Point Pattern Representation Using Imprecise, Incomplete, Nonmetric Information

Stephen H. Levine, John G. Kreifeldt and Ming-Chuen Chuang

Abstract—**A novel method is described for representing two- or three-dimensional patterns of *n* points utilizing imprecise, incomplete, nonmetric information. This information consists solely of a rank ordered list of interpoint distances determined from pairwise comparisons. Ideally each comparison should determine a longer and shorter distance, and a set of comparisons should include all possible pairs. Actual representation information is likely to be imprecise and incomplete. Methods are presented for maximizing the information obtained from imprecise, incomplete sets of comparisons through inferencing procedures. The sufficiency of the resulting information for precise pattern representation is demonstrated through its use in the reconstruction of the patterns using multidimensional scaling (MDS). Some surprising results are presented on the possible advantages of imprecision from the viewpoint of data requirements. A short appendix links the inferencing procedures developed in this paper to the mathematical concept of a semi-order.**

I. Introduction

There is a natural expectation that precise representation of a two or three dimensional pattern requires precise metric measurement information. A pattern consisting of n points might thus be described by the cartesian coordinates of those points or, alternatively, by a sufficient set of distances and angles between point locations. This paper will describe an alternate approach utilizing imprecise, incomplete, nonmetric information [1].

Algorithms have been developed, primarily within the field of psychology, which use an input set of comparisons between alternatives in order to construct a geometric configuration representing these alternatives as points in a multidimensional space. The distance between two alternatives in this spatial representation reflects their similarity; similar alternatives judged are close together. These algorithms are referred to as multidimensional scaling, MDS [2], [3].

In particular, one MDS algorithm contained in ALSCAL [4] permits constructions to be made from nonmetric pairwise comparisons of the type A is more similar to B than A is to C. (These are, in fact, just the relative comparisons that humans seem to be able to make, thus the interest of psychologists.) Specifically, these comparisons, rank ordered from the two least similar alternatives to the two most similar, provide the input information for ALSCAL.

The existence of this algorithm suggests that information sufficient to represent a pattern of points can be obtained based only on pairwise comparisons of interpoint distances. Information will be of the form "point A is closer to point B than point A is to point C." From this set of comparisons a rank ordered list of interpoint distances can be developed. This list represents an alternative pattern representation method, utilizing nonmetric measurement.

Moreover, this representation can be accurate using relatively imprecise information. Imprecise information results when a comparison fails to determine the longer of two interpoint distances. This occurs when the difference lies below a threshold value or minimum resolution level (known as a just noticeable difference (JND), when a human is making the comparisons) of the comparator. The comparator is said to exhibit *indifference* with regard to the two distances and the result is declared to be a "tie." However, many, if not all, of these ties can be broken through inference procedures, more so as the number of points increases. The greater the percentage of ties that are resolved the closer we come to achieving a complete rank ordered list and, in turn, an accurate representation.

While representational accuracy potentially increases with n, the number of points in the configuration, the number of possible pairwise interpoint distance comparisons increases as the fourth power of n. Fortunately, a complete set of these possible comparisons need not be directly made; the results of many can be inferred using algorithms that will be described in this paper. This is true even when the comparator threshold is relatively large. The role of inference methods in allowing the use of imprecise, incomplete information is a basic theme of the pattern representation method described in this paper; computing power is utilized to reduce the amount and preciseness of the information that must be acquired through direct measurement. Given the present increases in computing power and the decreases in computer size and cost, this gives promise of being an increasingly appealing tradeoff.

The sufficiency of imprecise, incomplete, nonmetric information in pattern representation can be demonstrated through the accurate reconstruction of two and three dimensional point configurations using MDS. Chuang [1], in a study that forms the basis for this paper, has shown that as the number of points increases this nonmetric information is sufficient to represent the object with arbitrarily increased accuracy, independent of any exact scale or rotation information.

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Three related characteristics of the comparison information, important to this representional method, should be distinguished. These are *precision, crispness*, and *correctness*. *Precision*, as noted, refers to the size of the threshold value. "Precise" comparators have zero thresholds, and therefore produce only two comparison outcomes, "larger" or "smaller." In contrast, imprecise comparators have nonzero thresholds and can produce a third outcome—"tie." *Crisp* comparators exhibit sharp, contiguous, nonoverlapping cutoffs between the various outcomes. Humans, by contrast, exhibit noncrisp, or fuzzy, behavior when making comparisons [5]. *Correct* comparators may produce ties but do not make mistakes. They therefore are consistent; the same comparison will always produce the same outcome. We will briefly consider the impact of inconsistencies on representation accuracy, an area demanding considerably more attention.

A large literature exists on pattern representation. Many representation schemes are based on the use of various series expansions, such as Karhunen–Loeve [6], Fourier [7], and Gabor [8] functions. The pattern specific information is then stored in the numerical values of the coefficients in the expansions. Alternative methods, widely used in computer graphics as well, decompose "pictures" into basic shapes, such as spherical surfaces [9]. The numerical parameters describing these shapes, i.e., the radius of the sphere, require require precise metric information.

Within the the broader context of pattern recognition. Watson [10] has noted that the perceptual process in general involves three stages; filtering, coding, and interpretation. A statement such as "a pattern consisting of a configuration of points" assumes that, unless the original pattern consisted only of points, substantial filtering of information has occurred. It further assumes that primitive analysis has already identified relevant characteristics, i.e., important points [11]. Filtering is also implied by the statement "utilizing imprecise, incomplete, nonmetric information" since the possibility of metric information is disregarded. The representation technique to be described involves a coding of the information that remains after this primitive analysis and filtering has been done.

The remaining sections of this paper will cover the following topics. Section II briefly describes the use of MDS in this research, and provides some useful definitions pertaining to data sets and rank ordered lists. Sections III and IV develop the algorithms for achieving rank ordered lists with complete and incomplete sets of data respectively. Section V describes the basic results obtained in pattern representation as demonstrated through reconstruction. These results relate the accuracy of representation to the number of points describing the pattern and to the preciseness of the comparators. Reconstructions were based on information obtained from both precise and imprecise comparators. A very brief discussion is included on results obtained using noncrisp comparators. A more detailed description can be found in [1]. Section VI is a brief discussion of several related topics. An Appendix relates the algorithms developed in this paper to semi-orders, a measurement concept developed in the realm of the social and biological sciences.

II. Rank Ordered Lists, Data Sets, and Inference

In this section we provide some background on MDS and data sets, particularly with regard to the role of rank ordered lists and their development using inference procedures. Inference procedures can achieve two basic benefits. First, they can in many instances determine relative rank when, due to lack of precision, direct measurement fails to do so. Second, they can in many instances determine relative rank in lieu of direct measurement. In both cases the determination of relative rank leads to an increase in representational accuracy.

A. A Brief Description of Multidimensional Scaling (MDS)

In using ALSCAL we begin with a rank ordered list of interpoint distances generated from a two (or three) dimensional pattern of n points from which we wish to reconstruct this pattern. (In applications in psychology there is no preexisting pattern, and the dimensionality of the data is unknown. The user is constructing a spatial representaion of nonspatial information.) ALSCAL begins by locating the n points at random in a two (or three) dimensional space and then determining a rank ordered list of interpoint distances based on this random configuration. The two lists can then be compared. They will exhibit differences and the measure of this difference is referred to as *stress*.

The algorithm then proceeds through a sequence of steps that reduces this stress by moving the points in the configuration it is developing. As the algorithm proceeds it will generally be able to reduce the stress to an arbitrarily small number. If the number of points is sufficiently large this arbitrarily small stress can only be reached by converging on the original pattern.

An incomplete list for the original pattern does not prevent the algorithm from proceeding but missing information cannot contribute to the stress and therefore can contribute to reduced accuracy of the reconstruction. MDS is designed, ideally, to use precise, complete information. In this paper the "enhancement" of imprecise, incomplete data by various inference techniques allows MDS to work well on less than ideal information, and therefore demonstrate the adequacy of this imprecise, incomplete information for pattern representation.

While ALSCAL evaluates reconstruction accuracy based on stress we are interested in a more meaningful measure. Intuitively, such a measure should describe the deviation of the points in the reconstruction, once it is appropriately scaled and positioned, from their location in the original configuration, normalized for the size of the pattern [1]. Location can be given as cartesian coordinates. To illustrate, Fig. 1 shows a reconstruction with 3% mean coordinate error. Far smaller errors are readily achieved, even with imprecise, incomplete information. Even 3% error would not be readily perceived by a human if the two configurations were not overlayed.

B. Data Sets and Rank Orderings

The information obtained directly from the pairwise comparisons that are made is the *measured data set*. As noted, additional information can be achieved through inference. This

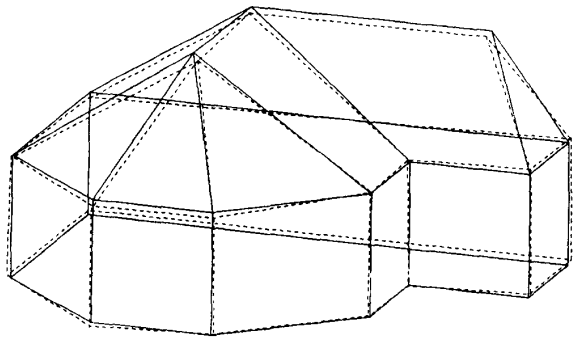

Fig. 1. Reconstruction of a configuration with 3% mean coordinate error. Note that this error, while perceptable with the configurations overlapped, would not be readily perceived if the configurations were separated.

indirectly obtained information is the *inferred data set*, and the combination of the two is simply the *data set*.

If information is available for every possible pairwise comparison a *complete data set* exists; a "tie" constitutes information as much as "longer than" or "shorter than." By contrast, an *incomplete data set* lacks any comparison information for one or more pairs of interpoint distances. Clearly, a measured data set can be complete or incomplete. If no ties exist the information constitutes a *precise data set*. Note that a precise data set need not be complete. If ties are present in the data it is referred to as an *imprecise data set*, and this too may be complete or incomplete.

Data sets provide the means for developing rank orderings among the interpoint distances, from "longest" to "shortest." Clearly, a *complete, precise data set* is necessary and sufficient for developing a *complete rank ordering*. Anything less will give rise to a *partial rank ordering*. While a complete rank ordering is not always required for an accurate pattern representation, in general the more complete the better the representation. However, obtaining a complete, precise data set is not always possible or, in some cases, even desirable. This will be demonstrated through MDS reconstructions.

We are now ready to discuss the specifics of inference procedures under different conditions of information completeness and preciseness. Interpoint distances will be referred to as line lengths since it is natural to visualize a pair of points as connected by a line segment. A pattern containing n points involves $m = n(n-1)/2$ interpoint distances or lines. A complete set of measurements requires $m(m-1)/2$ comparisons.

III. Complete Measured Data Sets

The data set obtained from a precise, complete set of comparator measurements will allow a complete rank ordering of line lengths. In this section we consider the role of inference in establishing a rank order with a complete but imprecise set of measurements. In particular, to what degree can inference help improve rank ordering?

A. Inference in a Complete Measured Data Set

When using a comparator with a threshold $d > 0$ to compare lines I and J, of lengths x_i and x_j respectively, three outcomes

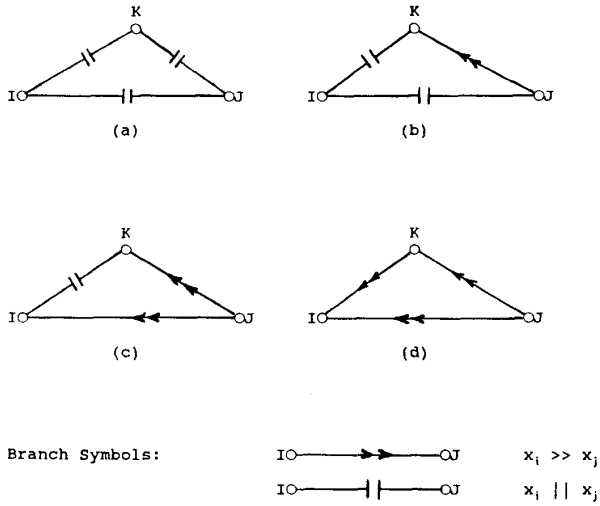

Fig. 2. The four possible outcomes of three pairwise comparisons.

are possible. These are:

(i) $x_i \gg x_j$, I is measurably longer than J, that is $x_i > x_j + d$;

(ii) $x_j \gg x_i$, J is measurably longer than I, that is $x_j > x_i + d$;

(iii) $x_i \| x_j$, I and J are not measurably different in length, that is, $x_j - d < x_i < x_j + d$.

The third outcome is not the same as saying that the two line segments are of equal length, an event that occurs with a probability of zero. (However, a line is by definition equal to itself, i.e., $x_i = x_i$.) If the first or second outcome occurs we shall consider the relative lengths to have been established by direct measurement, or directly established. If the third outcome occurs we will hope to resolve this "tie" indirectly through inference.

The basic mechanism of inference in a complete set of measurements is based on the pairwise comparisons of three lines, I, J, and K. The four possible outcomes of these measurements are illustrated by the modified digraphs shown in Fig. 2. The branch symbols correspond to the two outcomes noted above. The digraph of Fig. 2(b) is of particular interest. It illustrates the intransitivity associated with indifference; a tie between K and I and a tie between I and J does not always mean a tie between J and K. Later, we will list the rules of inference in detail; here we note that if J is measurably longer than K whereas I is not measurably different than K, then we can infer that J is longer than I, though possibly not measurably so. Remembering that the threshold is d, $x_j > x_k + d$ and $x_k - d < x_i < x_k + d$, from which we can conclude that $x_j > x_i$. Similar reasoning allows us to conclude that I is longer than K, and a complete rank ordering of the three lines is established. In contrast, Fig. 2(a) does not allow us to establish any relations through inference. Surprisingly, neither does Fig. 2(c), in spite of having fewer ties than Fig. 2(b). Fig. 2(d) has an already established complete rank ordering.

A complete measured data set for m lines can be represented as a complete digraph having m nodes using the branch

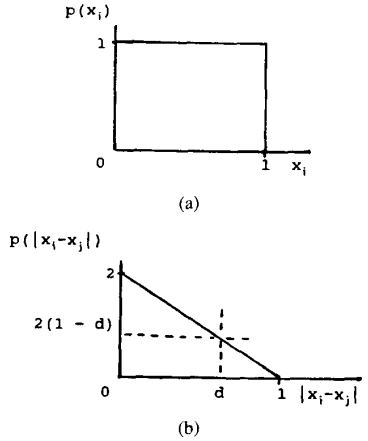

Fig. 3. Probability density functions for x_i and $|x_i - x_j|$.

symbols intoduced in Fig. 2. This complete digraph consists of $m!/(m-3)!3!$ triangular digraphs of the types shown in Fig. 2, and each branch occurs in $(m-2)$ of these. From the results above we note that a branch direction that has not been established through direct measurement may yet be established through inference if at least one of those $(m-2)$ triangular digraphs is of the type shown in Fig. 2(b). Intuitively, it seems that the larger the value of m the greater the possibility of this inference. At the same time the fraction of branches whose directions can not be established by direct measurement is important, and is determined by the size of the threshold. We will investigate this in detail.

B. Probability of Establishing Relative Lengths by Direct Measurement

We assume a set of m lines whose lengths are uniformly and randomly distributed between 0 and 1, and a minimum level of resolution, the threshold, given by d. Clearly, if $d = 0$, only Fig. 2(d) triangles occur and if $d > 1$ only Fig. 2(a) triangles occur. We are therefore interested in the range $0 < d < 1$. (In effect, d is scaled to the range of x_i.) Fig. 3 indicates the probability densities of x_i and of $|x_i - x_j|$. From Fig. 3(b) we can readily determine that with a threshold of d the probability of establishing the relative values of x_i and x_j by direct measurement, Pr(M), is

$$\Pr(\mathrm{M}) = \Pr(|x_i - x_j| > d) = (1-d)^2 \tag{1}$$

As expected, as d approaches 1, Pr(M) approaches 0.

C. Probability of Establishing Relative Values through Inference

Next we consider the case where $|x_i - x_j| < d$. Based on our previous discussion we know that given a third line segment K we will be able to establish the relative values of x_i and x_j if

(A) $|x_i - x_k| < d$ and $|x_j - x_k| > d$, or

(B) $|x_i - x_k| > d$ and $|x_j - x_k| < d$.

Either of these cases is equally likely so we will consider only Case A and evaluate

$$\Pr((|x_i - x_k| < d \quad \text{and} \quad |x_j - x_k| > d)/|x_i - x_j| < d).$$

We use the notation Pr(u/v) for the conditional probabilty event u occurs given that event v occurs. If we replace the conditional probability

$$\Pr(|x_i - x_k| < d/|x_i - x_j| < d)$$

by the unconditional probability

$$\Pr(|x_i - x_k| < d)$$

($|x_i - x_k|$ is not independent of $|x_i - x_j|$, but the error introduced is acceptably small), then

$$\begin{aligned}\Pr((|x_i - x_k| < d \text{ and } |x_j - x_k| > d)/|x_i - x_j| < d) \\ = \Pr(|x_i - x_k| < d)\Pr(|x_j - x_k| > d/(|x_i - x_k| < d \text{ and } \\ |x_i - x_j| < d)).\end{aligned} \tag{2}$$

Since

$$\Pr(|x_i - x_k| < d) = 1 - \Pr(|x_i - x_k| > d), \tag{3}$$

we obtain, using (1),

$$\Pr(|x_i - x_k| < d) = 1 - \Pr(\mathrm{M}) \tag{4a}$$

$$= 1 - (1-d)^2 \tag{4b}$$

For brevity, we define the conditional probability

$$\begin{aligned}&\Pr(\mathrm{A}) = \\ &\Pr(|x_j - x_k| > d/(|x_i - x_k| < d \quad \text{and} \quad |x_i - x_j| < d))\end{aligned} \tag{5}$$

and investigate the four possible cases.

Case 1: $x_i > d$ and $x_i < 1 - d$

In this case both x_j and x_k are uniformly distributed between $x_i - d$ and $x_i + d$, and the conditional probability density function of $|x_j - x_k|$ is shown in Fig. 4(a). (For brevity we have omitted repeating the conditional nature of these probabilities indicated in (5).) Clearly,

$$\Pr(\mathrm{A}) = 0.25 \tag{6}$$

Case 2: $x_i < d$ and $x_i < 1 - d$

In this case both x_j and x_k are uniformly distributed between 0 and $x_i + d$, and the probability density function of $|x_j - x_k|$ is shown in Fig. 4(b). We find that

$$\Pr(\mathrm{A}) = ((x_i/(x_i + d))^2 \tag{7}$$

Case 3: $x_i > d$ and $x_i > 1 - d$

In this case both x_j and x_k are uniformly distributed between $x_i - d$ and 1, and the probability density function of $|x_j - x_k|$ is shown in Fig. 4(c). Thus,

$$\Pr(\mathrm{A}) = ((1 - x_i)/(1 - x_i + d))^2 \tag{8}$$

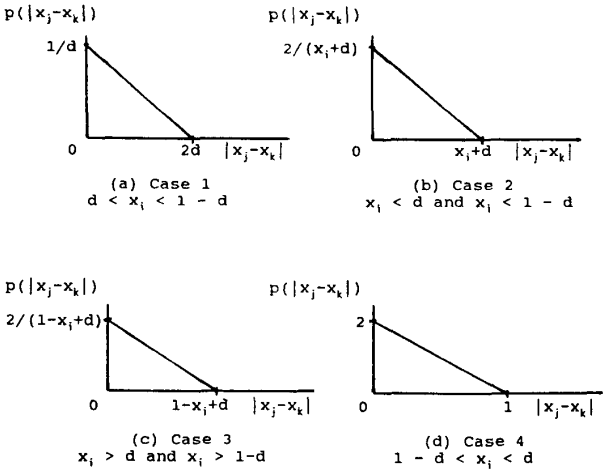

Fig. 4. The probability density function of $|x_j - x_k|$ for four cases relating x_i and d.

Case 4:$x_i < d$ and $x_i > 1 - d$

In this final case both x_j and x_k are uniformly distributed between 0 and 1, and the probability density function of $|x_j - x_k|$ is shown in Fig. 4(d). Therefore,

$$\text{Pr(A)} = (1 - d)^2. \tag{9}$$

Recalling that the probability of not establishing relative lengths by direct measurement is $1 - \text{Pr(M)}$, and that Case B was equally probable as Case A, we define $\text{Pr}(C)$ to be

$$\begin{aligned}\text{Pr(C)} &= (1 - \text{Pr(M)}) * (\text{Pr(A)} + \text{Pr(B)}) \\ &= 2 * (1 - \text{Pr(M)}) * \text{Pr(A)}\end{aligned} \tag{10}$$

where $\text{Pr}(B)$ is the equivalent expression to (5) for Case B. The event C corresponds to the lengths x_i and x_j not being measurably different but forming along with line K a triangle of the type shown in Fig. 2(b). The dependence of $\text{Pr}(C)$ on x_i can be eliminated by determining the expected value of $\text{Pr}(C)$ with respect to x_i. We will designate this expected value as $E\{\text{Pr}(C)\}$.

The probability that x_i and x_j can be distinguished by inference when they can not be distinguished by measurement, given a complete set of measurements among m line segments, is

$$\text{Pr(I/not M)} = 1 - [1 - E\{\text{Pr(C)}\}]^{m-2}. \tag{11}$$

$\text{Pr}(D)$, the probability that x_i and x_j can be distinguished either directly or, if not directly, by inference, is

$$\text{Pr(D)} = \text{Pr(M)} + (1 - \text{Pr(M)})\text{Pr(I/not M)}.$$

D. Determination of Pr(D) as a Function of d and m

Equation (12) was evaluated for various values of d and m. $E\{\text{Pr}(C)\}$ was approxximated by computing $\text{Pr}(C)$ for values of x_i ranging from 0 to 1, and averaging. The results of these evaluations are shown in Fig. 5, and compared to $\text{Pr}(M)$ as well. (Also, see Fig. 13.) Note the dramatic tradeoff

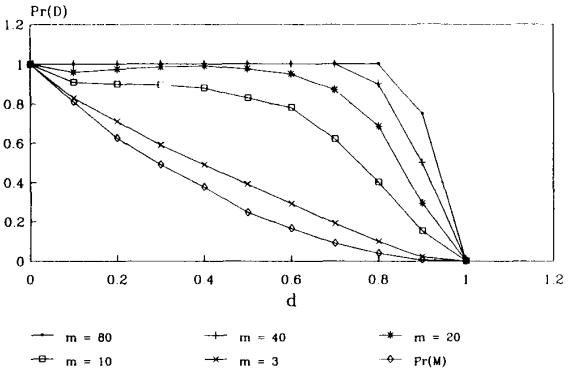

Fig. 5. $\text{Pr}(D)$ as a function of threshold d for several values of m. $\text{Pr}(D)$ is that probability that two lines segments can be rank ordered either directly by measurement or indirectly by inference.

between the requirement for a small threshold and the number of line segments being compared. For example, data from a comparator with a threshold of $d = 0.8$ will contain 96% "ties." This data might seem of little use in rank ordering lengths. However, for $m = 80$ its use leads to a near 100% unambiguous rank ordering of the length of the lines.

Another view of this tradeoff is of one between the quality and the quantity of data, where a large quantity of data requires, in turn, speed of computation. Lower quality data, as might come from less precise comparisons, is adequate if it is available in sufficient quantity and the computational power exists to carry out the inferencing required in an acceptable amount of time.

The results for $m = 40$ and $m = 80$ show a striking insensitivity to d over a substantial part of its range. The $m = 20$ curve displays a lower value of $\text{Pr}(D)$ for $d = 0.05$ than for $d = 0.4$, an initially rather surprising result. This is also true, though not so dramatically, for $m = 40$ and $m = 80$, and apparently results from the ability to infer a complete order in Fig. 2(b) and the inability to do likewise in Fig. 2(c). Note that a similar result appears in Fig. 13. This phenomena of more precision not necessarily leading to more information occurs elsewhere in our results.

IV. Incomplete Measured Data Sets

We have discovered that simple rules of inference can prove a powerful tool in determining rank by length among a set of lines when measurement information is restricted to pairwise comparisons even when the comparator has a nonzero threshold and therefore produces many ties. So far, however, we have only considered the case of a complete measured data set. We next explore the far broader use of inference in determining rank order among a set of m lines given that less than the complete set of $m(m-1)/2$ comparisons is available. We will first briefly explore the use of inference in the case of a zero threshold comparator, and then consider the more general case of a nonzero threshold. It will be necessary to significantly expand the set of relationships. We remind the reader that the importance of rank determination, and therefore

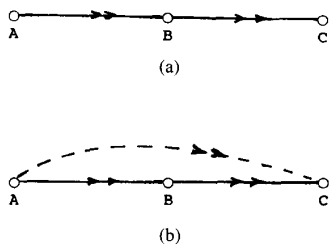

Fig. 6. Digraph representation of measured and inferred results for pairwise comparisons. (a) Measured results. (b) Measured plus inferred results.

of inference procedure, is increased representational accuracy. Inference procedures make explicit all the information that is implicit in a measured data set.

A. Inference with Precise, Incomplete Data

With a zero threshold a comparison of lines A and B can lead only to $x_a \gg x_b$ or $x_b \gg x_a$. The relationship $\gg$, "measurably longer than," is transitive, therefore the basic rule of inference is that $x_a \gg x_b$ and $x_b \gg x_c$ implies $x_a \gg x_c$. This rule is illustrated in Fig. 6.

Repeated use of this rule of inference allows all the information implied by a set of measurements to be determined. Fig. 7 illustrates how the four measurements

$$x_1 \gg x_2$$
$$x_2 \gg x_5$$
$$x_3 \gg x_1$$
$$x_5 \gg x_4$$

allow the remaining six pairwise comparisons to be inferred. (With $m = 5$ there are $5 * 4/2 = 10$ total comparisons possible.) The result is equivalent to what would have been obtained through the complete set of ten measurements. This inference procedure is easily developed as an algorithm based on the digraph concept of reachability [12], [13]. As an operational alternative to a digraph, the information gained by measurement can be stored in the measurement matrix M, a variant of an adjacency matrix [12], [13], where $m(i,j) = 1$ if $x_i \gg x_j$, as determined by measurement, and $m(i,j) = 0$ otherwise. For m lines, M is an $m \times m$ matrix. We then compute the inference matrix Q, where

$$Q = B[M + M^2 + M^3 + \cdots + M^{m-1}]. \qquad (13)$$

The (ij)th element of M^k is the number of different paths of length k from node i to node j. The possible path lengths, without cycles, range from 1, corresponding to direct measurement, to $m - 1$. The existence of even one path of whatever length from node i to node j implies line I is measurably longer than line J. $B[]$ is the matrix boolean operation; all nonzero matrix elements are set equal to one(1). Therefore $q(i,j) = 1$ means that $x_i \gg x_j$, determined either directly by measurement or indirectly through inference. Note that $q(i,j) = 0$ corresponds to either $x_i \ll x_j$, $x_i = x_i$, or to no knowledge of the relative lengths of I and J. More

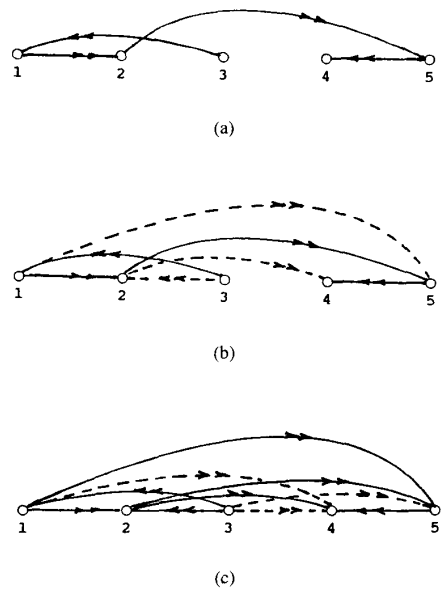

Fig. 7. Digraph representation of complete ordering according to length of five lines. (a) Measured results. (b) Measured results plus inferred results based on direct application of Fig. 2. (c) Measured results plus all inferred results based on continued application of Fig. 2.

complete notation would allow these outcomes to be separated but is not necessary at this point. In closing this section we also note that the rules of matrix multiplication plus the rules of the boolean operation together provide rules for simplifying parallel and serial connections of digraph branches. This will be elaborated in the next section.

B. Relationships for Imprecise, Incomplete Data

We consider the more general case where the data is obtained with a comparator having a nonzero threshold. Previously, a simple example of inference under this condition was presented. We now develop more general rules of inference. Consideration of a number of sample problems indicates that there are five different relationships between two lines that must be taken into account. The digraph branch associated with each of these relationships is given a different algebraic gain and the digraph model becomes a weighted digraph or network model [12], [13]. These relationships and associated gains are listed below, and the corresponding network branch symbols are indicated in Fig. 8.

i) $x_i \gg x_j$; I measurably longer than J; $x_i > x_j + d$; gain = $a1$.

ii) $x_i > x_j$; I longer than J, the difference may or may not be measurable; gain = $a2$.

iii) $x_i \not> x_j$; I longer than J, the difference is not measurable; $x_j < x_i < x_j + d$; gain = $a3$.

iv) $x_i \| x_j$; I and J not not measurably different in length; $x_j - d < x_i < x_j = d$; gain = b.

v) $x_i) | xj$; if I is longer than J the difference may or may not be measurable, if I is shorter than J the difference is not measurable; either $x_i > x_j$ or $x_j - d < x_i < x_j$; gain = c.

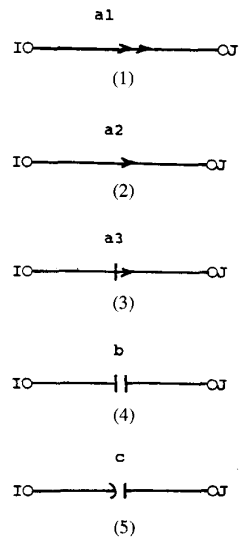

Fig. 8. The five branch types;their branch gains and symbols. (a1) $x_i >> x_j : I$ measureably longer than J. (a2) $x_i > x_j : I$ longer than J. (a3) $x_i| > x_j : I$ longer than J; difference is not measurable. (b) $x_i||x_j : I$ and J not measureably different in length. (c) $x_i > |x_j : I$ if I is longer than J difference may or may not be measurable; if J is longer than I difference is not measurable..

TABLE I
SERIES BRANCH REDUCTION
BRANCH INFERENCE ALGEBRA

	$a1$	$a2$	$a3$	$\underline{a1}$	$\underline{a2}$	$\underline{a3}$	b	c	$\underline{c}$
$a1$	$a1$	$a1$	$a1$	0	0	$a2$	$a2$	$a2$	0
$a2$		$a2$	$a2$	0	0	c	c	c	0
$a3$			$a2$	$\underline{a2}$	$\underline{c}$	b	c	c	0
$\underline{a1}$				$\underline{a1}$	$\underline{a1}$	$\underline{a1}$	$\underline{a2}$	0	$\underline{a2}$
$\underline{a2}$					$\underline{a2}$	$\underline{a2}$	$\underline{c}$	0	$\underline{c}$
$\underline{a3}$						$\underline{a2}$	$\underline{c}$	0	$\underline{c}$
b							0	0	0
c								0	0
$\underline{c}$									0

Notes: 1) 0 indicates no inference can be made. 2) Table is symmetrical since $ab = ba$, etc.

TABLE II
PARALLEL BRANCH REDUCTION BRANCH INFERENCE ALGEBRA

	$a1$	$a2$	$a3$	$\underline{a1}$	$\underline{a2}$	$\underline{a3}$	b	c	$\underline{c}$
$a1$	$a1$	$a1$	-	-	-	-	-	$a1$	-
$a2$		$a2$	$a3$	-	-	-	$a3$	$a2$	$a3$
$a3$			$a3$	-	-	-	$a3$	$a3$	$a3$
$\underline{a1}$				$a1$	$\underline{a1}$	-	-	-	$\underline{a1}$
$\underline{a2}$					$\underline{a2}$	$\underline{a3}$	$\underline{a3}$	$\underline{a3}$	$\underline{a2}$
$\underline{a3}$						$\underline{a3}$	$\underline{a3}$	$\underline{a3}$	$\underline{a3}$
b							b	b	b
c								c	b
$\underline{c}$									$\underline{c}$

Notes: 1) - indicates parallel combination cannot occur. 2) Table is symmetrical since $a + b = b + a$, etc.

Relationships (i) and (iv) can arise either through measurement or inference, whereas the other relationships arise only through inference. Relations (i), (ii), (iii), and (v) are directional or asymmetric; complementary relationships exist and, unlike in the previous section, must be accounted for. These are indicated by underscoring the gains; thus $\underline{a1}$ corresponds to I measurably shorter than J. Relationship (iv) is symmetric and is its own complement. Only relationship (i) and its complement are transitive.

C. Branch Inference Algebra

As in the case of the zero threshold analysis, we determine rules for the simplification of series and parallel branches in terms of their gains. The rules are somewhat more complex since with four asymmetric and one symmetric gain we have $9 \times 9 = 81$ combinations to consider in each case. These are summarized in Tables I and II. As an example, assume an $a3$ branch from I to J, meaning $x_j < x_i < x_j + d$, in series with an $\underline{a1}$ branch from J to K, $x_j < x_k - d$. Thus $x_i < x_j + d$ and $-x_k < -(x_j + d)$. Adding, we obtain $x_i - x_k < 0$ or $x_i < x_k$, which corresponds to an $\underline{a2}$ branch. As a second example assume an $a2$ branch in parallel with a b branch, both running from I to J. Then $x_i > x_j$ and $x_j - d < x_i < x_j + d$. These combine to $x_j < x_i < x_j + d$, an $a3$ branch. Note that the case of zero threshold could be handled as a special case where only branches of gain $a1$ (and $\underline{a1}$) are allowed.

D. Matrix Methods for Inferencing

The goal is to achieve a generalization of (13) allowing for nonzero thresholds. This can be done by constructing a measurement matrix in which the branches are represented by their gains. Note that the complements must be included so that all the algebraic rules of Tables I and II can be implemented. While it is possible to parallel equation (13) by first computing the matrix power series and then applying the branch inference algebra rules, to avoid the resultant long algebraic expressions it is more efficient to apply the reduction rules at each step. An even more efficient algorithm, requiring fewer matrix multiplications, follows.

(i) With M the measurement matrix determine the matrix

$$R(1) = (I + M) \tag{14}$$

where I is the $m \times m$ identity matrix.

(ii) Determine

$$R(2) = S[R(1)^2] \tag{15}$$

where $S[]$ is the branch algebra reduction operator, analogous to the Boolean operator of (13). It indicates the implementation of all the rules of Tables I and II.

(iii) Continue determining

$$R(i) = S[R(i-1)^2] \tag{16}$$

until either $R(i) = R(i-1)$ or $2^i \geq m$.

(iv) Then

$$Q = R(i). \tag{17}$$

This algorithm requires one additional algebraic rule, namely that 1 + (any algebraic expression) = 1. The gain 1 corresponds to a line segment being identical in length to itself.

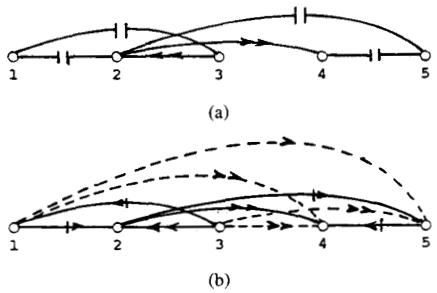

Fig. 9. Example 1—Measured and inferred results based on comparator with nonzero threshold. (a) Measured results. (b) Measured plus inferred results.

E. Matrix Inferencing: Example 1

Fig. 9(a) is a network indicating a system of five lines in for which six of the possible ten measurements were made. In only two of these measurements did the difference in length exceed the comparator threshold. This set of data can be represented by the matrix

$$M = \begin{bmatrix} 0 & b & b & 0 & 0 \\ b & 0 & \underline{a1} & a1 & b \\ b & a1 & 0 & 0 & 0 \\ 0 & \underline{a1} & 0 & 0 & b \\ 0 & b & 0 & b & 0 \end{bmatrix}. \tag{18}$$

From this we use the algorithm and inference rules described above to compute the sequence of matrices:

$$R(0) = \begin{bmatrix} 1 & b & b & 0 & 0 \\ b & 1 & \underline{a1} & a1 & b \\ b & a1 & 1 & 0 & 0 \\ 0 & \underline{a1} & 0 & 1 & b \\ 0 & b & 0 & b & 1 \end{bmatrix} \tag{19}$$

$$R(1) = \begin{bmatrix} 1 & a3 & \underline{a3} & a2 & 0 \\ \underline{a3} & 1 & \underline{a1} & a1 & a3 \\ a3 & a1 & 1 & a1 & a2 \\ \underline{a2} & \underline{a1} & \underline{a1} & 1 & \underline{a3} \\ 0 & \underline{a3} & \underline{a2} & a3 & 1 \end{bmatrix} \tag{20}$$

$$R(2) = \begin{bmatrix} 1 & a3 & \underline{a3} & a1 & a2 \\ \underline{a3} & 1 & \underline{a1} & a1 & a3 \\ a3 & a1 & 1 & a1 & a2 \\ \underline{a1} & \underline{a1} & \underline{a1} & 1 & \underline{a3} \\ \underline{a2} & \underline{a3} & \underline{a2} & a3 & 1 \end{bmatrix} \tag{21}$$

$$Q = R(2). \tag{22}$$

The Q matrix is represented by the digraph in Fig. 9(b), except that the loops of gain 1 from each node to itself are suppressed.

In this case the result is a complete digraph and a complete rank ordering of the lines. The jth line can be scored by adding the number of a's in the jth row and subtracting the number of a's in the jth column. These scores can then be arranged to rank the line lengths. For this example we obtain

Line	Score
3	+4
1	+2
2	0
5	−2
4	−4

with higher scores corresponding to longer lines. From the branch gain definitions note that only $a2$ and c represent incomplete information. A 0 gain would of course represent incomplete information as well. In this example additional information can be obtained only by direct comparison of line 1 and line 5. This will, however, not effect the rankings.

F. Matrix Inferencing: Example 2

Consider a second example, represented in Fig. 10(a) and by the measurement matrix

$$M = \begin{bmatrix} 0 & b & a1 & 0 & 0 & b \\ b & 0 & b & 0 & 0 & 0 \\ \underline{a1} & b & 0 & 0 & a1 & 0 \\ 0 & 0 & 0 & 0 & b & b \\ 0 & 0 & \underline{a1} & b & 0 & 0 \\ b & 0 & 0 & b & 0 & 0 \end{bmatrix} \tag{23}$$

This results in

$$Q = \begin{bmatrix} 1 & a3 & a1 & a1 & a1 & a3 \\ \underline{a3} & 1 & a3 & a2 & a1 & b \\ \underline{a2} & \underline{a3} & 1 & a3 & a1 & \underline{a3} \\ \underline{a1} & \underline{a2} & \underline{a3} & 1 & a3 & \underline{a3} \\ \underline{a1} & \underline{a1} & \underline{a1} & \underline{a3} & 1 & \underline{a1} \\ \underline{a3} & b & a3 & a3 & a1 & 1 \end{bmatrix} \tag{24}$$

and is represented by the digraph in Fig. 10(b). A ranking can be computed,

Line	Score
1	+5
2	+2
6	+2
3	−1
4	−3
5	−5

and this time line 2 and line 6 remain tied. Note that they are joined by an inferred b branch. Therefore, it would do no good to directly compare them; a tie would result. However, the $a2$ joining line 2 and line 4 represents incomplete information. If that measurement is made and line 2 is measurably longer than line 4 the tie would be resolved; line 2 would be longer than line 4, though not necessarily measurably so. If line 2 is not measurably longer than line 4 the tie between lines 2 and 6 cannot be resolved given the size of the comparator threshold.

V. Pattern Representation: Some Reconstruction Results

To determine the accuracy of pattern representation based on nonmetric information we utilized MDS to reconstruct the patterns. The reconstructions could then be compared to the original patterns to determine error.

A. Reconstructions Based on Precise, Complete Information

We initially explored the reconstruction accuracy of the nonmetric algorithms by reconstructing various 2-D and 3-D patterns based on precise, crisp, correct, complete information. Patterns ranging from 17 to 75 points were used. (75

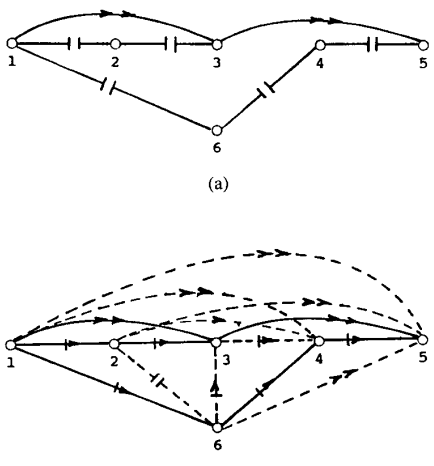

Fig. 10. Example 2—Measured and inferred results for comparator with nonzero threshold. (a) Measured results. (b) Measured plus inferred results.

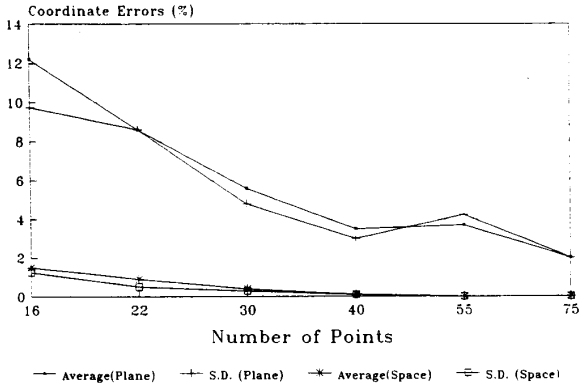

Fig. 11. The reconstruction accuracy of the nonmetric ALSCAL algorithms. Results for two configurations are shown, a three-dimensional random figure, labelled Space, and a planar, but otherwise random, configuration located in three-dimensional space, labelled Plane. In each case, as the number of points in the configuration increases the mismatch between the input and output configurations rapidly vanishes to zero.

points gives rise to almost 4 million comparisons.) Since the reconstruction is independent of scale, position and rotation, accuracy was determined by orienting and scaling the reconstruction until it had the best overlay with the original pattern.

Fig. 11 shows the average and the standard deviation of the percentage mismatch between the original and the reconstructed configurations for an increasing number of points in two three-dimensional test configurations. One is a random configuration labelled Space. The second is a planar configuration, labelled Plane. The MDS algorithm is not provided the information that the configuration is in fact planar. If this information were provided the reconstruction would naturally be more accurate [1]. The larger coordinate errors in Plane may result from coordinate errors normal to the plane of the planar configuration producing relatively little stress.

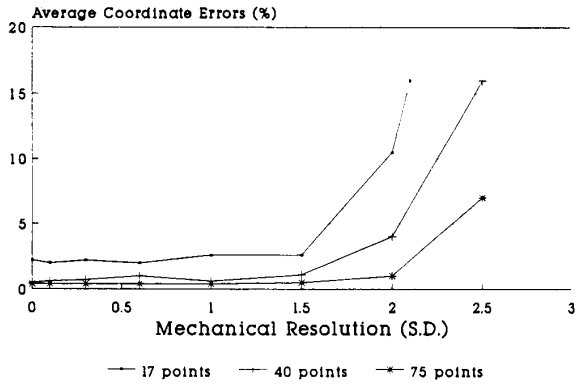

Fig. 12. Results indicating that the accuracy of reconstruction is virtually unaffected as the comparator becomes increasingly imprecise until large resolution levels (thresholds) are reached.

The results displayed in Fig. 11 indicate that 100% reconstruction accuracy can be approached with arbitrary closeness as more points are included in the configuration. This improvement results from the increasing constraint on the position of each point that occurs as the number of distance comparisons increases; eventually a point can be moved only a very small distance without producing a change in the rank ordered list.

B. Reconstructions Based on Imprecise, Complete Information

The previous examples were based on data obtained from precise comparisons. However, data obtained from imprecise comparisons, those with nonzero thresholds, contain "ties." In this research we are interested in how this threshold level affects reconstruction accuracy. In order to compensate for the "size" of the point configuration the threshold, or level of resolution, was standardized as a percentage of the standard deviation of the differences in interpoint distances to be compared. Zero percentage therefore corresponded to precise comparisons.

The frequency of ties depends not only on the resolution level but on the distribution of lengths as well. Tchebyshev's inequality insures that regardless of the distribution, resolution levels of 2 standard deviations must produce more than 75% tied comparisons. Even with this high percentage of ties we find that the MDS routine works well.

Fig. 12 indicates that rather large thresholds have virtually no effect on the accuracy of reconstruction until resolution levels approach 2 standard deviations. (Even then, increased numbers of points can reduce this error.) This surprising result is explained by the results of Section III, shown in Fig. 5. Recall that these results involved the roles of measurement and inference in determining the relative lengths of lines for complete but imprecise data. Fig. 13 presents simulation results complementing the theoretical results shown in Fig. 5. The scoring system used by Chuang [1] to input the results of the comparisons to the MDS algorithm implicitly carries out the required inferencing procedures.

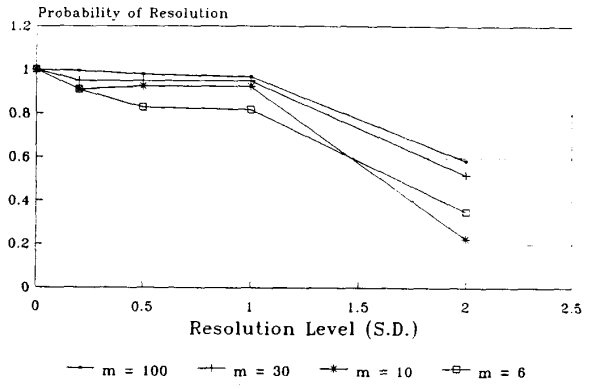

Fig. 13. The probability of resolving a comparison in a complete data set either by measurement or by inference. These results are based on simulations. Note that for larger numbers of comparisons this probability remains near unity until the comparator resolution level becomes relatively large. $m = 100$ corresponds to a configuration of approximately 15 points.

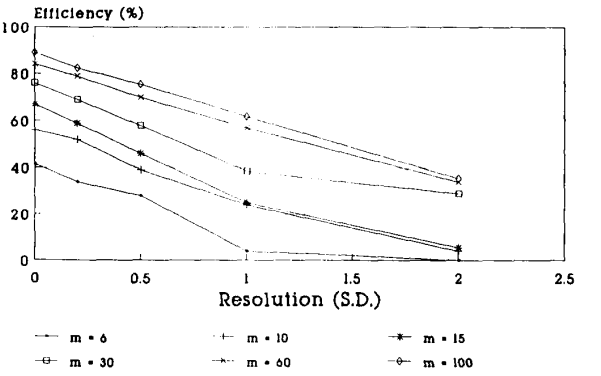

Fig. 14. The efficiency of data enhancement as a function of resolution level and the number of lines, m.

C. Reconstructions from Incomplete Measured Data

So far we have considered complete measured data sets. For a pattern made up of n points this is approximately $(1/8)n^4$ measurements. Fortunately, as we have seen, the significant amount of redundancy present in this data allows the outcomes of many of those comparisons not directly made to be inferred, using the procedures outlined, and thus the data set and the rank ordering made more complete. This process is referred to as *data enhancement*. Specifically, data enhancement involves:

1) wherever possible using inference procedures to make the data set more precise;
2) wherever possible using inference procedures to make the data set more complete;
3) whenever possible making additional measurements and repeating 1. and 2. until the data set is complete and precise or can be resolved no further.

At the completion of this procedure the data set is used to generate the rank ordered list required by MDS.

The effectiveness of the data enhancement algorithm was defined as an efficiency measure between 0% and 100%. For m interpoint distances the minimum possible number of direct comparisons that could possibly establish a complete rank ordering is $\min = m - 1$; the maximum number of comparisons possible is $\max = m(m-1)/2$. Efficiency was determined through simulations. A simulation was begun by making $m-1$ direct comparisons, the selection criteria being only that all the interpoint distances were represented, and then applying all the rules of inference to be described later. If a complete ranking was achieved this represented 100% efficiency. Generally, additional measurements, randomly chosen, were made one at a time, with addititional inferences made after each measurement, until either the complete rank ordering resulted or it was determined that no further improvement in rank ordering was possible. The required number of measurements was then linearly interpolated as an efficiency between 0%, if all $m(m-1)/2$ measurements were required, and 100%, if only the original $m-1$ were needed. The % efficiency E is given by

$$E = [(\max - X)/(\max - \min)] * 100 \qquad (25)$$

where X is the number of direct measurements made. Fig. 14 indicates these efficiencies as a function of m, the number of lines, and the comparator resolution level.

Recall that $m = 100$ corresponds to only about 15 points. Higher numbers of points will have somewhat higher efficiencies. A comparison of Figs. 12 and 14 indicates that for a fifteen point object and a resolution level of 0.5 standard deviations we can expect a 75% efficiency and a pattern representation sufficient for a reconstruction accuracy of about 98%.

D. Less than Perfect Reconstructions

To this point we have considered our goal to be a "perfect" representation of the pattern. This would be demonstrated, in practice, by the ability to reconstruct the pattern with virtually no error. However, the price to be paid in terms of information quality and quantity requirements may render this goal undersirable. Instead, a representation consistent with a specified reconstruction accuracy, say 97%, may be sufficient. What effect does this have on the number of direct comparisons that will actually be required, and what effect, if any, does the comparator threshold have on this number?

Simulation results revealed a rather unexpected and intriguing finding. Setting the reconstruction accuracy as 96.5 – 97.5%, the cycles of measurement and inference described previously were carried out. Fig. 15 summarizes the results. For $n = 40$ ($m = 780$), efficiencies of as high as 90% are obtained; roughly only 10% or 30,000 out of a possible 300,000 measurements need be made. In particular note the very surprising result that the highest efficiency is consistently achieved not with a perfectly precise comparator but one with a resolution limit of about 0.5 to 1.0 standard deviations. (Remember that standard deviations refers to variations in interpoint distances.) Thus, from the point of view of measurement efficiency, a "desirable" level of imprecision exists and is proportional to the size of the differences it is observing. This finding will be considered in the Discussion.

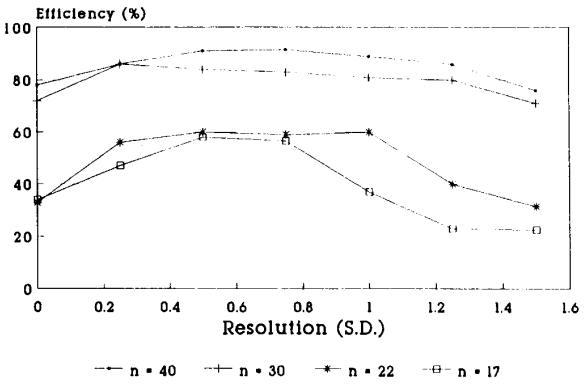

Fig. 15. Results indicating that meeting a goal of approximately 97% reconstruction accuracy with the fewest measurements requires an imprecise comparator with a resolution level of 0.5 to 1.0 standard deviations.

E. Reconstructions with Noncrisp Comparators

Finally, we considered pattern representation using noncrisp comparator data. In a crisp comparator if a difference exceeds the threshold, no matter by how small an amount, this result is correctly determined. No errors occur. Real data reflect errors due to noise, drift and other problems, and are therefore not crisp. We have made limited investigations of the effect of this noncrispness on pattern representation, again through reconstruction accuracy [1], [14], and found that it introduces a residual reconstruction error, one that is apparently not eliminated or even further reduced by an increase in the number of points. The relationship between the degree of noncrispness and the size of the residual error remains to be investigated. Our simulations produced a residual error of approximately 3%, that is the reconstructions were 97% accurate.

VI. Conclusion

This paper has described how patterns, specifically point configurations, may be represented as rank ordered lists of interpoint distances. Reconstructions of patterns from these lists using multidimensional scaling (MDS) demonstrated that adequate information is stored in these lists. However, reconstruction is often not the purpose of pattern representation. Instead, pattern classification or recognition may be the goal.

How might this be done? Typically, the rank ordered lists for a number of templates would be stored. These lists have the advantage of being independent of exact size or orientation. The stresses between the list describing the object to be classified and the templates could then be determined and classification based on the result. The major problem is establishing a correspondence of the points that define the observed pattern and the alternative template to which it will be compared.

The efficient representation of patterns is clearly an important component of viable machine vision systems. (See [15] and the articles that follow for recent work in this area.) The approach described here, with its use of nonmetric, imprecise, incomplete information suggests the possibility of a very different "technology" than that generally being pursued. Rather than being based on precise, generally expensive optical devices, it points toward the use of imprecise, inexpensive optical devices backed up by significant computing capability.

Speculation on this potential role of nonmetric measurement in machine vision systems arises from the recognition that the brain is unlikely to store information in any precise numerical form. In fact, most people are rather poor at estimating absolute numerical measures, such as the size of even a familiar object in common units. They are, however, far better at judging the relative measures of two objects, such as which stick is longer.

Interestingly, this ability to discriminate is not absolute; instead, the resolution threshold is itself relative to the size of the objects being discriminated. This discrimination phenomena, is known as "Weber's Ratio." The adjusting of resolution levels plays a central role in the analysis of images using the multiscale method, a technique being applied in medical imaging [16]. The advantage here is that larger resolution levels facilitate recognition of larger scale structures by simplifying them. There is evidence that human vision operates in this multiscale manner [17]. Generally, the existence of adjustable levels of resolution is reminiscent of the results obtained in Section V, Part D. Recall that the most efficient use of nonmetric measurement information for less than perfect representations dictated a level of imprecision proportional to the object sizes.

The present scheme envisions only three outcomes of a comparison. It might be profitable to consider additional categories, as an example comparisons with outputs "much shorter," "somewhat shorter," tie, "somewhat longer," and "much longer." Both the "somewhat" and "much" categories represent "measurable" differences, but neither is included in the other. These can be viewed as either fuzzy or crisp concepts [5]. This increase in comparison classifications might lead to more powerful inferencing algorithms and a reduced need for direct comparison measurements. As an example, if A is "much longer" than B and C is "somewhat longer" than B, we can conclude that A is longer than C, but not necessarily measurably longer. The applicability of these methods awaits further research.

Finally, while the work presented here was concerned with representational efficiency, that is, developing a useful rank ordered list from relatively few measurements, little was said about the sequence of measurements to make. For nonmetric data this remains a largely uninvestigated problem, though a start has been made for metric data [18].

Appendix

Relationship to Semi-Orders

Interest by psychologists in the phenomena of indifference, as measured by a threshold value or JND, led to the development of a mathematical relationship known as a semi-order. Scott and Suppes [19] provided a useful definition of this relationship (see also [13], [20]–[22]). The rules developed in

this paper incorporate this definition. Restated in terms of line segment lengths a relationship is a semi-order if for any four line segments x_1, x_2, x_3, x_4 the following axioms are satisfied.
Axiom 1. not $(x_1 \gg x_1)$
Axiom 2. $(x_1 \gg x_2$ and $x_3 \gg x_4)$ implies $(x_1 \gg x_4$ or $x_3 \gg x_2)$
Axiom 3. $(x_1 \gg x_2$ and $x_2 \gg x_3)$ implies $(x_1 \gg x_4$ or $x_4 \gg x_3)$

Axiom 1 is self evident. To examine axiom 2 we consider three cases. Case 1 is $x_3 \gg x_2$, in which case axiom 2 is satisfied. Case 2 is $x_2 \gg x_3$. But then $x_1 \gg x_2 \gg x_3 \gg x_4$ implies $x_1 \gg x_4$ and axiom 2 is satisfied. Finally, Case 3 is $x_2 \| x_3$. Then $x_1 \gg x_2$ and $x_2 \| x_3$ implies $x_1 > x_3$, and $x_1 > x_3$ and $x_3 \gg x_4$ implies $x_1 \gg x_4$. Again, axiom 2 is satisfied. Note that in terms of the branch algebra presented Case 3 can be summarized as

$$(a1)b(a1) = (a2)(a1) = a1.$$

Axiom 3 requires considering four cases. The first two cases involve line 4 and line 2 being measurably different, the second two cases involve the lines being not measurably different. Case 1 is $x_2 \gg x_4$. Then $x_1 \gg x_2 \gg x_4$ implies $x_1 \gg x_4$. Case 2 is $x_4 \gg x_2$; thus, $x_4 \gg x_2 \gg x_3$ implies $x_4 \gg x_3$. In each case axiom 3 is satisfied. Case 3 is $x_2| > x_4$. It follows that $x_1 \gg x_2| > x_4$ implies $x_1 \gg x_4$. Finally, Case 4 is $x_4| > x_2$ leading to $x_4| > x_2 \gg x_3$ implying $x_4 \gg x_3$. Again, axiom 3 is met in both cases.

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