Relic density of dark matter in brane world cosmology

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We investigate the thermal relic density of cold dark matter in the context of brane world cosmology. Since the expansion law in a high energy regime is modified from the one in the standard cosmology, if the dark matter decouples in such a high energy regime its relic number density is affected by this modified expansion law. We derive analytic formulas for the number density of dark matter. It is found that the resultant relic density is characterized by the "transition temperature" at which the modified expansion law in brane world cosmology is connecting with the standard one, and can be considerably enhanced compared to that in the standard cosmology, if the transition temperature is low enough.

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Recent various cosmological observations, especially Wilkinson Microwave Anisotropy Probe satellite [1], have established the Λ CDM cosmological model with a great accuracy, where the energy density in the present universe consists of about 73% of the cosmological constant (dark energy), 23% of nonbaryonic cold dark matter and just 4% of baryons. However, to clarify the identity of dark matter particle is still a prime open problem in cosmology and particle physics. Many candidates for dark matter have been proposed. Among them, the neutralino in supersymmetric models is a suitable candidate, if the neutralino is the lightest supersymmetric particle (LSP) and the R-parity is conserved [2].

In the case that dark matter is the thermal relic, we can estimate its number density by solving the Boltzmann equation (3),

$$\frac{dn}{dt} + 3Hn = -\langle\sigma v\rangle(n^2 - n_{EQ}^2), \tag{1}$$

with the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho, \tag{2}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter with $a(t)$ being the scale factor, n is the actual number density, n_{EQ} is the number density in thermal equilibrium, $\langle\sigma v\rangle$ is the thermal averaged product of the annihilation cross section σ and the relative velocity v, ρ is the energy density, and G is the Newton's gravitational constant. By using $\dot{\rho}/\rho = -4H = 4\dot{T}/T + \dot{g}_*/g_*$, where g_* is the effective total number of relativistic degrees of freedom, in terms of the number density to entropy ratio $Y = n/s$ and $x = m/T$, Eq. (1) can be rewritten as

$$\frac{dY}{dx} = -\frac{s\langle\sigma v\rangle}{xH}(Y^2 - Y_{EQ}^2) \tag{3}$$

if $\dot{g}_*/g_*$ is almost negligible as usual. As is well known, an approximate formula of the solution of the Boltzmann equation can be described as

$$Y(\infty) \simeq \frac{x_d}{\lambda(\sigma_0 + \frac{1}{2}\sigma_1 x_d^{-1})}, \tag{4}$$

with a constant $\lambda = xs/H = 0.26(g_{*S}/g_*^{1/2})M_P m$ for models in which $\langle\sigma v\rangle$ is approximately parametrized as $\langle\sigma v\rangle = \sigma_0 + \sigma_1 x^{-1} + \mathcal{O}(x^{-2})$, where $x_d = m/T_d$, T_d is the decoupling temperature and m is the mass of the dark matter particle, and $M_P \simeq 1.2 \times 10^{19}$ GeV is the Planck mass.

Recently, brane world models have been attracting a lot of attention as a novel higher dimensional theory. In these models, it is assumed that the standard model particles are confined on a "3-brane" while gravity resides in the whole higher dimensional spacetime. The model first proposed by Randall and Sundrum (RS) [4], the so-called RS II model, is a simple and interesting one, and its cosmological evolution have been intensively investigated [5]. In the model, our 4-dimensional universe is realized on the 3-brane with a positive tension located at the ultraviolet boundary of a five dimensional Anti de-Sitter spacetime. In this setup, the Friedmann equation for a spatially flat spacetime in the RS brane cosmology is found to be

$$H^2 = \frac{8\pi G}{3}\rho\left(1 + \frac{\rho}{\rho_0}\right) + \frac{C}{a^4}, \tag{5}$$

where

$$\rho_0 = 96\pi G M_5^6, \tag{6}$$

with M_5 being the five dimensional Planck mass, the third term with an integration constant C is referred to as "dark radiation", and we have omitted the four dimensional cosmological constant. The second term, proportional to ρ^2 and dark radiation, are new ingredients in brane world cosmology and lead to a nonstandard expansion law.

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In this paper, we investigate the brane cosmological effect for the relic density of dark matter due to this nonstandard expansion law. If the new terms in Eq. (5) dominate over the term in the standard cosmology at the free-zeout time of dark matter, they can cause a considerable modification for the relic abundance of dark matter as we will show later.

Before turning to our analysis, we give some comments here. First, the dark radiation term is severely constrained by the success of the Big Bang Nucleosynthesis (BBN), since the term behaves like an additional radiation at the BBN era [6]. Hence, for simplicity, we neglect the term in the following analysis. Even if we include nonzero C consistent with the BBN constraint, we cannot expect the significant effects from dark radiation since, at the era we will discuss, the contribution of dark radiation is negligible. The second term is also constrained by the BBN, which is roughly estimated as $\rho_0^{1/4} > 1$ MeV (or $M_5 > 8.8$ TeV). On the other hand, more severe constraint is obtained by the precision measurements of the gravitational law in submillimeter range. Through the vanishing cosmological constant condition, we find $\rho_0^{1/4} > 1.3$ TeV (or $M_5 > 1.1 \times 10^8$ GeV) discussed in the original paper by Randall and Sundrum [4]. However, note that this result, in general, is quite model dependent. For example, if we consider an extension of the model so as to introduce a bulk scalar field, the constraint can be moderated as discussed in [7]. Hence, hereafter we impose only the BBN constraint on ρ_0.

We are interested in the early stage of brane world cosmology where the ρ^2 term dominates, namely, $\rho^2/\rho_0 \gg \rho$. In this case the coupling factor of collision term in the Boltzmann equation is given by

$$\frac{s\langle\sigma v\rangle}{xH} \simeq \frac{\lambda}{x_t^2}\langle\sigma v\rangle, \tag{7}$$

with a temperature independent constant λ defined as

$$\frac{s}{xH} = \frac{s}{x\sqrt{\frac{8\pi G}{3}\rho\left(1+\frac{\rho}{\rho_0}\right)}} = \lambda\frac{x^{-2}}{\sqrt{1+\left(\frac{x_t}{x}\right)^4}}, \tag{8}$$

where a new temperature independent parameter x_t is defined as $x_t^4 = \frac{\rho}{\rho_0}|_{T=m}$. Note that the evolution of the universe can be divided into two eras. At the era $x \ll x_t$, the ρ^2 term in Eq. (5) dominates (the brane world cosmology era), while at the era $x \gg x_t$, the expansion law obeys the standard cosmological law (the standard cosmology era). In the following, we call the temperature defined as $T_t = m x_t^{-1}$ (or x_t itself) "transition temperature" at which the evolution of the universe changes from the brane world cosmology era to the standard cosmology era. We consider the case that the decoupling temperature of dark matter particle is higher than the transition temperature. In such a case, we can expect a considerable modification for the relic density of dark matter from the one in the standard cosmology.

At the brane world cosmology era the Boltzmann equation can be read as

$$\frac{dY}{dx} = -\frac{\lambda}{x_t^2}\sigma_n x^{-n}(Y^2 - Y_{EQ}^2). \tag{9}$$

Here, for simplicity, we have parametrized the average of the annihilation cross section times the relative velocity as $\langle\sigma v\rangle = \sigma_n x^{-n}$ with a (mass dimension 2) constant σ_n. Note that, in the right-hand side of the above equation, x^2 in the standard cosmology is replaced by the constant x_t^2. At the early time, dark matter particle is in the thermal equilibrium and Y tracks Y_{EQ} closely. To begin, consider the small deviation from the thermal distribution $\Delta = Y - Y_{EQ} \ll Y_{EQ}$. The Boltzmann equation leads to

$$\Delta \simeq -\frac{x_t^2(dY_{EQ}/dx)}{\lambda\sigma_n x^{-n}(2Y_{EQ}+\Delta)} \simeq \frac{x_t^2}{2\lambda\sigma_n x^{-n}}, \tag{10}$$

where we have used an approximation formula $Y_{EQ} = 0.145(g/g_{*S})x^{3/2}e^{-x}$ and $dY_{EQ}/dx \simeq -Y_{EQ}$. As the temperature decreases or equivalently x becomes large, the deviation relatively grows since Y_{EQ} is exponentially dumping. Eventually the decoupling occurs at x_d, roughly evaluated as $\Delta(x_d) \simeq Y(x_d) \simeq Y_{EQ}(x_d)$, or explicitly

$$\left.\frac{x_t^2}{2\lambda\sigma_n x^{-n}}\right|_{x=x_d} \simeq 0.145\left.\frac{g}{g_{*S}}x^{3/2}e^{-x}\right|_{x=x_d}. \tag{11}$$

At further low temperature, $\Delta \simeq Y \gg Y_{EQ}$ is satisfied and Y_{EQ}^2 term in the Boltzmann equation can be neglected so that

$$\frac{d\Delta}{dx} = -\frac{\lambda}{x_t^2}\sigma_n x^{-n}\Delta^2. \tag{12}$$

The solution is formally given by

$$\frac{1}{\Delta(x)} - \frac{1}{\Delta(x_d)} = \int_{x_d}^{x}\lambda\frac{\sigma_n x^{-n}}{x_t^2}. \tag{13}$$

For $n = 0$ (S-wave process), $n = 1$ (P-wave process), and $n > 1$ we find

$$\frac{1}{\Delta(x)} - \frac{1}{\Delta(x_d)} = \begin{cases} \frac{\lambda\sigma_0}{x_t^2}(x - x_d), \\ \frac{\lambda\sigma_1}{x_t^2}\ln\left(\frac{x}{x_d}\right), \\ \frac{\lambda\sigma_n}{x_t^2}\left(\frac{1}{n-1}\right)\left(\frac{1}{x_d^{n-1}} - \frac{1}{x^{n-1}}\right), \end{cases} \tag{14}$$

respectively. Note that $\Delta(x)^{-1}$ is continuously growing without saturation for $n \leq 1$. This is a very characteristic behavior of brane world cosmology, comparing the case in standard cosmology where $\Delta(x)$ saturates after decoupling and the resultant relic density is roughly given by

$Y(\infty) \simeq Y(x_d)$. For a large $x \gg x_d$ in Eq. (14), $\Delta(x_d)$ and x_d can be neglected. When x becomes large further and reaches x_t, the expansion law changes into the standard one, and then Y obeys the Boltzmann equation with the standard expansion law for $x \geq x_t$. Since the transition temperature is smaller than the decoupling temperature in standard cosmology (which is the case we are interested in), we can expect that the number density freezes out as soon as the expansion law changes into the standard one. Therefore the resultant relic density can be roughly evaluated as $Y(\infty) \simeq \Delta(x_t)$ in Eq. (14). In the following analysis, we will show that this expectation is in fact correct.

Now, we derive analytic formulas of the final relic density of dark matter in brane world cosmology. At low temperature where $\Delta \simeq Y \gg Y_{EQ}$ is satisfied, the Boltzmann equation is given by

$$\frac{d\Delta}{dx} = -\frac{\lambda}{\sqrt{x^4 + x_t^4}}(\sigma_n x^{-n})\Delta^2, \tag{15}$$

and the solution is formally described as

$$\frac{1}{\Delta(x)} - \frac{1}{\Delta(x_d)} = \lambda\sigma_n \int_{x_d}^{x} dy \frac{y^{-n}}{\sqrt{y^4 + x_t^4}}. \tag{16}$$

For $n = 0$, the integration is given by the elliptic integral of the first kind such as

$$\begin{aligned}\int_{x_d}^{x} dy \frac{1}{\sqrt{y^4 + x_t^4}} &= \int_{0}^{x} dy \frac{1}{\sqrt{y^4 + x_t^4}} - \int_{0}^{x_d} dy \frac{1}{\sqrt{y^4 + x_t^4}} \\ &= \frac{2 - \sqrt{2}}{x_t}\left\{F\left[\arctan(1 + \sqrt{2}) \right.\right.\\ &\quad \left.\times \frac{x_t + x}{x_t - x}, 2\sqrt{3\sqrt{2} - 4}\right] - F\left[\arctan\right. \\ &\quad \left.\left.\times(1 + \sqrt{2})\frac{x_t + x_d}{x_t - x_d}, 2\sqrt{3\sqrt{2} - 4}\right]\right\},\end{aligned} \tag{17}$$

where the elliptic integral of the first kind $F(\phi, k)$ is defined as

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}. \tag{18}$$

In the limit $x \rightarrow \infty$, we obtain (with appropriate choice of the phase ϕ in $F(\phi, k)$)

$$\begin{aligned}\int_{x_d}^{\infty} \frac{dy}{\sqrt{y^4 + x_t^4}} &= \frac{2 - \sqrt{2}}{x_t}\left\{-F\left[\arctan(1 + \sqrt{2}) \right.\right.\\ &\quad \left.\times \frac{x_t + x_d}{x_t - x_d}, 2\sqrt{3\sqrt{2} - 4}\right] \\ &\quad \left. + F\left[\pi - \arctan(1 + \sqrt{2}), 2\sqrt{3\sqrt{2} - 4}\right]\right\}.\end{aligned} \tag{19}$$

In the case of $x_d \ll x_t$, the integration gives $\simeq 1.85/x_t$, and we find the resultant relic density $Y(\infty) \simeq 0.54 x_t/(\lambda\sigma_0)$. Note that the density is characterized by the transition temperature x_t as we expected. By using the well known formula (4), for a given $\langle\sigma v\rangle$, we obtain the ratio of the energy density of dark matter in brane world cosmology ($\Omega_{(b)}$) to the one in standard cosmology ($\Omega_{(s)}$), such that

$$\frac{\Omega_{(b)}}{\Omega_{(s)}} \simeq 0.54\left(\frac{x_t}{x_{d(s)}}\right), \tag{20}$$

where $x_{d(s)}$ is the decoupling temperature in standard cosmology. Similarly, for $n = 1$ we find

$$\begin{aligned}\int_{x_d}^{\infty} dy \frac{y^{-1}}{\sqrt{y^4 + x_t^4}} &= \frac{1}{4x_t^2} \ln\left(\frac{\sqrt{x^4 + x_t^4} - x_t^2}{\sqrt{x^4 + x_t^4} + x_t^2}\right)\Bigg|_{x_d}^{\infty} \\ &\simeq -\frac{1}{4x_t^2} \ln\frac{\sqrt{x_d^4 + x_t^4} - x_t^2}{\sqrt{x_d^4 + x_t^4} + x_t^2}.\end{aligned} \tag{21}$$

Again, in the case of $x_d \ll x_t$, the integration gives $\simeq x_t^{-2}\ln(x_t/x_d) \simeq x_t^{-2}\ln x_t$, and we find the resultant relic density $Y(\infty) \simeq x_t^2/(\lambda\sigma_0 \ln x_t)$. Thus the ratio of the energy density of dark matter is found to be

$$\frac{\Omega_{(b)}}{\Omega_{(s)}} \simeq \frac{1}{2\ln x_t}\left(\frac{x_t}{x_{d(s)}}\right)^2. \tag{22}$$

We can obtain results for the case of $n > 1$ in the same manner. Now we have found that the relic energy density in brane world cosmology is characterized by the transition temperature and can be enhanced compared to the one in standard cosmology, if the transition temperature is lower than the decoupling temperature.

In summary, we have investigated the thermal relic density of cold dark matter in brane world cosmology. If the five dimensional Planck mass is small enough, the ρ^2 term in the modified Friedmann equation can be effective when dark matter is decoupling. We have derived the analytic formulas for the relic density and found that the resultant relic density can be enhanced. The enhancement factor is characterized by the transition temperature, at which the evolution of the universe changes from the brane world cosmology era to the standard cosmology era.

It would be interesting to apply our result to detailed numerical analysis of the relic abundance of supersymmetric dark matter (for example, the neutralino dark matter). Allowed regions obtained in the previous analysis in standard cosmology [8] would be dramatically modified if the transition temperature is small enough [9]. Furthermore, if the brane world cosmology era exists in the history of the universe, the modified expansion law affects many physics controlled by the Boltzmann equations in the early universe. For example, we can expect considerable modifications for the thermal production of gravitino [10], the thermal leptogenesis scenario [11], etc., Those are worth investigating.

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