Journal of Hydrology 395 (2010) 163–168

Contents lists available at ScienceDirect

Journal of Hydrology

journal homepage: www.elsevier.com/locate/jhydrol

Nonstationary stochastic analysis of flow in a heterogeneous unconfined aquifer subject to spatially-random periodic recharge

Ching-Min Chang, Hund-Der Yeh *

Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan

ARTICLE INFO

Article history:
Received 7 January 2010
Received in revised form 29 June 2010
Accepted 6 October 2010

This manuscript was handled by L. Charlet, Editor-in-Chief, with the assistance of Chong-yu Xu, Associate Editor

Keywords:
Stochastic analysis
Nonstationary spectral approach
Heterogeneous unconfined aquifers
Periodic random recharge

SUMMARY

The problem of two-dimensional groundwater flow in a heterogeneous unconfined aquifer under periodic random forcing in time is analyzed from a stochastic point of view. It is assumed that the periodic random forcing, namely random recharge, can be represented by a sinusoidal function in time. Analytical solutions are developed through the nonstationary spectral approach in conjunction with the principle of superposition. The results, namely the variances of hydraulic head and specific discharge, are expressed in terms of statistical properties of hydraulic parameters and recharge field. It can be concluded from the analytical results that the mean recharge rate and integral scale of spatially random recharge fluctuations play essential roles in enhancing the variability of hydraulic head and specific discharges.

1. Introduction

The change of water table in response to periodic forcing (e.g., seasonal recharge and tides) commonly occurs in many groundwater basins. The prediction of the aquifer response to periodic forcing is an intrinsic part of the procedure for determining a proposed management policy. Under realistic field conditions, there always exist uncertainties associated with heterogeneities in permeability and parameters that affect the flow process in nature and make the prediction uncertain. Not accounting for the heterogeneity effects (uncertainties) in the modeling process may result in significant errors in the prediction. Hence, there is a need to quantify the reliability (uncertainty) of the flow model.

One possible approach to quantify uncertainty due to the spatial heterogeneity of subsurface systems is to treat the natural heterogeneity in a stochastic sense. Within the stochastic framework, we seek the solution of the groundwater flow problem expressed in terms of the variance of spatial random hydraulic head fields (the reliability or the model error) to be anticipated in applying the flow model. The variance of spatial random fields may be also viewed as the characterization of large-scale spatial variability associated with predictions when the model is subject to spatial heterogeneity.

All the modeling efforts devoted to the issue of aquifer response to periodic forcing (e.g., Townley, 1995; Trefry, 1999; Smith, 2008) have been based on the deterministic approach. Heterogeneity has been shown to play an important role in the analysis of the behavior of groundwater flow (Dagan, 1989; Gelhar, 1993; Zhang, 2002; Rubin, 2003). The applications of stationary spectral transfer function to the analyses of temporal and spatial variations of groundwater quantity and quality subject to a time-varying source have been presented by Gelhar (1974) and Duffy and Gelhar (1985). On the other hand, Zhang and Li (2005) have adopted the nonstationary spectral method (Li and McLaughlin, 1991, 1995) to numerically study temporal scaling of the hydraulic head fluctuations due to natural groundwater recharge and discharge. However, no attempt has been made so far, to the best of our knowledge, to analytically investigate the effect of random heterogeneity on the reliability of the flow model prediction (or on the spatial variability of large-scale flow model) subject to spatially-random periodic recharge, which is the task undertaken here.

It is well known that the introduction of a forcing term, for example groundwater recharge, leads to nonuniformity in the mean gradient of hydraulic head, and results in nonstationary in the statistics of random hydraulic head and velocity fields (e.g., Hantush and Marino, 1994; Rubin and Bellin, 1994; Li and Graham, 1998, 1999; Destouni et al., 2001; Chang and Yeh, 2008; Trefry et al., 2010). As such, the aim of the present work is to apply the stochastic method to the analysis of the large-scale behavior of flow in a two-dimensional heterogeneous unconfined aquifer under spatially-random periodic recharge. In this study the nonstationary spectral approach (Li and McLaughlin, 1991, 1995), based on an unknown transfer function involving in Fourier–Stieltjes

* Corresponding author. Tel.: +886 3 5731910; fax: +886 3 5726050.
E-mail address: hdyeh@mail.nctu.edu.tw (H.-D. Yeh).

0022-1694/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jhydrol.2010.10.016

representations for the perturbed quantities, in conjunction with principle of superposition is used to develop closed-form expressions in characterizing variability of hydraulic head and specific discharges. The closed-form expressions, to the best of our knowledge, have never before been presented. The results will provide some basic understanding of the influence of heterogeneity in spatially random recharge on the large-scale flow process in a heterogeneous unconfined aquifer and a basis for judging reliability of large-scale flow models.

2. Statement of the problem

Under the Dupuit assumption of horizontal flow, the governing equation for water table fluctuations in an unconfined aquifer can be described by (e.g., Bear, 1979)

$$\frac{\partial}{\partial X_i}\left[K(\mathbf{X})\phi\frac{\partial\phi}{\partial X_i}\right]+R(\mathbf{X},t)=S_y\frac{\partial\phi}{\partial t}\quad i=1,2 \tag{1}$$

where ϕ is the hydraulic head, K is the hydraulic conductivity, S_y is the specific yield and R is a distributed recharge term. The variability of S_y is assumed negligible. To simplify the analysis we linearize (1) as

$$\frac{\partial}{\partial X_i}\left[T(\mathbf{X})\phi\frac{\partial\phi}{\partial X_i}\right]+R(\mathbf{X},t)=S_y\frac{\partial\phi}{\partial t} \tag{2}$$

where T denotes the transmissivity of the unconfined aquifer. Bear (1979, p. 115) noticed that the approximation involved in the linearization is justified in view of the relatively small changes in ϕ with respect to the total thickness of ϕ in most phreatic aquifers. In terms of $\ln T$, (2) can be rewritten as

$$\frac{\partial^2\phi}{\partial X_i^2}+\frac{\partial\ln T}{\partial X_i}\frac{\partial\phi}{\partial X_i}+\frac{R}{T}=\frac{S_y}{T}\frac{\partial\phi}{\partial t} \tag{3}$$

In the analysis that follows, the ϕ, $\ln T$ and R random fields in (3) are considered to be space functions. In addition, the natural logarithm of transmissivity ($\ln T$) is modeled as a second-order stationary random field with the known covariance function (or spectral density function). The log transmissivity and recharge fields are uncorrelated.

We express the ϕ, $\ln T$ and R random fields in (3) in terms of an ensemble mean and a small perturbation around the mean,

$$\phi(\mathbf{X},t)=<\phi>(\mathbf{X})+h(\mathbf{X},t)=H(\mathbf{X})+h(\mathbf{X},t) \tag{4a}$$

$$\ln T(\mathbf{X})=<\ln \mathbf{T}>+y(\mathbf{X})=Y+y(\mathbf{X}) \tag{4b}$$

$$R(\mathbf{X},t)=<R>+r(\mathbf{X},t)=\Re+r(\mathbf{X},t) \tag{4c}$$

where < > stands for the expected value operator. By substituting (4a)–(4c) into (3), taking the expected value and disregarding all products of perturbations, one obtains the first-order deterministic mean flow equation

$$\frac{\partial^2 H}{\partial X_i^2}+\frac{\Re}{e^Y}=\frac{S_y}{e^Y}\frac{\partial H}{\partial t} \tag{5}$$

Subtracting this mean equation from (3) and disregarding all products of perturbations, leads to a first-order equation describing the hydraulic head fluctuations

$$\frac{\partial^2 h}{\partial X_i^2}+\frac{\partial y}{\partial X_i}\frac{\partial H}{\partial X_i}-\frac{\Re}{e^Y}y+\frac{r}{e^Y}=\frac{S_y}{e^Y}\left(\frac{\partial h}{\partial t}-y\frac{\partial H}{\partial t}\right) \tag{6}$$

The analyses in this study are limited to small perturbations in hydraulic properties, assuming that the variance of log hydraulic conductivity is smaller than unit (weak heterogeneity) so that second-order terms (products of the perturbations) in the flow equation are negligible. However, Zhang and Winter (1999) found it to be accurate for the head variance solutions for the value of variance of log hydraulic conductivity as high as 4.38. A similar finding was reported in Gelhar (1993).

By regarding the $\partial H/\partial t$ term in (5) and (6) slowly varying in time and, for convenience, rotating the coordinate system so that X_1 is aligned with the mean flow, these simplify (5) and (6) considerably to

$$\frac{\partial^2 H}{\partial X_1^2}+\frac{\Re}{e^Y}=0 \tag{7}$$

$$\frac{\partial^2 h}{\partial X_i^2}-J\frac{\partial y}{\partial X_1}-\frac{\Re}{e^Y}y+\frac{r}{e^Y}=\frac{S_y}{e^Y}\frac{\partial h}{\partial t} \tag{8}$$

where $J=-\partial H/\partial X_1$. Eq. (8) is a random partial differential equation that relates output fluctuations in h to variations in inputs y and r. Upon solution of (8), one can develop expressions characterizing the variability of head and specific discharge fluctuations.

In attempting to arrive at the solution of (8), one must know the spatial behavior of the mean gradient $J(X_1)$, which is determined from solving (7) with the boundary conditions. It is straightforward to verify that the solution to (7) is

$$J(X_1)=-\frac{\partial H}{\partial X_1}=\frac{\Re}{e^Y}(X_1-X_0)+J_0 \tag{9}$$

where J_0 is the known value of reference mean head gradient at the arbitrary location $X_1=X_0$. Note that the solution of (7), (9) was given by Rubin and Bellin (1994, equation 5).

3. Solution for the head fluctuations

The approach followed is to solve the perturbation Eq. (8) to fully characterize the second moment of h fluctuations. This task will be performed using the nonstationary spectral approach (Li and McLaughlin, 1991, 1995) in conjunction with principle of superposition (e.g., Townley, 1995; Trefry, 1999; Smith, 2008).

The usefulness of the principle of superposition in the solution of a linear equation has long been recognized in groundwater hydrology. Based on this principle, a complex equation can be divided into sub-equations and the solution to the original equation is then obtained by summing the individual solution to each of the sub-equations. Following the approach of Townley (1995) we replace $h(\mathbf{X},t)$ in (8) by

$$h(\mathbf{X},t)=h_s(\mathbf{X})+h_\tau(\mathbf{X},t) \tag{10}$$

Separation of the steady-state and time-varying components yields following two differential equations:

$$\frac{\partial^2 h_s}{\partial X_i^2}-J\frac{\partial y}{\partial X_1}-\frac{\Re}{e^Y}y=0 \tag{11}$$

$$\frac{\partial^2 h_\tau}{\partial X_i^2}+\frac{r}{e^Y}=\frac{S_y}{e^Y}\frac{\partial h_\tau}{\partial t} \tag{12}$$

Based on the assumption of uniform mean flow, the Fourier–Stieltjes integral representations of statistically homogeneous processes is one of the most successful approaches in solving the small-perturbation expansion of the flow equation (e.g., Bakr et al., 1978; Mizell et al., 1982; Gelhar and Axness, 1983). Notice from (9) that the introduction of the effect of recharge results in nonuniform mean flow. This excludes the direct applicability of the stationary spectral method to solve (11). However, the solution of (11) can be obtained using the nonstationary spectral representation (Li and McLaughlin, 1991, 1995), which is expressed in terms of an unknown transfer function determined from the solution of a linearized flow equation.

By using this representation, the random fluctuation h_s in (11) is expressed in terms of two-dimensional wave number integral as

$$h_s(\boldsymbol{X}) = \int_{-\infty}^{\infty} \Phi_{hy}(\boldsymbol{X}, \boldsymbol{K}) dZ_y(\boldsymbol{K}) \tag{13}$$

where $\Phi_{hy}(\boldsymbol{X}, \boldsymbol{K})$ is an unknown transfer function, $dZ_y(\boldsymbol{K})$ is the complex Fourier amplitude of $\ln T$, $\boldsymbol{K} = (K_1, K_2)$ is the wave number vector and $K^2 = K_1^2 + K_2^2$. In addition, the stationarity of the $\ln T$ allows the following Fourier–Stieltjes representation

$$y(\boldsymbol{X}) = \int_{-\infty}^{\infty} \exp[i\boldsymbol{K} \cdot \boldsymbol{X}] dZ_y(\boldsymbol{K}) \tag{14}$$

Applying (13) and (14) into (11) we have the result

$$\frac{\partial^2 \Phi_{hy}}{\partial X_i^2} - iK_1 J(X_1) e^{i\boldsymbol{K}\cdot\boldsymbol{X}} - \frac{\Re}{e^Y} e^{i\boldsymbol{K}\cdot\boldsymbol{X}} = 0 \tag{15}$$

where $J(X_1)$ is defined in (9). The solution of (15) is then of the form (Chang and Yeh, 2008)

$$\Phi_{hy}(\boldsymbol{X}, \boldsymbol{K}) = -\frac{iK_1 K^2 J(X_1) - (K_1^2 - K_2^2)\Re/e^Y}{K^4} e^{i\boldsymbol{K}\cdot\boldsymbol{X}} \tag{16}$$

Inserting (16) into (13) yields the expression for the random fluctuations h_s

$$h_s(\boldsymbol{X}) = -\int_{-\infty}^{\infty} e^{i\boldsymbol{K}\cdot\boldsymbol{X}} \frac{iK_1 K^2 J - (K_1^2 - K_2^2)\Re/e^Y}{K^4} dZ_y(\boldsymbol{K}) \tag{17}$$

Assume that the recharge rate is in the form (e.g., Townley, 1995; Trefry, 1999; Smith, 2008)

$$r(\boldsymbol{X}, t) = r_s(\boldsymbol{X}) \cos(\omega t) \tag{18}$$

where r_s, a second-order stationary random space function, is the amplitude of the periodic fluctuations of recharge and ω is the angular frequency of fluctuations. The substitution of the following representations of $h_\tau(\boldsymbol{X}, t)$ and $r_s(\boldsymbol{X})$,

$$h_\tau(\boldsymbol{X}, t) = \int_{-\infty}^{\infty} e^{i\boldsymbol{K}\cdot\boldsymbol{X}} dZ_{h_\tau}(\boldsymbol{K}, t) \tag{19}$$

$$r_s(\boldsymbol{X}) = \int_{-\infty}^{\infty} e^{i\boldsymbol{K}\cdot\boldsymbol{X}} d_{Z_r}(\boldsymbol{K}) \tag{20}$$

respectively, into (12) and the use of uniqueness of the representations results in

$$\frac{d}{dt} dZ_{h_\tau}(\boldsymbol{K}, t) + \frac{e^Y K^2}{S_y} dZ_{h_\tau}(\boldsymbol{K}, t) = \frac{1}{S_y} \cos(\omega t) dZ_r(\boldsymbol{K}) \tag{21}$$

where dZ_{h_τ} and dZ_r are the complex Fourier amplitudes of h_τ and r_s processes, respectively.

The solution of (21) with $dZ_{h_\tau} = 0$ at $t = 0$ is

$$dZ_{h_\tau}(\boldsymbol{K}, t) = \frac{\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)}{(\beta^4 K^4 + \omega^2) S_y} dZ_r(\boldsymbol{K}) \tag{22}$$

where $\beta = (e^Y/S_y)^{1/2}$. With (22), the time-varying component of head fluctuation in (19) is of the form

$$h_\tau(\boldsymbol{X}, t) = \int_{-\infty}^{\infty} e^{iK\cdot X} \times \frac{\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)}{(\beta^4 K^4 + \omega^2) S_y} dZ_r(\boldsymbol{K}) \tag{23}$$

In the range of β that is likely to be interest, the transient exponential term in (23) vanishes away rapidly, becoming negligibly small.

4. Variance of head

Using (10), (17), and (23), and the representation theorem the general form of the head variance can be expressed as

$$\sigma_h^2(X_1, t) = \sigma_{h_s}^2(X_1) + \sigma_{h_\tau}^2(t) \tag{24}$$

where

$$\sigma_{h_s}^2(X_1) = \int_{-\infty}^{\infty} \frac{J^2 K^4 K_1^2 + (\Re/e^Y)^2 (K_1^2 - K_2^2)^2}{K^8} S_{yy}(\boldsymbol{K}) d\boldsymbol{K} \tag{25a}$$

$$\sigma_{h_\tau}^2(t) = \frac{1}{S_y^2} \int_{-\infty}^{\infty} \frac{[\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)]^2}{(\beta^4 K^4 + \omega^2)^2} S_{rr}(\boldsymbol{K}) d\boldsymbol{K} \tag{25b}$$

$S_{yy}(\boldsymbol{K})$ is the spectrum of $\ln T$ and $S_{rr}(\boldsymbol{K})$ is the spectrum of r fluctuations.

5. Variances of specific discharges

The first-order equation for the specific discharge perturbations derived from the Darcy equation is of the form (e.g., Gelhar, 1993; Rubin and Bellin, 1994)

$$q_i = e^Y \left[\delta_{i1} J(X_1) y - \frac{\partial h}{\partial X_i} \right] \tag{26}$$

where $q_i = Q_i - E[Q_i]$ and Q_i is the specific discharge. Using this expression, the spectral representations for the specific discharge perturbations, $\ln T$ perturbation field and head perturbation gradients,

$$\frac{\partial h}{\partial X_1} = \int_{-\infty}^{\infty} \frac{K^2 K_1 J - i2K_1 K_2^2 (\Re/e^Y)}{K^4} e^{iK\cdot X} dZ_y(\boldsymbol{K}) + \int_{-\infty}^{\infty} iK_1 \times \frac{\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)}{(\beta^4 K^4 + \omega^2) S_y} e^{iK\cdot X} dZ_r(\boldsymbol{K}) \tag{27}$$

$$\frac{\partial h}{\partial X_2} = \int_{-\infty}^{\infty} \frac{K^2 K_1 K_2 J + i(K_1^2 - K_2^2) K_2 (\Re/e^Y)}{K^4} e^{iK\cdot X} dZ_y(\boldsymbol{K}) + \int_{-\infty}^{\infty} iK_2 \times \frac{\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)}{(\beta^4 K^4 + \omega^2) S_y} e^{iK\cdot X} dZ_r(\boldsymbol{K}) \tag{28}$$

and invoking the uniqueness of the spectral representation gives the following Fourier amplitudes of the specific discharge fluctuations in the longitudinal and transverse directions, respectively,

$$dZ_{Q_1}(\boldsymbol{K}, t) = e^Y \left\{ \left[J\left(1 - \frac{K_1^2}{K^2}\right) + i2\left(\frac{\Re}{e^Y}\right)\frac{K_1 K_2^2}{K^4} \right] dZ_y(\boldsymbol{K}) - iK_1 \frac{\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)}{(\beta^4 K^4 + \omega^2) S_y} dZ_r(\boldsymbol{K}) \right\} \tag{29}$$

$$dZ_{Q_2}(\boldsymbol{K}, t) = -e^Y \left\{ \left[J\frac{K_1 K_2}{K^2} + i\left(\frac{\Re}{e^Y}\right)\frac{(K_1^2 - K_2^2) K_2}{K^4} \right] dZ_y(\boldsymbol{K}) + iK_2 \frac{\beta^2 K^2 \cos(\omega t) + \omega \sin(\omega t) - \beta^2 K^2 \exp(-\beta^2 K^2 t)}{(\beta^4 K^4 + \omega^2) S_y} dZ_r(\boldsymbol{K}) \right\} \tag{30}$$

By using the representation theorem with (29) and (30), the longitudinal and transverse specific discharge spectra become, respectively,

$$S_{Q_1Q_1}(\boldsymbol{K},t) = e^{2Y}\left\{\left[J^2\left(1-\frac{K_1^2}{K^2}\right)^2 + 4\left(\frac{\Re}{e^Y}\right)^2\frac{K_1^2K_2^4}{K^8}\right]S_{yy}(\boldsymbol{K}) + \frac{1}{S_y^2}\frac{K_1^2[\beta^2K^2\cos(\omega t)+\omega\sin(\omega t)-\beta^2K^2\exp(-\beta^2K^2t)]^2}{(\beta^4K^4+\omega^2)^2}S_{rr}(\boldsymbol{K})\right\} \tag{31}$$

$$S_{Q_2Q_2}(\boldsymbol{K},t) = e^{2Y}\left\{\left[J^2\frac{K_1^2K_2^2}{K^4} + \left(\frac{\Re}{e^Y}\right)^2\frac{(K_1^2-K_2^2)^2K_2^2}{K^8}\right]S_{yy}(\boldsymbol{K}) + \frac{1}{S_y^2}\frac{K_2^2[\beta^2K^2\cos(\omega t)+\omega\sin(\omega t)-\beta^2K^2\exp(-\beta^2K^2t)]^2}{(\beta^4K^4+\omega^2)^2}S_{rr}(\boldsymbol{K})\right\} \tag{32}$$

Then, the general form of the specific discharge variance can be found by integrating (31), (32) over the wave number domain as

$$\sigma_{Q_i}^2 = \int_{-\infty}^{\infty} S_{Q_iQ_i}(\boldsymbol{K},t)d\boldsymbol{K} \tag{33}$$

6. Closed-form solutions

To proceed with the evaluation of variances of head and specific discharge (Eqs. (24) and (33)) explicitly the forms of the log transmissivity and recharge spectra must be specified. For this analysis the Whittle-B spectrum (Mizell et al., 1982; Li and McLaughlin, 1995; Li and Graham, 1998, 1999) is considered representing the random log transmissivity or recharge fields, namely,

$$S_{yy}(\boldsymbol{K}) = \frac{3\sigma_y^2\alpha_y^2K^4}{\pi(K^2+\alpha_y^2)^4} \tag{34}$$

$$S_{rr}(\boldsymbol{K}) = \frac{3\sigma_r^2\alpha_r^2K^4}{\pi(K^2+\alpha_r^2)^4} \tag{35}$$

where $\alpha_y = 3\pi/(16\lambda_y)$, σ_y^2 and λ_y are the variance and integral scale of ln T, respectively, $\alpha_r = 3\pi/(16\lambda_r)$, σ_r^2 and λ_r are the variance and integral scale of the amplitude of recharge fluctuations, respectively.

In addition, to facilitate closed-form solutions we disregard the transient exponential term in (23) (i.e., $\exp(-\beta^2K^2t) \to 0$) so that

$$h_\tau(\boldsymbol{X},t) = \int_{-\infty}^{\infty} e^{i\boldsymbol{K}\cdot\boldsymbol{X}}\frac{\beta^2K^2\cos(\omega t)+\omega\sin(\omega t)}{(\beta^4K^4+\omega^2)S_y}dZ_r(\boldsymbol{K}) \tag{36}$$

6.1. Head variance

For the specified Whittle-B spectra for the y and r processes ((34) and (35)) the result of the integrations of (25a) and (25b) are, respectively,

$$\sigma_{h_s}^2(X_1) = \sigma_y^2\left[\frac{J^2(X_1)}{4\alpha_y^2} + \left(\frac{\Re}{e^Y}\right)^2\frac{1}{2\alpha_y^4}\right] = \sigma_y^2J_0^2\lambda_y^2\left[\left(\frac{16}{3\pi}\right)^2\frac{(\eta\xi_1+1)^2}{4} + \left(\frac{16}{3\pi}\right)^4\frac{\eta^2}{2}\right] \tag{37}$$

$$\begin{aligned}\sigma_{h_\tau}^2(t) = \frac{1}{4}\frac{\sigma_r^2}{\omega^2S_y^2}\frac{\rho^2}{(\rho^2+1)^5}\{&-(\rho^2+1)[-2+18\rho^2+18\rho^4-2\rho^6\\&+3\pi\rho(1-6\rho^2+\rho^4)-24\rho^2(\rho^2-1)\ln\rho]\\&+2\cos(2\omega t)[(\rho^2-1)(-1+45\rho^2+45\rho^4-\rho^6\\&+3\pi\rho(1-14\rho^2+\rho^4)-12\rho^2(3-10\rho^2+3\rho^4)\ln\rho]\\&+2\sin(2\omega t)[-14+54\rho^2+54\rho^4-14\rho^4\\&+6\pi\rho(3-10\rho^2+3\rho^4)+6(-1+15\rho^2-15\rho^4+\rho^6)\ln\rho]\}\end{aligned} \tag{38}$$

where $\eta = \Re\lambda_y/(eYJ_0)$, $\xi_1 = (X_1 - X_0)/\lambda_y$, $\rho = (16/(3\pi))^2\mu$ and $\mu = \omega S_y\lambda_r{}^2/e^Y$.

Fig. 1a shows how the steady-state component of head variance varies with recharge rate. As is predicted by (37), the head variance increases with the recharge rate. It implies that larger recharge leads to larger correlation between head fluctuations compared to the case of smaller recharge, which increases the head variability from the mean head surface. Fig. 1b illustrates the behavior of the dimensionless time-varying component of head variance in (38) as a function of dimensionless time. It can be clearly seen that the hydraulic head variability increases with λ_r for fixed values of e^Y and S_y at a specified time. The recharge profile with the smaller λ_r is rougher, while that with the larger λ_r is smaller. In other words, recharge fluctuations are either consistently above or below mean in the case of a larger λ_r, which produces more persistence of head fluctuations, and, therefore, leads to larger deviations of the head from the mean head surface.

6.2. Variance of specific discharge

The specific discharge variances in the longitudinal and transverse directions are obtained by substituting (31) and (32) into (33), respectively, and integrating them over the wave number domain with the specified spectra (34) and (35). The results are

$$\sigma_{Q_1}^2(X_1,t) = \sigma_{Q_{1s}}^2(X_1) + \sigma_{Q_{1\tau}}^2(X_1,t) \tag{39a}$$

$$\sigma_{Q_2}^2(X_1,t) = \sigma_{Q_{2s}}^2(X_1) + \sigma_{Q_{2\tau}}^2(X_1,t) \tag{39b}$$

where

$$\sigma_{Q_{1s}}^2(X_1) = \frac{1}{8}\sigma_y^2e^{2Y}\left[3J^2(X_1) + \left(\frac{\Re}{e^Y}\right)^2\frac{1}{\alpha_y^2}\right] = \frac{1}{8}\sigma_y^2J_0^2\lambda_y^2\left[3(\eta\xi_1+1)^2 + \left(\frac{16}{3\pi}\right)^2\eta^2\right] \tag{40}$$

$$\sigma_{Q_{2s}}^2(X_1) = \frac{1}{8}\sigma_y^2e^{2Y}\left[J^2(X_1) + \left(\frac{\Re}{e^Y}\right)^2\frac{1}{\alpha_y^2}\right] = \frac{1}{8}\sigma_y^2J_0^2\lambda_y^2\left[(\eta\xi_1+1)^2 + \left(\frac{16}{3\pi}\right)^2\eta^2\right] \tag{41}$$

$$\begin{aligned}\sigma_{Q_{1\tau}}^2 = \sigma_{Q_{2\tau}}^2 = \frac{1}{8}\frac{\sigma_r^2e^Y}{\omega S_y}\frac{\rho}{(\rho^2+1)^5}\{&(\rho^2+1)[1+15\rho^2+3\rho^4-11\rho^6\\&+12\pi\rho^3(\rho^2-1)+6(\rho^2-6\rho^4+\rho^6)\ln\rho]\\&+\cos(2\omega t)[1+44\rho^2-90\rho^4-116\rho^6+17\rho^8\\&+24\pi\rho^3(-2+5\rho^2-\rho^4)-6\rho^2(-3+35\rho^2-25\rho^4+\rho^6)\ln\rho]\\&+\rho\sin(2\omega t)[4-108\rho^2-36\rho^4+76\rho^6+3\pi\rho(-3+35\rho^2\\&-25\rho^4+\rho^6)-48\rho^2(2-5\rho^2+\rho^4)\ln\rho]\}\end{aligned} \tag{42}$$

It is clear from (40), (41) that for fixed input soil hydraulic parameters (namely, σ_y^2, e^Y and λ_y) the variability in the steady-state specific discharge is mainly controlled by the mean recharge rate and increases monotonically with it. Fig. 2 shows the behavior of the dimensionless time-varying component of longitudinal specific discharge variance in (42) as a function of dimensionless time. It indicates that the longitudinal specific variance increases with integral scale of recharge fluctuations for fixed values of e^Y and S_y at a specified time. This feature is a consequence of the increase in variability of hydraulic head with the persistence of recharge fluctuations (integral scale of recharge fluctuations). This enhanced variability of head increases the correlation function of the longitudinal specific discharge.

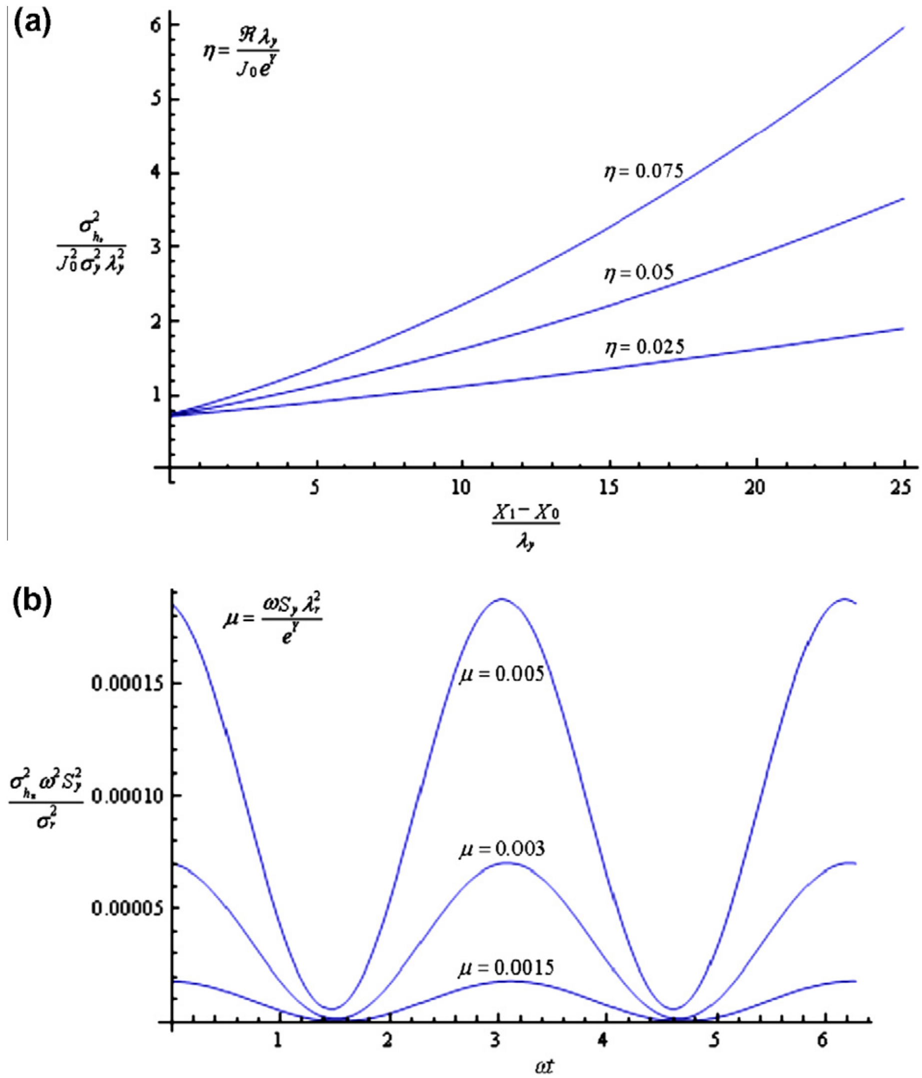

Fig. 1. (a) Dimensionless steady-state component of head variance in (37) as a function of dimensionless position for various η. (b) Dimensionless time-varying component of head variance in (38) as a function of dimensionless time for various μ.

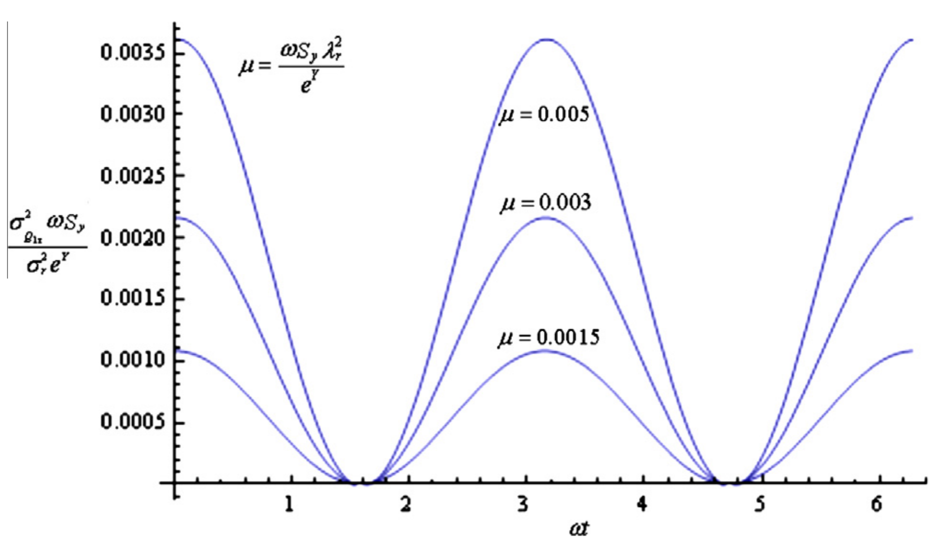

Fig. 2. Dimensionless time-varying component of longitudinal specific discharge variance in (42) as a function of dimensionless time for various μ.

7. Conclusions

This paper presents a stochastic analysis of the large-scale behavior of unsteady flow in a two-dimensional heterogeneous unconfined aquifer under spatially-random periodic recharge. Stochastic solutions of differential perturbation equation describing head fluctuations are developed through a nonstationary spectral approach in conjunction with the principle of superposition. The results, namely the variances of hydraulic head and specific discharge, which are used to characterize the variability of head and specific discharge, have been expressed in terms of statistical properties of hydraulic parameters and recharge field.

It was found that a larger mean recharge rate enhances the variation of the steady-state components of hydraulic head and the specific discharges about the means. Our results also indicate that the time-varying components of head and specific discharge variation increase with the persistence of recharge spatial distribution at a specified time. The findings presented here provide some basic understanding of the influence of heterogeneity in spatially random recharge on the large-scale flow process in a heterogeneous unconfined aquifer and a basis for judging reliability of large-scale flow models.

Acknowledgements

This work is particularly supported by "Aim for the Top University Plan" of the National Chiao Tung University and Ministry of Education, Taiwan. We are grateful to the Associate Editor and three anonymous reviewers for constructive comments that improved the quality of the work.

References

Bakr, A.A., Gelhar, L.W., Gutjahr, A.L., MacMillan, J.R., 1978. Stochastic analysis of spatial variability in subsurface flows: 1. Comparison of one- and three-dimensional flows. Water Resour. Res. 14 (2), 263–271.

Bear, J., 1979. Hydraulics of Groundwater. McGraw-Hill, New York, N.Y.

Chang, C.-M., Yeh, H.-D., 2008. Using the nonstationary spectral method to analyze asymptotic macrodispersion in uniformly recharged heterogeneous aquifers. J Hydrol. 350 (1–2), 93–99.

Dagan, G., 1989. Flow and Transport in Porous Formations. Springer-Verlag, New York.

Destouni, G., Simic, E., Graham, W.D., 2001. On the applicability of analytical methods for estimating solute travel time statistics in nonuniform groundwater flow. Water Resour. Res. 37 (9), 2303–2308.

Duffy, C.J., Gelhar, L.W., 1985. A frequency domain approach to water quality modeling in groundwater: theory. Water Resour. Res. 21 (8), 1175–1184.

Gelhar, L.W., 1974. Stochastic analysis of phreatic aquifers. Water Resour. Res. 10 (3), 539–545.

Gelhar, L.W., 1993. Stochastic Subsurface Hydrology. Prentice Hall, Englewood Cliffs, New Jersey.

Gelhar, L.W., Axness, C.L., 1983. Three-dimensional stochastic analysis of macrodispersion in aquifers. Water Resour. Res. 19 (1), 161–180.

Hantush, M.M., Marino, M.A., 1994. Two-dimensional stochastic analysis and optimal estimation inaquifers: Random leakage. Water Resour. Res. 30 (2), 559–569.

Li, L., Graham, W.D., 1998. Stochastic analysis of solute transport in heterogeneous aquifers subject to spatially random recharge. J Hydrol. 206 (1–2), 16–38.

Li, L., Graham, W.D., 1999. Stochastic analysis of solute transport in heterogeneous aquifers subject to spatiotemporal random recharge. Water Resour. Res. 35 (4), 953–971.

Li, S.-G., McLaughlin, D., 1991. A nonstationary spectral method for solving stochastic groundwater problems: Unconditional analysis. Water Resour. Res. 27 (7), 1589–1605.

Li, S.-G., McLaughlin, D., 1995. Using the nonstationary spectral method to analyze flow through heterogeneous trending media. Water Resour. Res. 31 (3), 541–551.

Mizell, S.A., Gutjahr, A.L., Gelhar, L.W., 1982. Stochastic analysis of spatial variability in two-dimensional steady groundwater flow assuming stationary and non-stationary heads. Water Resour. Res. 18 (4), 1053–1067.

Rubin, Y., 2003. Applied Stochastic Hydrogeology. Oxford University Press, New York.

Rubin, Y., Bellin, A., 1994. The effects of recharge on flow nonuniformity and macrodispersion. Water Resour. Res. 30 (4), 939–948.

Smith, A., 2008. Weakly nonlinear approximation of periodic flow in phreatic aquifers. Ground Water 46 (2), 228–238.

Townley, L.R., 1995. The response of aquifers to periodic forcing. Adv. Water Resour. 18 (3), 125–146.

Trefry, M.G., 1999. Periodic forcing in composite aquifers. Adv. Water Resour. 22 (6), 645–656.

Trefry, M.G., McLaughlin, D., Metcalfe, G., Lester, D., Ord, A., Regenauer-Lieb, K., Hobbs, B.E., 2010. On oscillating flows in randomly heterogeneous porous media. Phil. Trans. R. Soc. A 368, 197–216. doi:10.1098.

Zhang, D., 2002. Stochastic Methods for Flow in Porous Media: Coping with Uncertainties. Academic Press, San Diego, Calif.

Zhang, Y.-K., and Li, Z., 2005. Temporal scaling of hydraulic head fluctuations: nonstationary spectral analyses and numerical simulations, Water Resour. Res., 41(7), Article W07031, doi:10.1029/2004WR003797.

Zhang, D., Winter, C.L., 1999. Moment equation approach to single phase fluid flow in heterogeneous reservoirs. Soc. Petrol Eng. J. 4 (2), 118–127.