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Mutually independent bipanconnected property of hypercube ☆,☆☆

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ABSTRACT

A graph is denoted by G with the vertex set $V(G)$ and the edge set $E(G)$. A path $P = \langle v_0, v_1, \ldots, v_m \rangle$ is a sequence of adjacent vertices. Two paths with equal length $P_1 = \langle u_1, u_2, \ldots, u_m \rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_m \rangle$ from a to b are *independent* if $u_1 = v_1 = a$, $u_m = v_m = b$, and $u_i \neq v_i$ for $2 \leqslant i \leqslant m-1$. Paths with equal length $\{P_i\}_{i=1}^{n}$ from a to b are *mutually independent* if they are pairwisely independent. Let u and v be two distinct vertices of a bipartite graph G, and let l be a positive integer length, $d_G(u,v) \leqslant l \leqslant |V(G) - 1|$ with $(l - d_G(u,v))$ being even. We say that the pair of vertices u, v is (m,l)-*mutually independent bipanconnected* if there exist m mutually independent paths $\left\{P_i^l\right\}_{i=1}^{m}$ with length l from u to v. In this paper, we explore yet another strong property of the hypercubes. We prove that every pair of vertices u and v in the n-dimensional hypercube, with $d_{Q_n}(u,v) \geqslant n-1$, is $(n-1,l)$-mutually independent bipanconnected for every l, $d_{Q_n}(u,v) \leqslant l \leqslant |V(Q_n) - 1|$ with $(l - d_{Q_n}(u,v))$ being even. As for $d_{Q_n}(u,v) \leqslant n-2$, it is also $(n-1,l)$-mutually independent bipanconnected if $l \geqslant d_{Q_n}(u,v) + 2$, and is only (l,l)-mutually independent bipanconnected if $l = d_{Q_n}(u,v)$.

1. Introduction

For the graph definitions and notations we refer the reader to [10]. A graph is denoted by G with the vertex set $V(G)$ and the edge set $E(G)$. The simulation of one architecture by another is an important issue in interconnection networks. The problem of simulating one network by another is also called embedding problem. One particular problem of path embedding deals with finding all the possible length of paths in an interconnection network.

A path $P = \langle v_0, v_1, \ldots, v_m \rangle$ is a sequence of adjacent vertices. We also write $P = \langle v_0, \ldots, v_i, Q, v_j, \ldots, v_m \rangle$ where Q is a path $\langle v_i, \ldots, v_j \rangle$. A cycle $C = \langle v_0, v_1, \ldots, v_m, v_0 \rangle$ is a sequence of adjacent vertices where the first vertex is the same as the last one. The *length* of a path P (a cycle C respectively) is the number of edges in P (in C respectively).

A cycle of G is a *hamiltonian cycle* if it traverses all the vertices exactly once. A graph G is called a *hamiltonian graph* if G contains a hamiltonian cycle. There are many studies about the hamiltonian graphs [3,4,15]. A path of G is a *hamitonian path* if it contains all the vertices exactly once. A graph G is *hamitonian connected* if there exists a hamiltonian path between any two different vertices of G. A graph $G = (B \cup W, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets B and W such that every edge joins B with W. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected. A bipartite graph G is *hamiltonian laceable* if there exists a hamiltonian path joining any two vertices from different partite sets.

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A graph G is *pancyclic* [2] if G includes cycles of all lengths. If these cycles are restricted to even length, G is called a *bipancyclic graph*. The *distance* from x to y, written $d_G(x,y)$, is the least length among all paths from x to y in G. A graph is *panconnected* if, for any two different vertices x and y, there exists a path of length l joining x and y, for every l, $d_G(x,y) \leqslant l \leqslant |V(G)| - 1$. The concept of panconnected graphs is proposed by Alavi and Williamson [1]. Recently, there are many studies about pancyclicity and panconnectivity of graphs [5,6].

It is not hard to see that any bipartite graph with at least 3 vertices is not panconnected. Therefore, the concept of bipanconnected graphs is proposed. A bipartite graph is *bipanconnected* if, for any two different vertices x and y, there exists a path of length l joining from x to y, for every l, $d_G(x,y) \leqslant l \leqslant |V(G)| - 1$ and $(l - d_G(x,y))$ being even. There are many studies on bipanconnected graphs and bipancyclic graphs [7,11,13,18].

We introduce some terms defined recently. Two paths $P_1 = \langle u_1, u_2, \ldots, u_m\rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_m\rangle$ from a to b are *independent* [14] if $u_1 = v_1 = a$, $u_m = v_m = b$, and $u_i \neq v_i$ for $2 \leqslant i \leqslant m-1$. Paths with equal length $\{P_i\}_{i=1}^n$ from a to b are *mutually independent* [14] if they are pairwisely independent. Two cycles $C_1 = \langle u_1, u_2, \ldots, u_m, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_m, v_1\rangle$ beginning at x are *independent* if $u_1 = v_1 = x$ and $u_i \neq v_i$ for $2 \leqslant i \leqslant m$. Cycles with equal length $\{C_i\}_{i=1}^n$ beginning at x are *mutually independent* if every two cycles are independent. Two hamiltonian paths $P_1 = \langle u_1, u_2, \ldots, u_{|V(G)|}\rangle$ and $P_2 = \langle v_1, v_2, \ldots, v_{|V(G)|}\rangle$ are *independent beginning at x* [9] if $u_1 = v_1 = x$ and $u_i \neq v_i$ for $2 \leqslant i \leqslant |V(G)|$, denoted P_1: $x \to u_{|V(G)|}$ and P_2: $x \to v_{|V(G)|}$. Hamiltonian paths $\{P_i\}_{i=1}^n$ are *mutually independent hamiltonian paths beginning at x* [9] if any two of them are independent beginning at x.

An n-dimensional hypercube, denoted by Q_n, is a graph with 2^n vertices, and each vertex u can be distinctly labeled by an n-bit binary string, $u = u_{n-1}u_{n-2}, \ldots, u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. Let (u,v) be an edge in Q_n. If the binary labels of u and v differ in ith position, then the edge between them is said to be in ith dimension and the edge (u,v) is called an ith dimension edge. We use Q^0_{n-1} to denote the subgraph of Q_n induce by $\{u \in V(Q_n) | u_i = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{u \in V(Q_n) | u_i = 1\}$.$Q^0_{n-1}$ and Q^1_{n-1} are all isomorphic to Q_{n-1}.Q_n can be decomposed into Q^0_{n-1} and Q^1_{n-1} by dimension i, and Q^0_{n-1} and Q^1_{n-1} are $(n-1)$-dimensional subcubes of Q_n induced by the vertices with the ith bit position being 0 and 1 respectively. For each vertex u in $Q^i_{n-1}, i = \{0,1\}$, there is exactly one vertex in $Q^{|i-1|}_{n-1}$, denoted by $\bar{u}$, such that $(u,\bar{u})$ is an edge in Q_n. There are many studies on the hypercubes [9,13,16,17,19,20].

We now introduce a new concept. Let u and v be two distinct vertices of a bipartite graph G and let l be a positive integer length, $d_G(u,v) \leqslant l \leqslant |V(G) - 1|$ with $(l - d_G(u,v))$ being even. We say that the pair of vertices u, v is *(m, l)-mutually independent bipanconnected* if there exist m mutually independent paths $\left\{P^l_i\right\}_{i=1}^m$ with length l from u to v. In this paper,we explore yet another strong property of the hypercubes. We prove that every pair of vertices u and v in the n-dimensional hypercube, with $d_{Q_n}(u,v) \geqslant n-1$, is $(n-1,l)$-mutually independent bipanconnected for every l, $d_{Q_n}(u,v) \leqslant l \leqslant |V(Q_n) - 1|$ with $(l - d_{Q_n}(u,v))$ being even. As for $d_{Q_n}(u,v) \leqslant n-2$, it is also $(n-1,l)$-mutually independent bipanconnected if $l \geqslant d_{Q_n}(u,v) + 2$, and is only (l,l)-mutually independent bipanconnected if $l = d_{Q_n}(u,v)$. Our result strengthens a previous results of Sun et al. [19], and Li et al. [13]. Li et al. [13] proved that the hypercube Q_n is bipanconnected for $n \geqslant 2$. Sun et al. [19] proved that there are $n-1$ mutually independent hamiltonian paths in Q_n between every two vertices from different partite sets for $n \geqslant 4$. The number "$n-1$" in our result is tight as we have the following observation. Because each vertex of the hypercube Q_n has exactly n edges incident with it, we can expect at most $n-1$ mutually independent paths when the given two vertices are adjacent.

2. Preliminaries

In order to prove our claim, we need some previous results. The following results state that there exist $n-1$ mutually independent hamiltonian paths between two vertices. We shall strengthen the result by showing that there exist $n-1$ mutually independent paths of length l between two vertices, for every reasonable length l.

Theorem 1 [19]. *Let x and y be two vertices from different partite sets of Q_n, for $n \geqslant 4$. Then there exist $n-1$ mutually independent hamiltonian paths joining x to y.*

Theorem 2 [19]. *For $n \geqslant 4$, there are n mutually independent hamiltonian cycles beginning at any vertex x in Q_n.*

A hamiltonian laceable graph G is *hyper hamiltonian laceable* if for any vertex u, there is a hamiltonian path of $G - \{u\}$ between every pair of vertices in the opposite partite set of u.

Theorem 3 [12]. *For $n \geqslant 2$, the hypercube Q_n is hyper hamiltonian laceable.*

Lemma 1 [8]. *Let F_v be a set of faulty vertices in Q_n. For $n \geqslant 3$, if $|F_v| \leqslant n-2$, there exists a path of $Q_n - F_v$ with any odd length l, $3 \leqslant l \leqslant 2^n - 2|F_v| - 1$, between any two adjacent vertices.*

Lemma 2 [19]. *$Q_n - \{x,y\}$ is hamiltonian laceable, if x and y are any two vertices from different partite sets of Q_n with $n \geqslant 4$.*

Lemma 3 [9]. *In Q_n, $n \geqslant 2$, let u be any vertex, and $v_1, v_2, \ldots, v_{n-1}$ be any $n-1$ vertices in the opposite partite set of u. There exist $n-1$ mutually independent hamiltonian paths beginning at u of Q_n such that $\{P_i : u \to v_i\}_{i=1}^{n-1}$.*

3. Mutually independent bipanconnected property of hypercube

Lemma 4. *Let x and y be two vertices from different partite sets of Q_n with $n \geqslant 4$. There exists a path of every odd length from 1 to $2^n - 3$ joining any two adjacent fault-free vertices in $Q_n - \{x,y\}$.*

Proof. Let u, v be two adjacent fault-free vertices in $Q_n - \{x,y\}$. Because u and v are adjacent fault-free vertices, there exists a path of length 1 joining from u to v in $Q_n - \{x,y\}$. According to Lemma 1, there exists a path of every odd length from 3 to $2^n - 2|2| - 1 (=2^n - 5)$ joining u to v in $Q_n - \{x,y\}$. Then by Lemma 2, there exists a path of length $2^n - 3$ joining u to v in $Q_n - \{x,y\}$. Therefore, the lemma holds. □

Sun et al. [19] proved that any two hamiltonian path connecting 000 and 100 in Q_3 are not independent, in other words, there do not exist 2 mutually independent hamiltonian paths in Q_3 between 000 and 100. So, we will prove our theorem beginning from $n \geqslant 4$ for Q_n. We found that there are only d mutually independent paths with length d if $d_{Q_n}(u, v) = d$. In order to see this, we have the following lemma.

Lemma 5. *Let u and v be two vertices of Q_n with $d_{Q_n}(u, v) = d$, there are d and at most d mutually independent paths with length d joining from u to v.*

Proof. By the symmetric property of the hypercubes, we may assume that u is the vertex with n bits containing n 0's, and v is the vertex with n bits containing d 1's. In order to see the basic idea, we first give an example $n = 6$. In Q_6. Let $u = 000000$ and $v = 001111$ then $d_{Q_6}(u, v) = 4$. We can construct 4 mutually independent paths with length 4 between u and v.

$$P_0 = \langle u, 000001, 000011, 000111, v\rangle,$$
$$P_1 = \langle u, 000010, 000110, 001110, v\rangle,$$
$$P_2 = \langle u, 000100, 001100, 001101, v\rangle, \text{and}$$
$$P_3 = \langle u, 001000, 001001, 001011, v\rangle.$$

For general n, let $u = 0\ldots0 = 0^n$ and $v = 0\ldots01\ldots1 = 0^{n-d}1^d$, then $d_{Q_n}(u, v) = d$.

$$P_0 = \langle 0^n, 0^{n-1}1, 0^{n-2}1^2, \cdots, 0^{n-d+1}1^{d-1}, 0^{n-d}1^d\rangle,$$
$$P_1 = \langle 0^n, 0^{n-2}10, 0^{n-3}1^20, \cdots, 0^{n-d}1^{d-1}0, 0^{n-d}1^d\rangle,$$
$$P_2 = \langle 0^n, 0^{n-3}10^2, 0^{n-4}1^20^2, \cdots, 0^{n-d}1^{d-2}01, 0^{n-d}1^d\rangle,$$
$$P_3 = \langle 0^n, 0^{n-4}10^3, 0^{n-5}1^20^3, \cdots, 0^{n-d}1^{d-3}01^2, 0^{n-d}1^d\rangle,$$
$$\vdots$$
$$P_{d-2} = \langle 0^n, 0^{n-(d-1)}10^{d-2}, 0^{n-d}1^20^{d-2}, 0^{n-d}1^20^{d-3}1, 0^{n-d}1^20^{d-4}1^2 \cdots, 0^{n-d}1^201^{d-3}, 0^{n-d}1^d\rangle,$$
$$P_{d-1} = \langle 0^n, 0^{n-d}10^{d-1}, 0^{n-d}10^{d-2}1, 0^{n-d}10^{d-3}1^2, 0^{n-d}10^{d-4}1^3, \cdots, 0^{n-d}101^{d-2}, 0^{n-d}1^d\rangle.$$

$\{P_0, P_1, \ldots, P_{d-1}\}$ form d mutually independent paths with length d joining u to v. If there exists a $(d+1)$ th path P' with length d between u and v such that P' is mutually independent to the first d paths. So the first vertex after the beginning vertex u of P' has to be different from all those of P_i $i = 0$ to $d-1$. Without loss of generality, assume that the first vertex after the beginning vertex u of P' is $(x)^i = 0^i10^{n-i-1}$ for $0 \leqslant i \leqslant n-d-1$. It is easy to see that $d_{Q_n}((x)^i, v) = d+1$, since there are $d+1$ distinct bits between $(x)^i$ and v. Therefore, it is impossible to find out a $(d+1)$ th path with length d between u and v which is independent to $P_0, P_1, \ldots, P_{d-1}$. □

We now show our main result Theorem 5 below. Our proof is by induction on n, for Q_n. The base case is $n = 4$.

Theorem 4. *Let u and v be a pair of vertices of Q_4. If $d_{Q_4}(u, v) \geqslant 3$, Q_4 is (3, l)-mutually independent bipanconnected for every l, $d_{Q_4}(u, v) \leqslant l \leqslant 2^4 - 1$ with $(l - d_{Q_4}(u, v))$ being even. As for $d_{Q_4}(u, v) \leqslant 2$, it is also (3, l)-mutually independent bipanconnected if $l \geqslant d_{Q_4}(u, v) + 2$, and is only (l, l)-mutually independent bipanconnected if $l = d_{Q_4}(u, v)$.*

Proof. We know that $d_{Q_4}(u, v) \leqslant 4$. By the symmetric property of hypercubes, we may let $u = 0000$, and let v be 0001, 0110, 0111, and 1111 when $d_{Q_4}(u, v)$ is 1, 2, 3, and 4, respectively. In Table 1, we construct the required paths with length l such that $d_{Q_4}(u, v) \leqslant l \leqslant 2^4 - 1$ and $(l - d_{Q_4}(u, v))$ being even. □

We will use the notation P_i^k or R_i^k to denote a path i with length k.

Lemma 6. *Let u and v be two adjacent vertices of Q_n for $n \geqslant 4$. There exist $n-1$ mutually independent paths $\left\{P_i^l\right\}_{i=1}^{n-1}$ of Q_n with any odd length l, $3 \leqslant l \leqslant 2^n - 1$, joining from u to v.*

Table 1
The proof of Theorem 4.

Vertex v	Required length	Required pathes
v = 0001	l = 1	⟨0000,0001⟩
	l = 3	⟨0000,0100,0101,0001⟩
		⟨0000,0010,0011,0001⟩
		⟨0000,1000,1001,0001⟩
	l = 5	⟨0000,0100,0110,0111,0101,0001⟩
		⟨0000,0010,1010,1011,0011,0001⟩
		⟨0000,1000,1100,1101,1001,0001⟩
	l = 7	⟨0000,0100,0110,0010,0011,0111,0101,0001⟩
		⟨0000,0010,1010,1110,1111,1011,1001,0001⟩
		⟨0000,1000,1001,1011,1010,0010,0011,0001⟩
	l = 9	⟨0000,0100,0110,0010,1010,1011,0011,0111,0101,0001⟩
		⟨0000,0010,0011,0111,0110,0100,0101,1101,1001,0001⟩
		⟨0000,1000,1010,1110,1100,1101,1111,1011,0011,0001⟩
	l = 11	⟨0000,0100,0110,0010,1010,1110,1111,1011,0011,0111,0101,0001⟩
		⟨0000,0010,0011,0111,0110,0100,0101,1101,1100,1000,1001,0001⟩
		⟨0000,1000,1100,1101,1111,0111,0110,1110,1010,1011,0011,0001⟩
	l = 13	⟨0000,0100,0101,0111,0011,1011,1010,1000,1100,1101,1001,1011,0011,0001⟩
		⟨0000,0010,0110,0100,0101,0111,0011,1011,1010,1000,1100,1101,1001,0001⟩
		⟨0000,1000,1010,1011,1001,1101,1100,1110,1111,0111,0110,0100,0101,0001⟩
	l = 15	⟨0000,0100,0101,0111,0110,0010,1010,1000,1100,1110,1111,1011,1001,1011,0011,0001⟩
		⟨0000,0010,0110,0100,0101,0111,0011,1011,1010,1000,1100,1110,1111,1101,1001,0001⟩
		⟨0000,1000,1010,1011,1001,1101,1100,1110,1111,0111,0011,0010,0110,0100,0101,0001⟩
v = 0110	l = 2	⟨0000,0100,0110⟩
		⟨0000,0010,0110⟩
	l = 4	⟨0000,0001,0101,0100,0110⟩
		⟨0000,0010,0011,0111,0110⟩
		⟨0000,1000,1100,1110,0110⟩
	l = 6	⟨0000,0001,0101,0111,0011,0010,0110⟩
		⟨0000,0010,0011,0001,0101,0111,0110⟩
		⟨0000,1000,1100,1101,1111,1110,0110⟩
	l = 8	⟨0000,0001,0101,0111,0011,1011,1010,0010,0110⟩
		⟨0000,0010,0011,0001,0101,0111,1111,1110,0110⟩
		⟨0000,1000,1001,1011,1010,1110,1100,0100,0110⟩
	l = 10	⟨0000,0001,0101,1101,1001,1011,1111,0111,0011,0010,0110⟩
		⟨0000,0010,0011,1011,1111,1101,1001,0001,0101,0111,0110⟩
		⟨0000,1000,1001,0001,0101,0111,0011,0010,1010,1110,0110⟩
	l = 12	⟨0000,0001,0101,1101,1100,1000,1010,1110,1111,0111,0011,0010,0110⟩
		⟨0000,0010,0011,1011,1111,1110,1100,1101,1001,0001,0101,0111,0110⟩
		⟨0000,0100,1100,1000,1101,0001,0101,0111,0011,1011,1111,1110,0110⟩
	l = 14	⟨0000,0001,0101,0100,1100,1000,1001,1101,1111,1110,1010,1011,0011,0010,0110⟩
		⟨0000,0100,1100,1110,1111,1011,1010,0010,0011,0001,1001,1101,0101,0111,0110⟩
		⟨0000,1000,1001,1011,1010,0010,0011,0001,0101,0111,1111,1110,1100,0100,0110⟩
v = 0111	l = 3	⟨0000,0001,0101,0111⟩
		⟨0000,0100,0110,0111⟩
		⟨0000,0010,0011,0111⟩
	l = 5	⟨0000,0001,0101,0100,0110,0111⟩
		⟨0000,0010,0011,0001,0101,0111⟩
		⟨0000,0100,0110,0010,0011,0111⟩
	l = 7	⟨0000,0001,0101,0100,0110,0010,0011,0111⟩
		⟨0000,0010,0011,0001,0101,0100,0110,0111⟩
		⟨0000,0100,0110,0010,0011,0001,v0101,0111⟩
	l = 9	⟨0000,0001,0101,1101,1100,0100,0110,0010,0011,0111⟩
		⟨0000,0010,0011,1011,1001,0001,0101,0100,0110,0111⟩
		⟨0000,0100,0110,1110,1010,0010,0011,0001,0101,0111⟩
	l = 11	⟨0000,0001,0101,1101,1111,1110,1100,0100,0110,0010,0011,0111⟩
		⟨0000,0010,0011,1011,1010,1000,1001,0001,0101,0100,0110,0111⟩
		⟨0000,0100,0110,0010,0011,0001,0101,1101,1100,1110,1111,0111⟩
	l = 13	⟨0000,0001,0101,1101,1111,1011,1010,1110,1100, 0100,0110,0010,0011,0111⟩
		⟨0000,0010,0011,1011,1010,1110,1100,1000,1001,0001,0101,0100,0110,0111⟩
		⟨0000,0100,0110,1110,1100,1101,0101,0001,0011,0010,1010,1011,1111,0111⟩
	l = 15	⟨0000,0001,0101,1101,1111,1011,1001,1000,1010,1110,1100,0100,0110,0010,0011,0111⟩
		⟨0000,0010,0011,1011,1010,1110,1111,1101,1100,1000,1001,0001,0101,0100,0110,0111⟩
		⟨0000,0100,0110,1110,1100,1101,0101,0001,0011,0010,1010,1000,1001,1011,1111,0111⟩
v = 1111	l = 4	⟨0000,1000,1001,1011,1111⟩
		⟨0000,0100,1100,1101,1111⟩
		⟨0000,0010,0110,1110,1111⟩
		⟨0000,0001,0011,0111,1111⟩

Table 1 (*continued*)

Vertex v	Required length	Required pathes
	$l = 6$	⟨0000,0001,1001,1101,1100,1110,1111⟩
		⟨0000,0010,1010,1000,1001,1101,1111⟩
		⟨0000,0100,0110,0010,0011,0111,1111⟩
	$l = 8$	⟨0000,0001,1001,1000,1100,1110,1010,1011,1111⟩
		⟨0000,0010,1010,1011,1001,1101,1100,1110,1111⟩
		⟨0000,0100,0110,0010,0011,0001,0101,0111,1111⟩
	$l = 10$	⟨0000,0001,0101,0100,1100,1000,1001,1011,1010,1110,1111⟩
		⟨0000,0010,0011,0001,1001,1011,1010,1000,1100,1101,1111⟩
		⟨0000,1000,1001,1101,0101,0100,0110,0010,0011,0111,1111⟩
	$l = 12$	⟨0000,0001,0011,0111,0110,0100,1100,1000,1001,1011,1010,1110,1111 ⟩
		⟨0000,0010,0110,0100,0101,0001,1001,1011,1010,1000,1100,1101,1111⟩
		⟨0000,0100,0101,0001,0011,0111,0110,1110,1100,1101,1001,1011,1111⟩
	$l = 14$	⟨0000,0001,0101,0100,0110,0111,0011,1011,1001,1101,1100,1000,1010,1110,1111⟩
		⟨0000,0010,0011,0111,0101,0100,0110,1110,1100,1000,1010,1011,1001,1101,1111⟩
		⟨0000,1000,1001,1011,1010,1110,1100,1101,0101,0001,0011,0010,0110,0111,1111⟩

Proof. We choose a dimension to divide the hypercube Q_n into two subcubes Q^0_{n-1} and Q^1_{n-1} such that u is a black vertex in Q^0_{n-1} and v a white vertex in Q^1_{n-1}. Notice that $\bar{u} = v$. According to Theorem 2, there exist $n-1$ mutually independent hamiltonian cycles $\{C_i\}_{i=1}^{n-1}$ in Q^0_{n-1} beginning at u. For each k, $1 \leqslant k \leqslant 2^{n-1}-1$, let $C_i = \langle u, R_i^k, x_{i,k}, x_{i,k+1}, \ldots, x_{i,2^{n-1}-1}, u\rangle$ for $1 \leqslant i \leqslant n-1$, where $R_i^k = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,k}\rangle$ and $|R_i^k| = k$. Let $S_i^k = \langle \bar{x}_{i,k}, \ldots, \bar{x}_{i,2}, \bar{x}_{i,1}, \bar{u}\rangle$ for $1 \leqslant i \leqslant n-1$. Combine R_i^k and S_i^k, we let $P_i^{2k+1} = \langle u, R_i^k, x_{i,k}, \bar{x}_{i,k}, S_i^k, \bar{u} = v\rangle$, $1 \leqslant k \leqslant 2^{n-1}-1$, for $1 \leqslant i \leqslant n-1$. Then P_i^{2k+1} is a path joining u to v with length $2k+1$. Since $1 \leqslant k \leqslant 2^{n-1}-1$ so $3 \leqslant 2k+1 \leqslant 2^n-1$. Therefore, there exist $n-1$ mutually independent paths $\left\{P_i^l\right\}_{i=1}^{n-1}$ with any odd length l, $3 \leqslant l \leqslant 2^n-1$, joining from u to v. See Fig. 1 for illustration. □

Lemma 7. *Let u and v be two vertices from the same partite set of Q_n for $n \geqslant 4$. There exist $n-1$ mutually independent paths $\left\{P_i^l\right\}_{i=1}^{n-1}$ of Q_n with any even length l, $d_{Q_n}(u,v)+2 \leqslant l \leqslant 2^n-2$, joining from u to v.*

Proof. We prove the statement by induction on n. By Theorem 4, the statement holds for $n = 4$. Suppose that the result holds for Q_{n-1}, $n \geqslant 5$. Without loss of generality, let u and v be two black vertices of Q_n. We may choose a dimension to divide the hypercube Q_n into two subcubes Q^0_{n-1} and Q^1_{n-1} such that $u \in Q^0_{n-1}$ and $v \in Q^1_{n-1}$. Therefore, $\bar{u}$ and $\bar{v}$ are two white vertices in Q^1_{n-1} and Q^0_{n-1}, respectively. Assume that $d_{Q_n}(u,v) = d$ and d is even, then it is easy to see that $d_{Q_n}(u,\bar{v}) = d_{Q_n}(u,v) - 1 = d-1$. According to the length l of the paths, we divide the proof into the following three cases. In each case, the length l is assumed to be an even number. We shall find $n-1$ mutually independent paths with length l joining from u to v.

Case 1. For even length l and $d+2 \leqslant l \leqslant 2^{n-1}$.

By induction hypothesis, there exist $n-2$ mutually independent paths $\left\{R_i^k\right\}_{i=1}^{n-2}$ of Q^0_{n-1} with odd length k, $d+1 \leqslant k \leqslant 2^{n-1}-1$, between u and $\bar{v}$. For $1 \leqslant i \leqslant n-2$, we let $R_i^k = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}, \bar{v}\rangle$. Now, for each l between $d+2$ and 2^{n-1}, we show how to construct the $n-1$ mutually independent paths with length l. Let

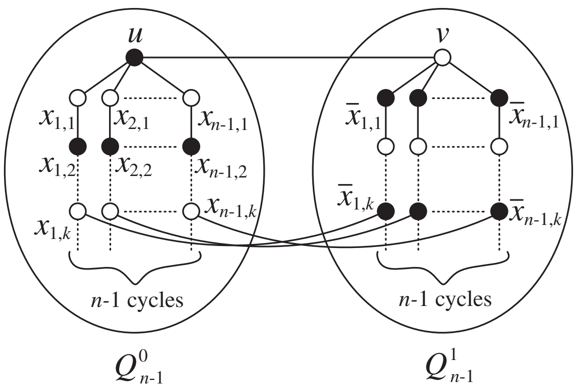

Fig. 1. Illustration for Lemma 6.

$P_1^{k+1} = \langle u, x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, \bar{v}, v \rangle$, $P_i^{k+1} = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}, \bar{x}_{i,k-1}, v \rangle$ for $2 \leqslant i \leqslant n-2$, and $P_{n-1}^{k+1} = \langle u, \bar{u}, \bar{x}_{1,1}, \bar{x}_{1,2}, \ldots, \bar{x}_{1,k-1}, v \rangle$, $d+2 \leqslant k+1 \leqslant 2^{n-1}$. Set $l = k + 1$. So, $\left\{P_i^l\right\}_{i=1}^{n-1}$ form $n-1$ mutually independent paths with each even length l, $d + 2 \leqslant l \leqslant 2^{n-1}$, joining from u to v.

Case 2. For even length l and $2^{n-1} + 2 \leqslant l \leqslant d + 2^{n-1} - 2$.

According to induction hypothesis, there exist $n-2$ mutually independent paths $\{R_i\}_{i=1}^{n-2}$ of Q_{n-1}^0 with odd length $d-1$ between u and $\bar{v}$. Without loss of generality, we write $R_i = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,d-2}, \bar{v} \rangle$ for $1 \leqslant i \leqslant n-2$. For each m, $2 \leqslant m \leqslant d-2$ and m is even, by Lemma 3, there exist $n-2$ mutually independent hamiltonian paths $\{S_i\}_{i=1}^{n-2}$ of Q_{n-1}^1 beginning at v such that $\{S_i : v \to \bar{x}_{i,m}\}_{i=1}^{n-2}$. For $1 \leqslant i \leqslant n-2$, let $P_i^{m+2^{n-1}} = \langle u, x_{i,1}, x_{i,2} \ldots, x_{i,m}, \bar{x}_{i,m}, (S_i)^{-1}, v \rangle$, $2^{n-1} + 2 \leqslant m + 2^{n-1} \leqslant d + 2^{n-1} - 2$. Set $l = m + 2^{n-1}$. We have the first $n-2$ mutually independent paths with each even length l, $2^{n-1} + 2 \leqslant l \leqslant d + 2^{n-1} - 2$ joining u to v.

Finally, we construct the $(n-1)$ th path joining u to v. Let z be any white vertex in Q_{n-1}^1. By Lemma 4, there exists a path T^k of $Q_{n-1}^1 - \{v, z\}$ with any odd length k, $1 \leqslant k \leqslant d-3$, joining $\bar{u}$ to $\bar{x}_{n-1,1}$, and by Theorem 3, there exists a hamiltonian path U of $Q_{n-1}^0 - \{u\}$ between $x_{n-1,1}$ to $\bar{v}$. Let $P_{n-1}^{k+2^{n-1}+1} = \langle u, \bar{u}, T^k, \bar{x}_{n-1,1}, x_{n-1,1}, U, \bar{v}, v \rangle$, $2^{n-1} + 2 \leqslant k + 2^{n-1} + 1 \leqslant d + 2^{n-1} - 2$. Set $l = k + 2^{n-1} + 1$. So, $\left\{P_i^l\right\}_{i=1}^{n-1}$ form $n-1$ mutually independent paths with each even length l, $2^{n-1} + 2 \leqslant l \leqslant d + 2^{n-1} - 2$, joining from u to v.

Case 3. For even length l and $d + 2^{n-1} - 2 \leqslant l \leqslant 2^n - 2$.

Again, by induction hypothesis, there exist $n-2$ mutually independent paths $\{R_i^m\}_{i=1}^{n-2}$ between u and $\bar{v}$ in Q_{n-1}^0 with odd length m, $d-1 \leqslant m \leqslant 2^{n-1} - 1$. Let $R_i^m = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,m-1}, \bar{v} \rangle$ for $1 \leqslant i \leqslant n-2$. By Lemma 3, there exist $n-2$ mutually independent hamiltonian paths $\{S_i\}_{i=1}^{n-2}$ of Q_{n-1}^1 beginning at v such that $\{S_i : v \to \bar{x}_{i,m-1}\}_{i=1}^{n-2}$. Let $P_i^{m+2^{n-1}-1} = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,m-1}, \bar{x}_{i,m-1}, (S_i)^{-1}, v \rangle$ for $1 \leqslant i \leqslant n-2$, $d + 2^{n-1} - 2 \leqslant m + 2^{n-1} - 1 \leqslant 2^n - 2$. Set $l = m + 2^{n-1} - 1$. We have the first $n-2$ mutually independent paths with each even length l, $d + 2^{n-1} - 2 \leqslant l \leqslant 2^n - 2$ joining u to v.

Finally, we construct the $(n-1)$ th path joining u to v. Assume that z is any white vertex in Q_{n-1}^1. According to Lemma 4, there exists a path T^k of $Q_{n-1}^1 - \{v, z\}$ with any odd length k, $d-3 \leqslant k \leqslant 2^{n-1} - 3$, joining $\bar{u}$ to $\bar{x}_{n-1,1}$, and by Theorem 3, there exists a hamiltonian path U of $Q_{n-1}^0 - \{u\}$ between $x_{n-1,1}$ to $\bar{v}$. Let $P_{n-1}^{k+2^{n-1}+1} = \langle u, \bar{u}, T^k, \bar{x}_{n-1,1}, x_{n-1,1}, U, \bar{v}, v \rangle$, $d + 2^{n-1} - 2 \leqslant k + 2^{n-1} + 1 \leqslant 2^n - 2$. Set $l = k + 2^{n-1} + 1$. So, $\left\{P_i^l\right\}_{i=1}^{n-1}$ form $n-1$ mutually independent paths with each even length l, $d + 2^{n-1} - 2 \leqslant l \leqslant 2^n - 2$, joining from u to v. □

Lemma 8. *Let u and v be two nonadjacent vertices from different partite sets of Q_n for $n \geqslant 4$. There exist $n-1$ mutually independent paths $\left\{P_i^l\right\}_{i=1}^{n-1}$ of Q_n with any odd length l, $d_{Q_n}(u, v) + 2 \leqslant l \leqslant 2^n - 1$, joining from u to v.*

Proof. We prove the statement by induction on n. By Theorem 4, the statement holds for $n = 4$. Suppose that the result holds for Q_{n-1}, $n \geqslant 5$. Without loss of generality, let u be a black vertex and v a white vertex in Q_n. We may choose a dimension to divide the hypercube Q_n into two subcubes Q_{n-1}^0 and Q_{n-1}^1 such that $u \in Q_{n-1}^0$ and $v \in Q_{n-1}^1$. Hence, $\bar{u}$ is a white vertex in Q_{n-1}^1 and $\bar{v}$ is a black vertices in Q_{n-1}^0. Assume that $d_{Q_n}(u, v) = d$ and d is odd, then it is easy to see that $d_{Q_n}(u, \bar{v}) = d_{Q_n}(u, v) - 1 = d - 1$. According to the length l of the paths, we divide the proof into the following four cases. In each case, the length l is assumed to be an odd number. We shall find $n-1$ mutually independent paths with length l joining from u to v.

Case 1. For odd length l and $d + 2 \leqslant l \leqslant 2^{n-1} - 1$.

By induction hypothesis, there exist $n-2$ mutually independent paths $\left\{R_i^k\right\}_{i=1}^{n-2}$ between u and $\bar{v}$ in Q_{n-1}^0 with even length k, $d + 1 \leqslant k \leqslant 2^{n-1} - 2$. For $1 \leqslant i \leqslant n-2$, we let $R_i^k = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}, \bar{v} \rangle$. Now, we show how to construct the required $n-1$ mutually independent paths. Set $P_1^{k+1} = \langle u, x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, \bar{v}, v \rangle$, $P_i^{k+1} = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}, \bar{x}_{i,k-1}, v \rangle$ for $2 \leqslant i \leqslant n-2$, and $P_{n-1}^{k+1} = \langle u, \bar{u}, \bar{x}_{1,1}, \bar{x}_{1,2}, \ldots, \bar{x}_{1,k-1}, v \rangle$, $d + 2 \leqslant k + 1 \leqslant 2^{n-1} - 1$. Let $l = k + 1$. So, $\left\{P_i^l\right\}_{i=1}^{n-1}$ form $n-1$ mutually independent paths with each odd length l, $d + 2 \leqslant l \leqslant 2^{n-1} - 1$, joining from u to v.

Case 2. For odd length l and $2^{n-1} + 1 \leqslant l \leqslant d + 2^{n-1} - 2$.

According to the induction hypothesis, there exist $n-2$ mutually independent paths $\{R_i\}_{i=1}^{n-2}$ of Q_{n-1}^0 with even length $d-1$ between u and $\bar{v}$. We write $R_i = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,d-2}, \bar{v} \rangle$ for $1 \leqslant i \leqslant n-2$. Assume that $1 \leqslant m \leqslant d-2$ and m is odd, by Lemma 3, there exist $n-2$ mutually independent hamiltonian paths $\{S_i\}_{i=1}^{n-2}$ of Q_{n-1}^1 beginning at

v such that $\{S_i : v \rightarrow \bar{x}_{i,m}\}_{i=1}^{n-2}$. Now, consider the first $n-2$ mutually independent paths. For $1 \leqslant i \leqslant n-2$, let $P_i^{m+2^{n-1}} = \langle u, x_{i,1}, x_{i,2} \ldots, x_{i,m}, \bar{x}_{i,m}, (S_i)^{-1}, v \rangle$, $2^{n-1}+1 \leqslant m+2^{n-1} \leqslant d+2^{n-1}-2$. Set $l = m + 2^{n-1}$.
We have the first $n-2$ mutually independent paths with each odd length l, $2^{n-1}+1 \leqslant l \leqslant d+2^{n-1}-2$ joining u to v. Finally, we construct the $(n-1)$ th path. Let z be any black vertex in $Q_{n-1}^1 - \{\bar{x}_{n-1,1}\}$, and w be any white vertex in $Q_{n-1}^0 - \{x_{n-1,1}\}$. By Lemma 4, there exists a path T^k of $Q_{n-1}^1 - \{v, z\}$ with any odd length k, $1 \leqslant k \leqslant d-2$, joining $\bar{u}$ to $\bar{x}_{n-1,1}$, and by Lemma 2, there exists a hamiltonian path U of $Q_{n-1}^0 - \{u, w\}$ between $x_{n-1,1}$ to $\bar{v}$. Let $P_{n-1}^{k+2^{n-1}} = \langle u, \bar{u}, T^k, \bar{x}_{n-1,1}, x_{n-1,1}, U, \bar{v}, v \rangle$, $2^{n-1}+1 \leqslant k+2^{n-1} \leqslant d+2^{n-1}-2$. Set $l = k + 2^{n-1}$. Hence, $\left\{P_i^l\right\}_{i=1}^{n-1}$ form $n-1$ mutually independent paths with each odd length l, $2^{n-1}+1 \leqslant l \leqslant d+2^{n-1}-2$, joining from u to v.

Case 3. For odd length l and $d+2^{n-1}-2 \leqslant l \leqslant 2^n-3$.
Again, by induction hypothesis, there exist $n-2$ mutually independent paths $\{R_i^m\}_{i=1}^{n-2}$ of Q_{n-1}^0 with even length m, $d-1 \leqslant m \leqslant 2^{n-1}-2$, between u and $\bar{v}$. Without loss of generality, let $R_i^m = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,m-1}, \bar{v} \rangle$ for $1 \leqslant i \leqslant n-2$. By Lemma 3, there exist $n-2$ mutually independent hamiltonian paths $\{S_i\}_{i=1}^{n-2}$ of Q_{n-1}^1 beginning at v such that $\{S_i : v \rightarrow \bar{x}_{i,m-1}\}_{i=1}^{n-2}$. Let $P_i^{m+2^{n-1}-1} = \langle u, x_{i,1}, x_{i,2}, \ldots, x_{i,m-1}, \bar{x}_{i,m-1}, (S_i)^{-1}, v \rangle$ for $1 \leqslant i \leqslant n-2$, $d+2^{n-1}-2 \leqslant m+2^{n-1}-1 \leqslant 2^n-3$. Set $l = m+2^{n-1}-1$. We have the first $n-2$ mutually independent paths with each odd length l, $d+2^{n-1}-2 \leqslant l \leqslant 2^n-3$ joining u to v. Then, we construct the $(n-1)$ th path.
Assume that z is any black vertex in $Q_{n-1}^1 - \{\bar{x}_{n-1,1}\}$, and w is any white vertex in $Q_{n-1}^0 - \{x_{n-1,1}\}$. According to Lemma 4, there exists a path T^k of $Q_{n-1}^1 - \{v, z\}$ with any odd length k, $d-2 \leqslant k \leqslant 2^{n-1}-3$, joining $\bar{u}$ to $\bar{x}_{n-1,1}$, and by Lemma 2, there exists a hamiltonian path U of $Q_{n-1}^0 - \{u, w\}$ between $x_{n-1,1}$ to $\bar{v}$. Let $P_{n-1}^{k+2^{n-1}} = \langle u, \bar{u}, T^k, \bar{x}_{n-1,1}, x_{n-1,1}, U, \bar{v}, v \rangle$, $d+2^{n-1}-2 \leqslant k+2^{n-1} \leqslant 2^n-3$. Set $l = k+2^{n-1}$. Therefore, $\left\{P_i^l\right\}_{i=1}^{n-1}$ form $n-1$ mutually independent paths with each odd length l, $d+2^{n-1}-2 \leqslant l \leqslant 2^n-3$, joining from u to v.

Case 4. For odd length $l = 2^n - 1$.
This case was proved by Theorem 1. □

By Theorem 4, Lemmas 5–8, we have the following theorem.

Theorem 5. *Let u and v be any pair of vertices of Q_n. For $d_{Q_n}(u, v) \geqslant n-1$, Q_n is $(n-1, l)$-mutually independent bipanconnected for every l, $d_{Q_n}(u, v) \leqslant l \leqslant 2^n-1$ with $(l - d_{Q_n}(u, v))$ being even. As for $d_{Q_n}(u, v) \leqslant n-2$, it is also $(n-1, l)$-mutually independent bipanconnected if $l \geqslant d_{Q_n}(u, v) + 2$, and is only (l, l)-mutually independent bipanconnected if $l = d_{Q_n}(u, v)$.*

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