*$ , we have

$$\alpha_1^* := \frac{2}{1 - \cos(\frac{2}{n}[\frac{n}{2}]\pi)}.$$

Thus,  $\lambda_N(\alpha, 1)$  is given as asserted.

The least assertion of the theorem is obvious. We have just completed the proof of the theorem.  $\square$

Figures 1(a)–1(d) show the calculated eigenvalues  $\lambda_2(\alpha, 1)$  and  $\lambda_N(\alpha, 1)$  of the coupling matrix  $C(\alpha, 1)$  as a function of wavelet parameter  $\alpha$ . The critical wavelet parameter  $\alpha_1$  and  $\alpha_1^*$  are also marked to distinguish the indices  $m$  of the eigencurves  $\lambda_m^+(\alpha, 1)$  and  $\lambda_m^-(\alpha, 1)$  in Theorems 2.1 and 2.2.

We are now in a position to state one of our main results, which are to identify the “optimal” wavelet constant. To this end, we begin with letting  $f(\alpha) := \frac{\lambda_N(\alpha, 1)}{\lambda_2(\alpha, 1)}$  and



$g(\alpha) := \frac{\lambda_N(\alpha, 1) - \lambda_2(\alpha, 1)}{\lambda_N(\alpha, 1) + \lambda_2(\alpha, 1)}$  for the wavelet parameter  $\alpha$  and defining  $m_i$  to be  $\tan^2 \frac{\pi}{in}$  for each positive integer  $i$ .

**Theorem 2.3** Let  $N = 2n$  be a multiple of four. The best choice of the wavelet constant  $\alpha$  for local synchronization of (1.1) is

- (1)  $\alpha = \alpha_{\min} = 1$  for  $n = 2$ .  
 (2)  $\alpha = \alpha_{\min} = \frac{-\tan^4 \frac{\pi}{n} + 4 \tan^2 \frac{\pi}{n} + 4}{2 \tan^2 \frac{\pi}{n} (\tan^2 \frac{\pi}{n} + 2)}$  for  $n \geq 4$ .

Moreover,  $\alpha_{\min} \in [\alpha_1^*, \alpha_1] = [1, \frac{1}{\sin^2 \frac{\pi}{n}}]$ . Such wavelet constant  $\alpha_{\min}$  gives the fastest convergence rate of initial values toward the synchronous manifold. Furthermore, its corresponding synchronization interval  $N_{N,f}(\alpha_{\min})$  can be explicitly obtained as follows:

$$N_{N,f}(\alpha_{\min}) = \left( \frac{1 - e^{-L_{\max}}}{\lambda_2(\alpha_{\min})}, \frac{1 + e^{-L_{\max}}}{\lambda_N(\alpha_{\min})} \right).$$

*Proof:* To prove the theorem, it is sufficient to show that the minimum of  $g(\alpha)$  occurs at  $\alpha_{\min}$ . By the Proposition 2.1, it is equivalent to showing that the minimum of  $f(\alpha)$  occurs at  $\alpha_{\min}$ .

- (1) From Theorems 1.3, and 2.2, we have the second and the least eigencurves for  $n = 2$  are

$$\lambda_2(\alpha, 1) = -2 \text{ for } \alpha \geq 0.$$

$$\lambda_N(\alpha, 1) = \begin{cases} \lambda_0^-(\alpha, 1) = -4, & 0 \leq \alpha \leq \alpha_1^*; \\ \lambda_{\frac{n}{2}}^-(\alpha, 1) = -2\alpha - 2, & \alpha \in [\alpha_1^*, \infty). \end{cases}$$

Obviously, the minimum of  $f(\alpha)$  occurs at  $\alpha_{\min} = \alpha_1^* = 1$ .

- (2) Clearly, the minimums of  $f(\alpha)$  on  $[0, \alpha_1^*]$  and  $[\alpha_1, \infty)$  occur at  $\alpha = \alpha_1^*$  and  $\alpha = \alpha_1$ , respectively. To complete the proof of the theorem, we need to calculate the minimum of  $f(\alpha)$  on  $[\alpha_1^*, \alpha_1]$ . Using the notation  $m_i$ , we rewrite  $f(\alpha)$  as follows:

$$f(\alpha) = \frac{2(m_1 + 1)(\alpha + 1)}{\alpha m_1 + 2m_1 + 2 - \sqrt{(\alpha m_1 - 2)^2 + 4m_1}}.$$

To find the critical point of  $f$  is equivalent to solving the following equation:

$$(\alpha m_1 + 2m_1 + 2)\sqrt{(\alpha m_1 - 2)^2 + 4m_1} - (\alpha m_1 - 2)^2 - 4m_1 - (\alpha + 1)m_1 \left[ \sqrt{(\alpha m_1 - 2)^2 + 4m_1} - (\alpha m_1 - 2) \right] = 0.$$

Some calculations would yield that

$$(m_1 + 2)^2 [(\alpha m_1 - 2)^2 + 4m_1] = [4m_1 - (\alpha m_1 - 2)(m_1 + 2)]^2 \\ 2(\alpha m_1 - 2)(m_1 + 2) = -m_1^2 - 4.$$

Hence, the critical point of  $f$  is

$$\alpha = \frac{-m_1^2 + 4m_1 + 4}{2m_1(m_1 + 2)} =: \alpha_{\min}.$$

It is easy to check that  $\alpha_{\min} \in (\alpha_1^*, \alpha_1)$  is indeed the minimizer of  $f(\alpha)$  on  $(0, \infty)$ .  $\square$

The ratio curves for the second and least eigencurve with  $N = 8$  are shown in Fig. 2. Here, we note that the theoretically predicted synchronization intervals which are in agreement with numerically produced synchronization intervals, see Figs. 3(a)–3(b) for  $N = 8$ .

Next, we consider the case that the number of the oscillators is not a multiple of four.

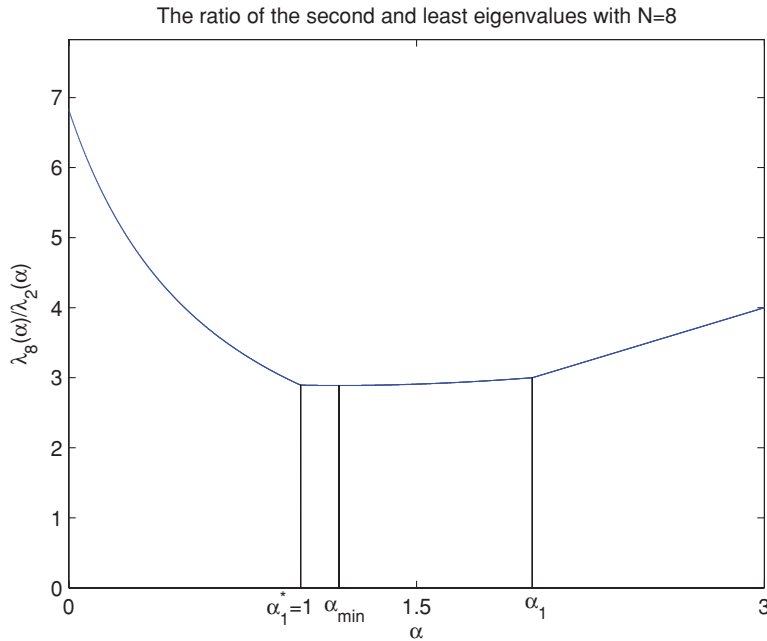


FIG. 2. (Color online) The ratio curves for the second and least eigencurve with  $N = 8$ .

**Theorem 2.4:** Let  $N = 2n$  be not a multiple of four. The best choice of the wavelet constant  $\alpha_{\min}$  for local synchronization of (1.1) lies in the interval  $\left[\frac{2}{1 - \cos(\frac{2}{n}[\frac{n}{2}]\pi)}, \tilde{\alpha}_c\right] = [\alpha_1^*, \tilde{\alpha}_c]$ , where  $\tilde{\alpha}_c$  is defined in Theorem 2.1. Moreover, its corresponding synchronization interval  $N_{N,f}(\alpha_{\min})$  can be explicitly obtained as follows:

$$N_{N,f}(\alpha_{\min}) = \left( \frac{1 - e^{-L_{\max}}}{\lambda_2(\alpha_{\min})}, \frac{1 + e^{-L_{\max}}}{\lambda_N(\alpha_{\min})} \right).$$

*Proof:* From Theorems 2.1, and 2.2, the minimum of  $f(\alpha)$  on  $[0, \alpha_1^*]$  obviously occurs at  $\alpha = \alpha_1^*$ . Next, we find the minimum of the function  $\ln g(\alpha)$  on  $[\tilde{\alpha}_c, \infty)$ . Some direct calculations yield that

$$\frac{d \ln g(\alpha)}{d\alpha} = \frac{g'(\alpha)}{g(\alpha)} = \frac{(1 + 2m_2)\alpha - 4m_2}{((\alpha - 2m_2)^2 + 4m_2)(\alpha + 2 + 2m_2)}.$$

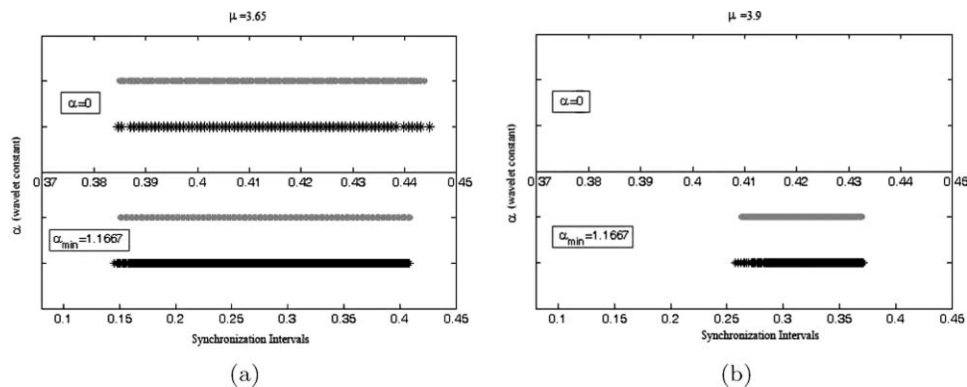


FIG. 3. Two typical synchronization intervals for coupled logistic map with  $\mu = 3.65$  and  $3.9$  are shown. Solid (bold) lines are synchronization intervals obtained by computer simulation. Gray lines are synchronization intervals predicted by our theorems. All are scaled for clear visualization.

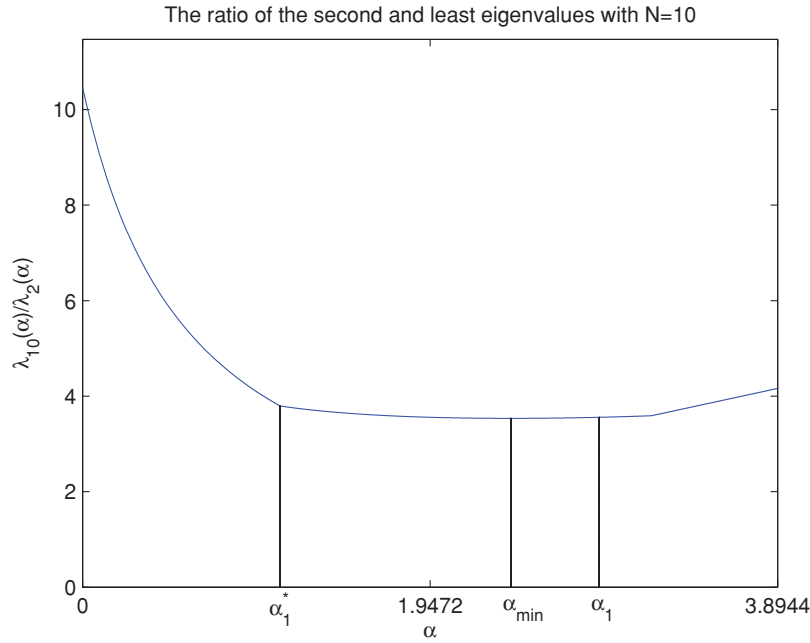


FIG. 4. (Color online) The ratio curves for the second and least eigencurve with  $N = 10$ .

Consequently, the critical point  $\bar{\alpha} (= \frac{4m_2}{1+2m_2})$  satisfies the following inequalities:

$$\bar{\alpha} \leq 1 \leq \frac{(1+m_2)^2}{4m_2} = \frac{1}{\sin^2 \frac{\pi}{n}} = \alpha_1 \leq \alpha_1.$$

Hence, we have  $\frac{d \ln g(\alpha)}{d\alpha} \geq 0$ , for  $\alpha \geq \tilde{\alpha}_c$ . And, the minimum of the function  $g(\alpha)$  on  $\alpha \geq \tilde{\alpha}_c$  occurs at  $\alpha = \tilde{\alpha}_c$ . Thus, the assertion of the theorem now follows.  $\square$

*Remark 2.1:* Since the behavior of  $g(\alpha)$  on  $[\alpha_1, \tilde{\alpha}_c]$  is complicated, we are unable to identify the minimum point of  $g$  on  $[\alpha_1, \tilde{\alpha}_c]$ . In Fig. 5, we pick  $\alpha = \alpha_1$  as our choice of the wavelet constant. And, its corresponding synchronization interval is  $N_{N,f}(\alpha_1)$ .

The ratio curves for the second and least eigencurve with  $N = 10$  are shown in Fig. 4. Note that the theoretically predicted synchronization intervals are in agreement with numerically produced synchronization intervals, see Figs. 5(a)–5(b) for  $N = 10$ .

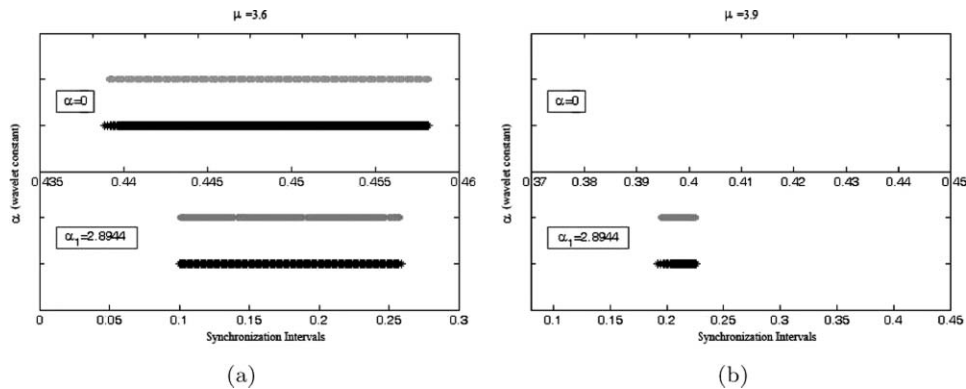


FIG. 5. Two typical synchronization intervals for coupled logistic map with  $\mu = 3.6$  and  $3.9$  are shown. Solid (bold) lines are synchronization intervals obtained by computer simulation. Gray lines are synchronization intervals predicted by our theorems. All are scaled for clear visualization.

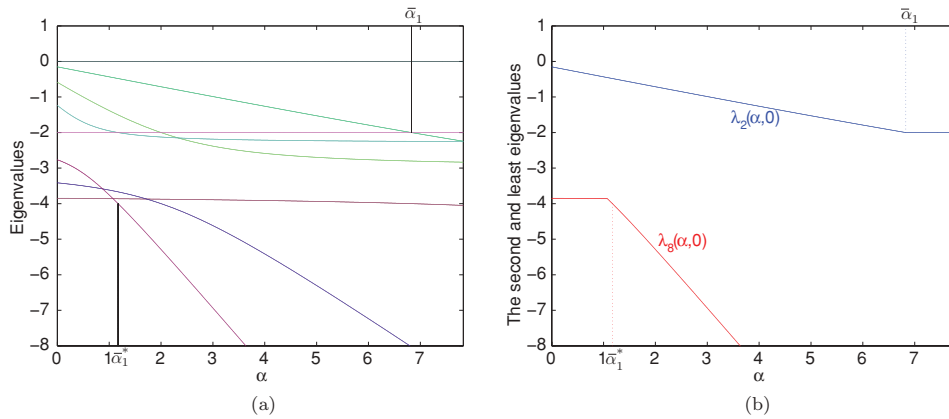


FIG. 6. (Color online) All eigenvalues of  $C(\alpha, 0)$  for  $N = 8$ . The second and least eigenvalues for  $N = 8$ .

### III. MAIN RESULTS FOR $C(\alpha, 0)$

In this section, we shall verify the effectiveness of the wavelet transform method applying on (1.1) for  $A$  being the nearest neighbor coupling with Neumann boundary conditions. To this end, in addition to study the least eigencurve  $\lambda_N(\alpha, \beta)$  of  $C(\alpha, 0)$ , we need to further explore some additional properties of its second eigencurve  $\lambda_2(\alpha, \beta)$ , which were not studied in Ref. 10. We begin with noting  $t = \cos(\frac{m\pi}{n})$ , then we have

$$\begin{aligned} \lambda_m^\pm(\alpha, 0) &= \frac{1}{2} \left\{ \alpha(t-1) - 4 \pm [(t-1)^2\alpha^2 + 4(t^2-1)\alpha + 8(1+t)]^{1/2} \right\} \\ &=: \lambda_t^\pm(\alpha, 0). \end{aligned}$$

It should be noted that  $\lambda_i^+(\alpha, \beta)$  (respectively,  $\lambda_i^-(\alpha, \beta)$ ) may intersect with  $\lambda_j^+(\alpha, \beta)$  (respectively,  $\lambda_j^-(\alpha, \beta)$ ) for  $i \neq j$ . Hence,  $\lambda_i(\alpha, \beta)$  may consist of pieces from various  $\lambda_m^+(\alpha, \beta)$  or  $\lambda_m^-(\alpha, \beta)$ ,  $m = 0, 1, 2, \dots, n-1$ . See Theorem 1.3, for example. This disorder structure of  $\lambda_m^\pm(\alpha, \beta)$ ,  $m = 0, 1, 2, \dots, n-1$ , makes the analytical identification of  $\lambda_i(\alpha, \beta)$  a nontrivial matter. We study some properties of the least eigencurve  $\lambda_N(\alpha, 0)$  of  $C(\alpha, 0)$ . Some direct calculations would yield the following proposition.

*Proposition 3.1:* (i) In the  $\alpha - \lambda$  plane,  $\lambda_t^-(\alpha, 0)$  intersect with  $\lambda = -4 + k$  at  $\alpha_{t,k}^*$ , where

$$\alpha_{t,k}^* = \frac{2 - 2t + k^2 - 4k}{(1-t)(1-t-k)} > 0 \text{ for } k \leq 0.$$

(ii) For  $-1 \leq t < 1$ , ( $m = 1, 2, \dots, n-1$ ).

$$\lim_{\alpha \rightarrow \infty} \lambda_t^-(\alpha, 0) = \lim_{\alpha \rightarrow \infty} \lambda_m^-(\alpha, 0) = -\infty.$$

To better understand the interwining properties of the second largest and the least eigenvalues of  $C(\alpha, 0)$ , we compute numerically all eigenvalues of  $C(\alpha, 0)$  for  $0 \leq \alpha \leq 8$  with  $N = 8$ , and identify their corresponding  $\lambda_2(\lambda, 0)$  and  $\lambda_8(\alpha, 0)$ . Such computation results are illustrated in Fig. 6.

**Theorem 3.1:** Let  $N = 2n$  be any positive even integer. The dimension of each block matrix in  $C(\alpha, 0)$  is  $2 \times 2$ . Let  $\lambda_N(\alpha, 0)$  be the least eigenvalue of  $C(\lambda, 0)$  then  $\lambda_N(\alpha, 0) = \lambda_{n-1}^-(\alpha, 0)$  for  $\alpha \geq \frac{1}{\cos^2(\frac{\pi}{2n})} =: \bar{\alpha}_1^*$ .

*Proof:* For  $k \leq 0$ , we consider the following inequality:

$$\begin{aligned} (1-t)^2(1-t-k)^2 \frac{d\alpha_{t,k}^*}{dt} &= 2(1-t)^2 - 8k(1-t) + 6k^2 - 2tk^2 - k^3 \\ &> 2(1-t)^2 - 8k(1-t) + 2k^2(1-t) - k^3 > 0. \end{aligned}$$

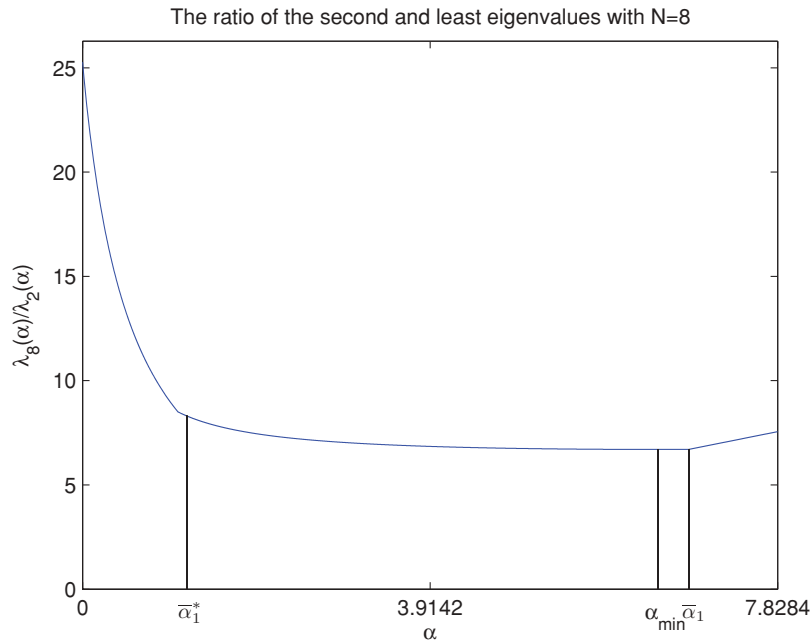


FIG. 7. (Color online) The ratio curves for the second and least eigencurve with  $N = 8$ .

Thus,  $\frac{d\alpha_{t,k}^*}{dt} > 0$ , whenever  $\lambda = -4 + k$ ,  $k \leq 0$ . Upon using Proposition 3.1, we conclude that for  $1 < m \leq n - 1$ , the portion of the graphs of  $\lambda_m^-(\alpha, 0)$  lying below the line  $\lambda = -4$  do not intersect each other. Moreover, let  $k = 0$ ,  $t = \cos(\frac{(n-1)\pi}{n})$  in  $\alpha_{t,k}^*$ , we have

$$\bar{\alpha}_1^* := \frac{1}{\cos^2(\frac{\pi}{2n})}$$

Thus,  $\lambda_N(\alpha, 0)$  is given as asserted.

We are now in a position to state one of our main results, which are to identify the “optimal” wavelet constant. To this end, we begin with letting  $f(\alpha) := \frac{\lambda_N(\alpha, 0)}{\lambda_2(\alpha, 0)}$  and  $g(\alpha) := \frac{\lambda_N(\alpha, 0) - \lambda_2(\alpha, 0)}{\lambda_N(\alpha, 0) + \lambda_2(\alpha, 0)}$  for the wavelet parameter  $\alpha$ .

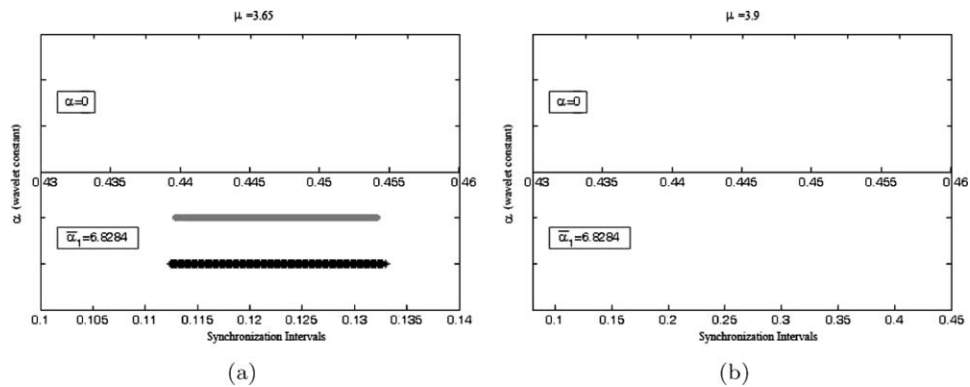


FIG. 8. Two typical synchronization intervals for coupled logistic map with  $\mu = 3.65$  and  $3.9$  are shown. Solid (bold) lines are synchronization intervals obtained by computer simulation. Gray lines are synchronization intervals predicted by our theorems. All are scaled for clear visualization.

**Theorem 3.2:** Let  $N = 2n$  be any even number and  $N \geq 4$  where  $N \times N$ , ( $N = 2n, n \in \mathbb{N}$ ) is the dimension of the matrix  $C(\alpha, 0)$ . The best choice of coupling strength for local synchronization of (1.1) via wavelet transform method exists at  $\alpha_{\min} \in [0, \frac{1}{\sin^2(\frac{\pi}{2n})}] = [0, \bar{\alpha}_1]$ . In the sense that such  $\alpha_{\min}$  make a coupling strength to give the fastest convergence rate of initial values toward the synchronous manifold. Moreover, its corresponding synchronization interval  $N_{N,f}(\alpha_{\min})$  can be explicitly obtained as follows:

$$N_{N,f}(\alpha_{\min}) = \left( \frac{1 - e^{-L_{\max}}}{\lambda_2(\alpha_{\min})}, \frac{1 + e^{-L_{\max}}}{\lambda_N(\alpha_{\min})} \right).$$

*Proof:* To prove the theorem, it is sufficient to show that there exists  $\alpha_{\min} \in [0, \frac{1}{\sin^2(\frac{\pi}{2n})}]$  such that  $g(\alpha)$  attains its minimum. By Proposition 2.1, we may show that  $\alpha = \alpha_{\min}$  attains the minimum of  $f(\alpha)$ . From Theorems 1.5 and 3.1, we have the second and the least eigencurves are

$$\lambda_2(\alpha, 0) = \begin{cases} \lambda_1^+(\alpha, 0), & 0 \leq \alpha \leq \bar{\alpha}_1 := \frac{1}{\sin^2(\frac{\pi}{2n})}; \\ \lambda_n^+(\alpha, 0) = -2, & \alpha \in [\bar{\alpha}_1, \infty). \end{cases}$$

$$\lambda_N(\alpha, 0) = \lambda_1^-(\alpha, 0) \text{ for } \alpha \geq \bar{\alpha}_1.$$

Obviously, the minimum of  $f(\alpha)$  on  $[\bar{\alpha}_1, \infty)$  occurs at  $\alpha = \bar{\alpha}_1$ . Thus, the assertion of the theorem now follows.

*Remark 3.1:* Since the behavior of  $g(\alpha)$  on  $[0, \bar{\alpha}_1]$  is complicated, we are unable to identify the minimum point of  $g$  on  $[0, \bar{\alpha}_1]$ . In Fig. 6, we pick  $\alpha = \bar{\alpha}_1$  as our choice of the wavelet constant. And, its corresponding synchronization interval is  $N_{N,f}(\bar{\alpha}_1)$ .

The ratio curves for the second and least eigencurve with  $N = 8$  are shown in Fig. 7. Finally, it should be noted that the theoretically predicted synchronization intervals are in agreement with numerically produced synchronization intervals, see Figs. 8(a)–8(b) for  $N = 8$ .

#### IV. CONCLUSIONS

By adjusting the wavelet constant of the wavelet transform method, its was reported in Ref. 12 that the method can greatly increase the applicable ranges of coupling strengths, the parameters of the individual chaotic map, and the number of nodes for local synchronization of CMLs. The theory of wavelet transform method on chaotic synchronization of coupled map lattices was analytically studied in this paper. In particular, we are able to find explicitly the optimal or near optimal wavelet constant. Consequently, rigorous proof for the work done in Ref. 12 is established in this paper. We conclude our paper for suggesting some possible future work. It is of interest to develop new techniques to find the explicit eigenvalue formula for  $C(\alpha, \beta)$ , where  $\beta \neq 0$  or 1. Note that our techniques given in Ref. 11 fail to carry over to Robin boundary conditions. It is also very worthwhile to pursue the cases when the coupling matrix  $A$  of the CMLs or coupled chaotic systems is nonsymmetric.

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