Lifetime and compactness of range for super-Brownian motion with a general branching mechanism

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Abstract

Let X be a super-Brownian motion with a general (time-space) homogeneous branching mechanism. We study a relation between lifetime and compactness of range for X. Under a restricted condition on the branching mechanism, we show that the set X survives is the same as that the range of X is unbounded. (For α-branching super-Brownian motion, $1 < \alpha \leqslant 2$, similar results were obtained earlier by Iscoe (1988) and Dynkin (1991).) We also give an interesting example in that case X dies out in finite time, but it has an unbounded range. © 1997 Elsevier Science B.V.

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0. Introduction

A super-Brownian motion $X = (X_t, P_\mu)$ is a branching measure-valued Markov process. It can be obtained as a continuous limit of branching Brownian particle systems. Among them the most well-studied is the α-branching super-Brownian motion. Here α, $1 < \alpha \leqslant 2$, is the branching mechanism of X, and it is related to the distribution of number of offspring for each particle in the particle system setting. Many deep results have been obtained in the past decade on the sample path behavior of this process. In particular it is easy to show that, P_μ-a.s., the process X dies out in finite time (which means that there exists a finite random variable $\mathscr{L}$ satisfying $X_t = 0$ for all $t \geqslant \mathscr{L}$). Moreover the range of X is, P_μ-a.s., compact. (For $\alpha = 2$, see Iscoe, 1988; Dynkin, 1991 for general α.)

In this paper we consider a super-Brownian motion $X = (X_t, P_\mu)$ with a general (time–space) homogeneous branching mechanism ψ as given in (1.6). Our objective in

this work is to establish a relation between lifetime and compactness of range for the process X. In Section 1, we first recall a passage from Brownian particle systems to super-Brownian motions, and review a relation between a super-Brownian motion and the Cauchy problem for the corresponding p.d.e. In terms of ψ, we then estimate in Section 2 the probability that X survives. For every bounded function (function always means positive function) f, we study in Section 3 the asymptotic behavior of $\langle f, X_t \rangle$. (We write $\langle f, v \rangle$ for the integral of f with respect to the measure v.) In particular, we obtain that the total mass process of X converges, a.s., to either 0 or ∞. In Section 4, we show that under a restricted condition on ψ, the set that X survives is the same as that X has an unbounded range. Finally, we give an interesting example in which X has an unbounded range, even though it dies out eventually.

1. Super-Brownian motion with a general branching mechanism

A branching Brownian particle system $Z = \{Z_t, t \geq 0\}$ is a probabilistic model of a system of particles living in the space $\mathbb{R}^d$, dying and producing at their death time and death location random number of offspring. We assume that

(i) each particle has an exponentially distributed lifetime;

(ii) at its death time and death location, a particle is replaced by a random number of offspring;

(iii) during its lifetime, the law of the motion of a particle is governed by the law of the standard Brownian motion;

(iv) all particle lifetimes, motions, and branching are independent of one another, and the particles alive at time t are indistinguishable.

For fixed $t > 0$, $Z_t(B)$ is the number of particles at time t in a set $B \subset \mathbb{R}^d$. Then, Z_t is a point measure on $\mathbb{R}^d$ and the distribution of Z_t is determined by the three parameters q, k and φ. Here, q is a random measure on $\mathbb{R}^d$ describing the initial distribution of the particle system, k the killing rate for each particle and φ a generating function describing the distribution of number of offspring. We call Z a branching Brownian particle system with parameters (q, k, φ).

Let μ be a fixed finite measure on $\mathbb{R}^d$ and write q_μ for the Poisson random measure with the intensity μ. If $Z = (Z_t, P_{q_\mu})$ is the Brownian particle system with parameters (q_μ, k, φ), then for every positive bounded function f, we have

$$P_{q_\mu} e^{-\langle f, Z_t \rangle} = e^{-\langle u_t, \mu \rangle}, \tag{1.1}$$

where u satisfies

$$u(t,x) + \Pi_x \left[\int_0^t k \tilde{\varphi}(u(t-s, \xi_s)) \, ds \right] = \Pi_x [1 - e^{-f(\xi_t)}]. \tag{1.2}$$

Here $u_t(x) = u(t,x)$, $\tilde{\varphi}(z) = \varphi(1-z) - 1 + z$ and $\xi = (\xi_t, \Pi_x)$ is a d-dimensional Brownian motion. (We write PW for the expected value of the random variable W with respect to the probability measure P.)

Let $b_1 \in \mathbb{R}$, $b_2 \geqslant 0$, m a measure on $[1, \infty)$, and n a measure on $(0, 1)$. We assume further that

$$\int_1^\infty um(\mathrm{d}u) < \infty$$

and

$$\int_0^1 u^2 n(\mathrm{d}u) < \infty.$$

Set $c_1 = |b_1|$, $c_2 = 2b_2 + \int_0^1 u^2 n(\mathrm{d}u)$ and $c_3 = m([1, \infty))$. For every $\beta > 0$, let Z^β be a branching Brownian particle system with parameters $(q_{\mu/\beta}, k_\beta, \varphi_\beta)$, where

$$k_\beta = (c_1 + c_2)\frac{1}{\beta} + c_3\beta,$$

$$\varphi_\beta = \frac{1}{k_\beta}\left\{ c_3\beta + \frac{c_1 + c_2}{\beta}z + b_1(1 - z) + \left(c_1 + \frac{b_2}{\beta}\right)(1 - z)^2 \right.$$
$$\left. + \beta\left[\int_1^\infty (\mathrm{e}^{-u(1-z)/\beta} - 1)m(\mathrm{d}u) + \int_0^1 \left(\mathrm{e}^{-u(1-z)/\beta} - 1 + \frac{u(1 - z)}{\beta}\right)n(\mathrm{d}u)\right] \right\}.$$

As $\beta \downarrow 0$, then Z_t^β converges to a measure-valued diffusion X_t, the super-Brownian motion. Moreover, for every positive bounded function f on $\mathbb{R}^d$, (1.1) and (1.2) imply that the function

$$v_f(t, x) = -\log P_x \mathrm{e}^{-\langle f, X_t \rangle}, \tag{1.3}$$

(we write P_x for P_{δ_x}) satisfies the integral equation

$$v(t, x) + \Pi_x\left[\int_0^t \psi(v(t - s, \xi_s))\,\mathrm{d}s\right] = \Pi_x f(\xi_t), \tag{1.4}$$

where

$$\psi(z) = b_1 z + b_2 z^2 + \int_0^1 (\mathrm{e}^{-uz} - 1 + uz)n(\mathrm{d}u) + \int_1^\infty (\mathrm{e}^{-uz} - 1)m(\mathrm{d}u). \tag{1.5}$$

(For more detail, see, Dynkin, 1993.) Set $a = b_1 - \int_1^\infty um(\mathrm{d}u)$, $b = b_2$ and $l(\mathrm{d}u) = m(\mathrm{d}u) + n(\mathrm{d}u)$. Then (1.5) becomes

$$\psi(z) = az + bz^2 + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz)l(\mathrm{d}u), \tag{1.6}$$

where $a \in \mathbb{R}$, $b \geqslant 0$ and $\int_0^\infty \min(u, u^2)l(\mathrm{d}u) < \infty$. We call X a super-Brownian motion with parameter ψ. Throughout this paper $X = (X_t, P_\mu)$ always denotes a super-Brownian motion with the parameter ψ of the form (1.6) and we write

$$\psi(z) = az + \phi(z), \tag{1.7}$$

where

$$\phi(z) = bz^2 + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz)l(\mathrm{d}u). \tag{1.8}$$

Note that it follows from (1.3) and (1.4) that

$$P_\mu \langle f, X_t \rangle = e^{-at} \Pi_\mu f(\xi_t). \tag{1.9}$$

Therefore, we say X is subcritical if $a > 0$, critical if $a = 0$, and supercritical if $a < 0$. Moreover, by Jensen's inequality and (1.9), we have the estimate

$$v_f(t, x) \leqslant -\log e^{-P_x \langle f, X_t \rangle} = e^{-at} \Pi_x f(\xi_t). \tag{1.10}$$

If f is a bounded measurable function on $\mathbb{R}^d$, then v_f satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u - \psi(u) \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{1.11}$$

Moreover, if f is continuous, then $v_f(t, x) \to f(x)$ as $t \downarrow 0$. (In the case $a \geqslant 0$, see Dynkin, 1993 for a proof and with a little change, the proof there also works for the general cases.) If f is a constant function λ, then $v_f(t, x)$ is independent of x, and we will write $v_\lambda(t)$ instead of $v_f(t, x)$.

If $a \geqslant 0$, set $z_0 = 0$; otherwise, put $z_0 = \max\{z > 0 \mid \psi(z) = 0\}$. (We take $z_0 = \infty$ if $\{z > 0 \mid \psi(z) > 0\} = \emptyset$.) Note that $\psi'(z) = a + 2bz + \int_0^\infty u(1 - e^{-uz}) l(\mathrm{d}u)$ is strictly increasing in z. If $0 < z_0 < \infty$, $\psi(z) < 0$ for $0 < z < z_0$, and $\psi(z) > 0$ for $z > z_0$. If $z_0 = \infty$, then $\psi(z) < 0$ and $\psi(z)$ is decreasing in z.

Proposition 1.1. *Assume $0 \leqslant z_0 < \infty$. We have $v_{z_0}(t) = z_0$ for all $t \geqslant 0$, and if $0 < \lambda_1 < z_0 < \lambda_2$, then*

$$\lambda_1 \leqslant v_{\lambda_1}(t) < z_0 < v_{\lambda_2}(t) \leqslant \lambda_2, \quad \forall t > 0.$$

Proof. To prove the first statement, we assume $v_{z_0}(t_0) < z_0$ for some $t_0 > 0$. Let $\tilde{t_0} = \sup\{t \leqslant t_0, v_{z_0}(t) \geqslant z_0\}$. We have, by assumption, $0 \leqslant \tilde{t_0} < t_0$. Therefore, $v_{z_0}(t) < z_0$ for all $\tilde{t_0} < t < t_0$, and $v_{z_0}(\tilde{t_0}) = z_0$. Since, $v_{z_0}'(t) = -\psi(v_{z_0}(t))$ and $\psi(z) \leqslant 0$ for $z < z_0$, $v_{z_0}(t)$ is increasing in $(\tilde{t_0}, t_0)$. Therefore, we have $v_{z_0}(t) \geqslant v_{z_0}(\tilde{t_0}) = z_0$ for all $\tilde{t_0} < t < t_0$. The contradiction implies that $v_{z_0}(t) \geqslant z_0$ for all $t > 0$. Since $v_{z_0}'(t) = -\psi(v_{z_0}(t)) < 0$, we have $v_{z_0}(t) = z_0$.

Assume that $0 < \lambda_1 < z_0 < \lambda_2$. Note that (1.9) implies that $P_\mu \{X_t \neq 0\} > 0$ for all t and μ. On $\{X_t \neq 0\}$, we have $\langle \lambda_1, X_t \rangle < \langle z_0, X_t \rangle < \langle \lambda_2, X_t \rangle$. Combining with (1.3) gives $v_{\lambda_1}(t) < z_0 < v_{\lambda_2}(t)$. Moreover, the fact that $\psi(z) \leqslant 0$ for $z \leqslant z_0$ and (1.4) imply that $\lambda_1 \leqslant v_{\lambda_1}(t)$. Similar arguments imply that $v_{\lambda_2}(t) \leqslant \lambda_2$. □

2. Survival probability of X

We first study the total mass process $\langle 1, X_t \rangle$. For every $c > 0$, the function

$$v_c(t) = -\log P_x e^{-\langle c, X_t \rangle}, \tag{2.1}$$

satisfies the ODE

$$v_c'(t) + \psi(v_c(t)) = 0 \quad \text{for all } t > 0, \tag{2.2}$$

and $v_c(0) = c$. (The study of the total mass process via the ODE (2.2) goes back to Kawazu and Watanabe, 1991.) Clearly, for every $t > 0$, the function $v_c(t)$ is increasing in c and so the limit $v_\infty(t) = \lim_{c \to \infty} v_c(t)$ exists. If $z_0 = \infty$, then $\psi(z) < 0$ for all z, and thus $v'_c(t) > 0$ for all $t > 0$. In that case we have $v_\infty(t) = \infty$.

Proposition 2.1. *Assume* $0 \leqslant z_0 < \infty$. *Then* $v_\infty(t) = \infty$ *for all* $t > 0$ *if and only if* ψ *satisfies the following condition*

$$\int_z^\infty \frac{1}{\psi(u)} \, du = \infty, \quad \forall z > z_0. \tag{2.3}$$

Proof. Assume $c > z_0$. Dividing both sides of (2.2) by $\psi(v_c(t))$ gives

$$\frac{v'_c(t)}{\psi(v_c(t))} + 1 = 0.$$

Integrating from 0 to t, we have

$$\int_0^t \frac{v'_c(s)}{\psi(v_c(s))} \, ds + t = 0. \tag{2.4}$$

Put $v_c(s) = u$ and $v'_c(s) \, ds = du$. Then (2.4) becomes

$$\int_c^{v_c(t)} \frac{1}{\psi(u)} \, du + t = 0. \tag{2.5}$$

To prove the sufficiency, we assume ψ satisfies the condition (2.3) and $v_\infty(t_0) < \infty$ for some $t_0 > 0$. By Proposition 1.1, we have $v_\infty(t_0) > z_0$. Letting $c \uparrow \infty$ in (2.5) gives

$$\int_\infty^{v_\infty(t_0)} \frac{1}{\psi(u)} \, du + t_0 = 0, \tag{2.6}$$

which contradicts the assumption. Therefore, $v_\infty(t) = \infty$ for every $t > 0$.

To prove the necessity, we assume that $v_\infty(t) = \infty$ for all $t > 0$ and $c_2 = \int_{c_1}^\infty 1/\psi(u) \, du < \infty$ for some $c_1 > z_0$. It follows from (2.5) that

$$0 < 2c_2 = \int_{v_c(2c_2)}^c \frac{1}{\psi(u)} \, du \leqslant \int_{v_c(2c_2)}^\infty \frac{1}{\psi(u)} \, du \quad \text{for all } c > z_0. \tag{2.7}$$

Since $v_\infty(2c_2) = \infty$, we can pick up a constant $c > z_0$ such that $v_c(2c_2) \geqslant c_1$. Then we have, by (2.7), that

$$2c_2 \leqslant \int_{c_1}^\infty \frac{1}{\psi(u)} \, du = c_2.$$

We get a contradiction and thus $\int_z^\infty \frac{1}{\psi(u)} \, du = \infty$ for all $z > z_0$. $\square$

Proposition 2.2. *If* ψ *does not satisfy condition* (2.3), *then* $v_\infty(t)$ *converges to* z_0 *as* t *goes to* ∞.

Proof. Assume $c > z_0$. Since $v_c(t) > z_0$ for all $t > 0$, we get $v'_c(t) = -\psi(v_c(t)) < 0$. Therefore, $v_c(t)$ is decreasing in t and so is $v_\infty(t)$. Set $z_1 = \lim_{t \to \infty} v_\infty(t)$. Clearly, we

have $z_1 \geqslant z_0$. Letting $c \to \infty$ in (2.5) gives

$$t = \int_{v_\infty(t)}^{\infty} \frac{1}{\psi(u)} \, du,$$

and thus

$$\infty = \int_{z_1}^{\infty} \frac{1}{\psi(u)} \, du.$$

Since ψ does not satisfy condition (2.3), we have $z_1 = z_0$. □

Lemma 2.3. *Let $g(u)$ be a continuous function. Assume that $g(u) > 0$ for all $u \geqslant z_1 > 0$ and there exist two constant $c_1 > 0$ and $c_2, c_2 \geqslant z_1$, such that $g(u) \geqslant 1/c_1 u$ for all $u \geqslant c_2$. Then for every $z > z_1$, we have*

$$\int_z^{\infty} \frac{1}{g(u)} \, du = \infty,$$

if and only if

$$\int_z^{\infty} \frac{1}{g(u) + u} \, du = \infty.$$

Proof. For $u > c_2$, we have $1 \geqslant g(u)/(g(u) + u)$ and, by assumption,

$$\frac{g(u)}{g(u) + u} = \frac{1}{1 + \frac{u}{g(u)}} \geqslant \frac{1}{1 + c_1} > 0.$$

The conclusion follows from the comparison test for integral. □

Definition 1. A function $g : [0, \infty) \to [0, \infty)$ is said to satisfy the condition (L) if for every $z > 0$,

$$\text{(L)} \quad \int_z^{\infty} \frac{1}{g(u)} \, du = \infty.$$

Proposition 2.4. *Assume $0 \leqslant z_0 < \infty$. The function ϕ satisfies the condition (L) if and only if the function ψ satisfies the condition (2.3).*

Proof. If $b > 0$, it is clear that neither ψ satisfies (2.3) nor ϕ satisfies condition (L). Therefore, we consider the case $b = 0$. Without loss of generality, we can assume $|a| = 1$. Note that $(\phi(z)/z) \geqslant c > 0$ for z sufficiently large (see (4.5)). If $a = 1$, our result follows from Lemma 2.3. Assume $a = -1$. The necessity is clear. To prove the sufficiency, we assume $\int_z^{\infty} \frac{1}{\phi} \, du = \infty$. Set $g(z) = \psi(z) = \phi(z) - z$. In this case $(\phi(z)/z) \geqslant c > 1$ (see, e.g., Sheu, 1994, Lemma 4.2), g satisfies the conditions in Lemma 2.3. Lemma 2.3 implies that $\phi(z) = g(z) + z$ satisfies condition (L). □

Let $\mathscr{L} = \sup\{t \geqslant 0, X_t \neq 0\}$ be the lifetime of the super-Brownian motion X. We say X survives if $\mathscr{L} = \infty$.

Theorem 2.5. *Let μ be a finite measure on $\mathbb{R}^d$ and assume $0 \leqslant z_0 < \infty$. If ϕ satisfies condition* (L), *then X survives, $P_\mu - a.s.$; otherwise, we have*

$$P_\mu[X \text{ survives}] = 1 - e^{-z_0\langle 1, \mu \rangle}. \tag{2.8}$$

Proof. Note that

$$P_\mu[X_t \neq 0] = 1 - P_\mu[X_t = 0] = 1 - e^{-v_\infty(t)\langle 1, \mu \rangle},$$

and

$$P_\mu[X \text{ survives}] = \lim_{t \to} P_\mu[X_t \neq 0].$$

Our conclusion follows from Propositions 2.1, 2.2 and 2.4. □

It is clear from a similar argument as above that if $z_0 = \infty$, then X survives, P_μ-a.s. In that case it is easy to see that the range of X is unbounded a.s. Therefore, from this point on, we consider only the case $0 \leqslant z_0 < \infty$. The following result is of independent interest. It was first obtained by Evans and O'Connell (1994) in the case $\psi(z) = az + bz^2$.

Theorem 2.6. *Assume ϕ does not satisfy condition* (L) *and $a < 0$. Then the super-Brownian motion X conditioned on extinction has the same law as super-Brownian motion with parameter $\tilde{\psi}(z)$, where $\tilde{\psi}(z) = \psi(z + z_0)$.*

Proof. The proof is the same in spirit as that of Evans and O'Connell's (1994). For every positive bounded function f, we have

$$\begin{aligned} P_\mu[e^{-\langle f, X_t \rangle} \mid \mathscr{L} < \infty] &= \frac{1}{P_\mu[\mathscr{L} < \infty]} \quad P_\mu[e^{-\langle f, X_t \rangle}, \quad P_{X_t}[\mathscr{L} < \infty]] \\ &= e^{z_0\langle 1, \mu \rangle} P_\mu e^{-\langle f + z_0, X_t \rangle} \\ &= e^{-\langle v_{f+z_0}(t) - z_0, \mu \rangle}, \end{aligned}$$

where $v_{f+z_0}(t)(x) = v_{f+z_0}(t, x)$ satisfies the integral equation

$$v_{f+z_0}(t, x) + \Pi_x \left[\int_0^t \psi(v_{f+z_0}(t - s, \xi_s)) \, ds \right] = \Pi_x f(\xi_t) + z_0. \tag{2.9}$$

Set $u(t, x) = v_{f+z_0}(t, x) - z_0$. Then u satisfies

$$u(t, x) + \Pi_x \left[\int_0^t \tilde{\psi}(u(t - s, \xi_s)) \, ds \right] = \Pi_x f(\xi_t). \tag{2.10}$$

Our result follows from the uniqueness of the solutions of (2.10). □

3. Asymptotic behavior of $\langle f, X_t \rangle$

We first study the asymptotic behavior of the total mass $Y_t = \langle 1, X_t \rangle$ of X.

Proposition 3.1. *For every finite measure μ, $Y_\infty = \lim_{t\to\infty} Y_t$ exists P_μ-a.s.*

Proof. By the Markov property, we have

$$P_\mu[e^{-Y_{t+s}}|\mathscr{F}_s] = P_{X_s}e^{-Y_t} = e^{-v(t)Y_s}, \tag{3.1}$$

where $v(t) = v_1(t)$. If $a \geqslant 0$, the Eq. (2.2) implies that $v(t) \leqslant 1$ and thus, by (3.1), e^{-Y_t} is a submartingale. If $0 < z_0 < \infty$, similar arguments as above show that $e^{-z_0 Y_t}$ is a martingale. Our claim now follows from the martingale convergence theorem. □

Theorem 3.2. *For every finite measure μ, we have $P_\mu[0 < Y_\infty < \infty] = 0$ and*

$$P_\mu[Y_\infty = 0] = 1 - P_\mu[Y_\infty = \infty] = e^{-z_0\langle 1, \mu\rangle}. \tag{3.2}$$

Proof. For every $\lambda > z_0$, set $w_\lambda(x) = -\log P_x e^{-\lambda Y_\infty}$. It follows from the Markov property that

$$w_\lambda(x) = -\log P_x P_{X_t} e^{-\lambda Y_\infty} = -\log P_x e^{-\langle w_\lambda, X_t\rangle}.$$

Since $w_\lambda \leqslant \lambda$, $w_\lambda(x)$ satisfies the p.d.e.

$$\frac{\partial w}{\partial t} = \Delta w - \psi(w). \tag{3.3}$$

Since w_λ is independent of t and x, (3.3) implies that $\psi(w_\lambda) = 0$. For every $\lambda > z_0$, we have, by Proposition 1.1, $w_\lambda \geqslant z_0$ and thus, $w_\lambda = z_0$. For every $\lambda > z_0$, write

$$e^{-z_0\langle 1, \mu\rangle} = P_\mu e^{-\lambda Y_\infty} = P_\mu[Y_\infty = 0] + P_\mu[e^{-\lambda Y_\infty}, 0 < Y_\infty < \infty]. \tag{3.4}$$

If $\lambda_2 > \lambda_1 > z_0$, we have $e^{-\lambda_1 Y_\infty} > e^{-\lambda_2 Y_\infty}$ on the set $\{0 < Y_\infty < \infty\}$. Thus, we obtain from (3.4) that $P_\mu[0 < Y_\infty < \infty] = 0$. Moreover, we have

$$P_\mu[Y_\infty = 0] = \lim_{\lambda\to\infty} P_\mu e^{-\lambda Y_\infty} = e^{-z_0\langle 1, \mu\rangle}. \quad □$$

Theorem 3.3. *For every positive bounded nonzero function f, $\langle f, X_t\rangle$ converges weakly to Y_∞. Moreover, if $a \geqslant 0$, it converges to 0 a.s.*

Proof. In the cases $a \geqslant 0$ the result follows easily from Theorem 3.2. We consider the case $a < 0$. For every $\lambda > 0$, we have

$$
\begin{aligned}
e^{-z_0\langle 1,\mu\rangle} &= \lim_{t\to\infty} P_\mu[e^{-\lambda\langle f, X_t\rangle}, Y_\infty = 0] \leqslant \liminf_{t\to\infty} P_\mu e^{-\lambda\langle f, X_t\rangle} \\
&\leqslant \limsup_{t\to\infty} P_\mu e^{-\lambda\langle f, X_t\rangle} = \limsup_{t\to\infty} e^{-\langle v_{\lambda f}(t), \mu\rangle} \\
&= e^{-\lim_{t\to\infty} \inf\langle v_{\lambda f}(t), \mu\rangle} \leqslant e^{-\langle \lim_{t\to\infty} \inf v_{\lambda f}(t), \mu\rangle}.
\end{aligned} \tag{3.5}
$$

If $\lambda \sup_x f(x) < z_0$, $v_{\lambda f}(t, x) \leqslant v_{z_0}(t, x) = z_0$ and therefore as $t \to \infty$, $v_{\lambda f}(t, x)$ converges to z_0. (See, e.g. Aronson and Weinberger, 1978, Corollary 3.1.) Then (3.5) implies that for λ sufficiently small, we have

$$\lim_{t\to\infty} P_\mu e^{-\lambda\langle f, X_t\rangle} = e^{-z_0\langle 1, \mu\rangle}. \tag{3.6}$$

For any $\lambda > \lambda_0 > 0$, we have

$$\mathrm{e}^{-z_0 \langle 1, \mu \rangle} \leqslant \lim_{t \to \infty} P_\mu \mathrm{e}^{-\lambda \langle f, X_t \rangle} \leqslant \lim_{t \to \infty} P_\mu \mathrm{e}^{-\lambda_0 \langle f, X_t \rangle}. \tag{3.7}$$

Our result follows by taking λ_0 sufficiently small in (3.7) and then by (3.6). □

4. Finite lifetime and compactness of range

The range $\mathscr{R}$ of the super-Brownian motion X is the smallest closed set in $\mathbb{R}^d$ satisfying $S_t \subset \mathscr{R}$ for every $t \geqslant 0$, where S_t is the support of X_t.

Definition 2. A function $g : [0, \infty) \to [0, \infty)$ is said to satisfy the condition (R) if for every $z > 0$,

$$\text{(R)} \quad \int_z^\infty \frac{1}{\sqrt{\int_0^u g(s)\,\mathrm{d}s}}\,\mathrm{d}u = \infty.$$

The following is quoted from Sheu (1994).

Theorem 4.1. *Let μ be a finite measure. If ϕ satisfies condition* (R), *then P_μ-a.s., the range $\mathscr{R}$ is not compact; otherwise, we have*

$$P_\mu[\mathscr{R} \text{ is compact}] = \mathrm{e}^{-z_0 \langle 1, \mu \rangle}. \tag{4.1}$$

In view of Theorems 2.5 and 4.1 it is interesting to compare condition (L) with condition (R).

Proposition 4.2. (A) *If ϕ satisfies condition* (L), *then it also satisfies condition* (R).
　(B) *If ϕ satisfies condition* (R) *and $\int_0^1 u l(\mathrm{d}u) < \infty$, then ϕ also satisfies condition* (L).

Proof. (A) Assume ϕ satisfies condition (L). Since $\phi(t)$ is increasing in t, we get $\int_0^t \phi(s)\,\mathrm{d}s \leqslant t\phi(t)$. To show ϕ satisfies condition (R), it suffices to verify that for every $z > 0$, we have

$$\int_z^\infty \frac{1}{\sqrt{s\phi(s)}}\,\mathrm{d}s = \infty. \tag{4.2}$$

Note that

$$\frac{\phi(s)}{\sqrt{s\phi(s)}} = \sqrt{\frac{\phi(s)}{s}},$$

$$\frac{\phi(s)}{s} = bs + \int_0^\infty \frac{\mathrm{e}^{-us} - 1 + us}{s}\,l(\mathrm{d}u). \tag{4.3}$$

By the comparison theorem for integrals, it suffices to check that $\liminf_{s\to\infty} \phi(s)/s = c > 0$. This is trivial in the case $b > 0$. Assume $b = 0$. Then for any $\varepsilon > 0$, we have

$$\frac{\phi(s)}{s} = \int_0^\infty \frac{e^{-us} - 1 + us}{s} l(\mathrm{d}u) \geqslant \int_\varepsilon^\infty \left(u - \frac{1}{s} \right) l(\mathrm{d}u)$$

$$= \int_\varepsilon^\infty ul(\mathrm{d}u) - \frac{1}{s} l([\varepsilon, \infty)). \tag{4.4}$$

Choosing $\varepsilon > 0$ such that $l([\varepsilon, \infty)) > 0$ and letting $s \to \infty$ gives $\liminf_{s\to\infty} \phi(s)/s > 0$.

(B) Assume ϕ satisfies condition (R). Clearly, the condition (R) implies $b = 0$. Then, by l'Hôpital's rule,

$$\lim_{s\to\infty} \left(\frac{\sqrt{\int_0^s \phi(u)\,\mathrm{d}u}}{\phi(s)} \right)^2 = \lim_{s\to\infty} \frac{\int_0^s \phi(u)\,\mathrm{d}u}{\phi^2(s)} = \lim_{s\to\infty} \frac{1}{2\phi'(s)}. \tag{4.5}$$

Since

$$\lim_{s\to\infty} \phi'(s) = \lim_{s\to\infty} \int_0^\infty u(1 - e^{-us})l(\mathrm{d}u) = \int_0^\infty ul(\mathrm{d}u) < \infty,$$

the comparison theorem implies that ϕ satisfies condition (L).   □

Theorem 4.3. (A) *If ϕ satisfies condition* (L), *then*

$$P_\mu[\, X \text{ survives}] = P_\mu[\mathscr{R} \text{ is not compact}] = 1.$$

(B) *If ϕ does not satisfy condition* (L) *and $\int_0^\infty ul(\mathrm{d}u) < \infty$, then we have*

$$\{X \text{ survives}\} = \{\mathscr{R} \text{ is not compact}\}, \quad P_\mu - a.s.$$

Proof. (A) follows easily from Theorems 2.5, 4.1 and 4.2. To prove (B), we set

$$E = \{X \text{ survives}\},$$

$$F = \{\mathscr{R} \text{ is not compact}\}.$$

First, we check that $P_\mu(G) = 0$, where $G = E \backslash F$. Suppose that $P_\mu(G) > 0$. Then there exists a ball B_n centered at 0 with radius n such that

$$P_\mu[\mathscr{R} \subset B_n, L = \infty] > 0. \tag{4.6}$$

Take $f = 1_{B_n^c}$. Then we have, by Theorem 3.3 and then by (4.6),

$$e^{-z_0\langle 1,\mu\rangle} = \lim_{t\to\infty} P_\mu e^{-\langle f, X_t\rangle} \geqslant e^{-z_0\langle 1,\mu\rangle} + P_\mu[\mathscr{R} \subset B_n, L = \infty] > e^{-z_0\langle 1,\mu\rangle}.$$

The contradiction gives $P_\mu(G) = 0$. Since $P_\mu(E) = P_\mu(F) = 1 - e^{-z_0\langle 1,\mu\rangle}$, an elementary reasoning shows that $P_\mu(G) = 0$ implies that $P_\mu(E \triangle F) = 0$.   □

If ϕ satisfies condition (L) and $\int_0^1 ul(\mathrm{d}u) < \infty$, then the range of X is the total space $\mathbb{R}^d$. In fact, we prove a much stronger result. We write S_t for the support of X_t.

Theorem 4.4. *If ϕ satisfies condition* (L) *and* $\int_0^1 ul(\mathrm{d}u) < \infty$, *then for every* $t > 0$, *we have*, P_μ-*a.s.*, $S_t = \mathbb{R}^d$ (*cf. Theorem 9, in Bertoin et al., 1996*).

Proof. If ϕ satisfies condition (L), then $b = 0$. Moreover, we have

$$\psi(z) = az + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz) l(\mathrm{d}u)$$

$$\leqslant \left(a + \int_0^\infty ul(\mathrm{d}u) \right) z = cz, \tag{4.7}$$

where $c = a + \int_0^\infty ul(\mathrm{d}u)$. Note that for every bounded continuous function f, we have

$$P_\mu[\langle f, X_t \rangle = 0] = \lim_{\lambda \to \infty} P_\mu \mathrm{e}^{-\lambda \langle f, X_t \rangle} = \lim_{\lambda \to \infty} \mathrm{e}^{-\langle v_\lambda(t), X_t \rangle}, \tag{4.8}$$

where $v_{\lambda f}(t, x)$ satisfies the Eq. (1.11) with $v_{\lambda f}(t, x) \to \lambda f(x)$ as $t \downarrow 0$. We consider first the case that $a \geqslant 0$. Set $w_{\lambda f}(t, x) = \lambda \Pi_x f(\tilde{\xi}_t)$, where $\tilde{\xi}_t$ is a Brownian motion with the constant killing rate c. Then $w_{\lambda f}$ satisfies

$$\frac{\partial w}{\partial t} = \Delta w - cw \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d,$$

with $w_{\lambda f}(t, x) \to \lambda f$ as $t \downarrow 0$. It follows from the comparison principle (see, e.g., Dynkin, 1992) that $v_{\lambda f}(t, x) \geqslant w_{\lambda f}(t, x)$. Clearly, $w_{\lambda f}(t, x) \uparrow \infty$, as $\lambda \uparrow \infty$ and so is $v_{\lambda f}(t, x)$. If $a < 0$, then $v_{\lambda f} \geqslant u_{\lambda f}$, where $u_{\lambda f}$ is the unique solution of

$$\frac{\partial u}{\partial t} = \Delta u - \phi(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d,$$

with $u_{\lambda f}(t, x) \to \lambda f(x)$ as $t \downarrow 0$. In the previous case we obtained that $u_{\lambda f} \to \infty$, and thus $v_{\lambda f} \to \infty$ as $\lambda \to \infty$. Therefore, it follows from (4.8) that $P_\mu[\langle f, X_t \rangle = 0] = 0$ for all bounded continuous function f. This implies that for every $y \in \mathbb{R}^d$, we have $P_\mu[S_t \cap \{y\} \neq \emptyset] = 1$ and thus $S_t = \mathbb{R}^d, P_\mu$-a.s. $\quad\square$

Remark. If X is a 2-branching super-stable process (which means the underlying process is a symmetric stable process of index $\beta \in (0, 2)$), then Perkins (1990) observed that $S_t \neq \emptyset$ implies that $S_t = \mathbb{R}^d, P_\mu$-a.s., for all $t > 0$. Also see Evans and Perkins (1991) for much more general cases than just stable spatial motion and finite variance branching.

5. Examples

For every nonnegative integer β, set $l_\beta(\mathrm{d}u) = 1_{0 \leqslant u \leqslant 1}(u) u^{-2} |\log u|^\beta \, \mathrm{d}u$. We first estimate $\int_0^1 u^2 l_\beta(\mathrm{d}u)$. For $\beta = 0, 1, 2, \ldots$, we have

$$\int_0^1 u^2 l_\beta(\mathrm{d}u) = \int_0^1 |\log u|^\beta \, \mathrm{d}u, \quad (\log u = v, \ \mathrm{d}u = \mathrm{e}^v \mathrm{d}v)$$

$$= \int_{-\infty}^0 \mathrm{e}^v |v|^\beta \, \mathrm{d}v = \int_0^\infty \mathrm{e}^{-w} w^\beta \mathrm{d}w < \infty.$$

Note that

$$\int_0^1 u l_\beta(\mathrm{d}u) = \int_0^1 \frac{1}{u} |\log u|^\beta \mathrm{d}u = \int_0^\infty v^\beta \, \mathrm{d}v = \infty.$$

Put $\psi_\beta(z) = \int_0^1 (\mathrm{e}^{-uz} - 1 + uz) l_\beta(\mathrm{d}u)$. Then

$$\psi_\beta(z) = \int_0^1 (\mathrm{e}^{-uz} - 1 + uz) u^{-2} |\log u|^\beta \, \mathrm{d}u$$

$$= \sum_{k=2}^\infty \frac{(-1)^k z^k}{k!} \int_0^1 u^{k-2} |\log u|^\beta \, \mathrm{d}u$$

$$= \sum_{k=2}^\infty \frac{(-1)^k z^k}{k!} \int_{-\infty}^0 \mathrm{e}^{v(k-1)} |v|^\beta \, \mathrm{d}v$$

$$= \sum_{k=2}^\infty \frac{(-1)^k z^k}{k!} \int_0^\infty \mathrm{e}^{-s} \frac{s^\beta}{(k-1)^{\beta+1}} \, \mathrm{d}s$$

$$= \Gamma(\beta+1) \sum_{k=2}^\infty \frac{(-1)^k z^k}{k!(k-1)^{\beta+1}}$$

$$= \Gamma(\beta+1) z \sum_{k=1}^\infty \frac{(-1)^{k+1} z^k}{(k+1)! k^{\beta+1}}, \tag{5.1}$$

where $\Gamma(\beta+1) = \int_0^\infty \mathrm{e}^{-s} s^\beta \, \mathrm{d}s$. Applying l'Hôpital's rule repeatedly $\beta+1$ times gives

$$\lim_{z\to\infty} \left(\sum_{k=1}^\infty \frac{(-1)^{k+1} z^k}{(k+1)! k^{\beta+1}} \Big/ (\log z)^{\beta+1} \right) = \lim_{z\to\infty} \frac{1}{(\beta+1)!} \sum_{k=1}^\infty \frac{(-1)^{k+1} z^k}{(k+1)!}$$

$$= \frac{1}{(\beta+1)!} \lim_{z\to\infty} \frac{1}{z}(\mathrm{e}^{-z} - 1 + z)$$

$$= \frac{1}{(\beta+1)!}. \tag{5.2}$$

Therefore, (5.1) and (5.2) imply that

$$\psi_\beta(z) \sim z(\log z)^{\beta+1} \quad \text{as } z \to \infty. \tag{5.3}$$

(We write $f(z) \sim g(z)$ as $z \to \infty$ if there exist two constant, $0 < c_1 < c_2 < \infty$, such that $c_1 \leqslant f(z)/g(z) \leqslant c_2$ for z sufficiently large.) It follows from (5.3) that $\psi_0(z)$ satisfies condition (L) and ψ_β, $\beta = 1, 2, \ldots$, does not satisfy condition (L). To check condition (R), notice that

$$\lim_{z\to\infty} \left(\frac{z(\log z)^{(\beta+1)/2}}{\sqrt{\int_0^z \psi_\beta(s) \, \mathrm{d}s}} \right)^2 = \lim_{z\to\infty} \frac{z^2 (\log z)^{\beta+1}}{\int_0^z \psi_\beta(s) \, \mathrm{d}s}$$

$$= \lim_{z\to\infty} \frac{2z(\log z)^{\beta+1} + (\beta+1)z(\log z)^\beta}{\psi_\beta(z)} = c > 0. \tag{5.4}$$

Therefore, $\psi_\beta(x)$ satisfies condition (R) if and only if $z(\log z)^{(\beta+1)/2}$ satisfies condition (L). Thus, $\psi_\beta(z)$ satisfies condition (R) if and only if $\beta = 0$ or 1.

Remark. The function $\psi_1(z)$ satisfies condition (R), but it does not satisfy condition (L). It follows from Theorem 2.5 and 4.1 that the super-Brownian motion with parameter ψ_1 does have unbounded range, even though it dies out eventually.

Acknowledgements

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