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An Improved Phase Error Tolerance in a Quantum Search Algorithm

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As the matching condition in the Grover search algorithm is transgressed due to inevitable errors in phase inversions, it causes a reduction in the maximum probability of success. With a given degree of maximum success, we have derived a generalized and improved criterion for the tolerated error and the corresponding size of the quantum database under the inevitable gate imperfections. The vanishing inaccurancy of this condition has also been shown. Moreover, a concise formula for evaluating a minimum number of iterations is also presented in this work.

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Grover's quantum search algorithm [1] provides a quadratic speedup over its classical counterpart, and it has been proven to be optimal for searching a marked element with minimum oracle calls [2]. This is achieved by applying the Grover kernel to a uniform superposition state, which is obtained by applying the Walsh-Hadamard transformation on an initial state at specific operating steps, such that the probability amplitude of the marked state is amplified to the desired value. Grover's kernel is composed of phase rotations and Walsh-Hadamard transformations. The phase rotations include two kinds of operations: π-inversion of the marked state and π-inversion of the initial state. It has been shown that the phases, π, can be replaced by two angles, ϕ and θ, under a phase matching criterion, which is the necessary condition for quantum searching with certainty. In other words, the relation between ϕ and θ will affect the degree of success of the quantum search algorithm. There have been several studies concerned with the effect of imperfect phase rotations. In their paper [3], Long *et al.* have found that the tolerated angle difference between two phase rotations, δ, due to systematic errors in phase inversions, with a given expected degree of success $P_{\max}$, is about $2/\sqrt{NP_{\max}}$, where N is the size of the database. Hoyer [4] has shown that after some number of iterations of the Grover kernel, depending on N and the unperturbed θ, it will give a solution with error probability $O(1/N)$ under a tolerated phase difference $\delta \sim O(1/\sqrt{N})$. The same result is also derived by Biham *et al.* [5]. On the other hand, a similar conclusion, $\delta \sim O(1/N^{2/3})$, is presented by Pablo-Norman and Ruiz-Altaba [6].

The result of Long *et al.* [3] is based on the approximate Grover kernel and certain assumptions, including large N and small δ. However, we found that the main inaccuracy comes from the approximate Grover kernel. Since all parameters in the Grover kernel connect with each other intimately, any reduction in the structure of Grover's kernel would destroy this penetrative relation, so accumulative errors emerge from the iterations of a quantum search. Although this assumption leads their study to a proper result, it cannot be applied to general cases, e.g. any set of two angles in phase rotations satisfying a phase matching condition [7, 8]. In what follows, we will get rid of the approximation to the Grover kernel, then derive an improved criterion for tolerated error in the phase rotation and the required number of qubits for preparing a database. Besides, a concise formula for evaluating the minimum number of iterations to achieve a maximum probability will also be acquired. By this formula, when evaluating the actual maximum probability, one can find that the derived criterion for tolerated error is nearly exact.

The Grover kernel is composed of two unitary operators G_τ and G_η, given by

$$\begin{aligned} G_\tau &= I + (e^{i\phi} - 1)\,|\tau\rangle\,\langle\tau|\;, \\ G_\eta &= I + (e^{i\theta} - 1) W\,|\eta\rangle\,\langle\eta|\,W^{-1}\,, \end{aligned} \tag{1}$$

where W is the Walsh-Hadamard transformation, $|\tau\rangle$ is the marked state, $|\eta\rangle$ is the initial state, and ϕ and θ are two phase angles. It can also be expressed in a matrix form, as long as an orthonormal set of basis vectors is chosen. The orthonormal set is

$$|I\rangle = |\tau\rangle \;\text{ and }\; |\tau_\perp\rangle = (W\,|\eta\rangle - W_{\tau\eta}\,|\tau\rangle)/l\,, \tag{2}$$

where $W_{\tau\eta} = \langle\tau|\,W\,|\eta\rangle$ and $l = (1 - |W_{\tau\eta}|^2)^{1/2}$. Letting $W_{\tau\eta} = \sin(\beta)$, we can write, from (2),

$$|s\rangle = W\,|\eta\rangle = \sin(\beta)\,|\tau\rangle + \cos(\beta)\,|\tau_\perp\rangle\;, \tag{3}$$

and the Grover kernel can now be written as

$$\begin{aligned} G &= -\,G_\eta G_\tau \\ &= -\begin{bmatrix} e^{i\phi}(1 + (e^{i\theta} - 1)\sin^2(\beta)) & (e^{i\theta} - 1)\sin(\beta)\cos(\beta) \\ e^{i\phi}(e^{i\theta} - 1)\sin(\beta)\cos(\beta) & 1 + (e^{i\theta} - 1)\cos^2(\beta) \end{bmatrix}. \end{aligned} \tag{4}$$

After m iterations, the operator G^m can be expressed as

$$G^m = (-1)^m e^{im(\frac{\phi+\theta}{2})} \begin{bmatrix} e^{imw}\cos^2(x) + e^{-imw}\sin^2(x) & e^{-i\frac{\phi}{2}} i\sin(mw)\sin(2x) \\ e^{i\frac{\phi}{2}} i\sin(mw)\sin(2x) & e^{imw}\sin^2(x) + e^{-imw}\cos^2(x) \end{bmatrix}, \tag{5}$$

where the angle w is defined by

$$\cos(w) = \cos(\frac{\phi - \theta}{2}) - 2\sin(\frac{\phi}{2})\sin(\frac{\theta}{2})\sin^2(\beta)\,, \tag{6}$$

or

$$\sin(w) = \sqrt{(\sin(\frac{\theta}{2})\sin(2\beta))^2 + (\sin(\frac{\phi - \theta}{2}) + 2\sin(\frac{\theta}{2})\cos(\frac{\phi}{2})\sin(\beta))^2}\,, \tag{7}$$

and the angle x is defined by

$$\sin(x) = \sin(\frac{\theta}{2})\sin(2\beta)/\sqrt{l_m}\,, \tag{8}$$

where

$$\begin{aligned} l_m &= (\sin(w) + \sin(\frac{\phi-\theta}{2}) + 2\cos(\frac{\phi}{2})\sin(\frac{\theta}{2})\sin^2(\beta))^2 + (\sin(\frac{\theta}{2})\sin(2\beta))^2 \\ &= 2\sin(w)(\sin(w) + \sin(\frac{\phi-\theta}{2}) + 2\cos(\frac{\phi}{2})\sin(\frac{\theta}{2})\sin^2(\beta))\,. \end{aligned}$$

More details can be found in the study [8]. Then the probability of finding a marked state is

$$\begin{aligned} P &= 1 - |\langle\tau_\perp|\, G^m\, |s\rangle|^2 \qquad (9) \\ &= 1 - (\cos(mw)\cos(\beta) - \sin(mw)\sin(\frac{\phi}{2})\sin(2x)\sin(\beta))^2 \\ &\quad - \sin^2(mw)(\cos(\frac{\phi}{2})\sin(2x)\sin(\beta) - \cos(2x)\cos(\beta))^2\,. \end{aligned}$$

Moreover, from the equation $\partial P/\partial(\cos(mw)) = 0$, the minimum number of iterations for obtaining the maximum probability, $P_{\max}(\cos(m_{\min}w))$, is found:

$$m_{\min}(\beta,\phi,\theta) = \frac{\cos^{-1}(\sqrt{\frac{b-2a}{2b}})}{w}\,, \tag{10}$$

where

$$\begin{aligned} a &= \sin(2x)\cos(2\beta) + \cos(2x)\cos(\frac{\phi}{2})\sin(2\beta), \\ b &= (2 + \sin^2(2x) + (3\sin^2(2x) - 2)\cos(4\beta) - 2\sin^2(2x)\cos(\phi)\sin^2(2\beta)) \\ &\quad + 2\sin(4x)\cos(\frac{\phi}{2})\sin(4\beta)\,. \end{aligned}$$

For a sure-success search problem the phase condition $\phi = \theta$, with iterations, $m_{\min} = (\pi/2 - \sin^{-1}(\sin(\phi/2)\sin(\beta))/w$, is required. However, when effects of imperfect phase inversions are considered, the search is not certain, then a new condition for the phase error, $\delta = \phi - \theta$, and the size of the database should be derived in order to accomplish the search with a reduced maximum probability. Now, we suppose the database is large, i.e., if $\sin(\beta) << 1$, and the phase error δ is small, where $|\delta| << 1$, one will have the following approximation, viz.,

$$\begin{aligned}\cos(w) &= \cos(\frac{\delta}{2}) - 2\sin(\frac{\theta}{2}+\frac{\delta}{2})\sin(\frac{\theta}{2})\sin^2(\beta)\\ &\approx 1-(\frac{\delta^2}{8}+2\beta^2\sin^2(\frac{\theta}{2}))\,,\\ \sin(w) &= (1-\cos^2(w))^{1/2}\\ &\approx \frac{(\delta^2+16\beta^2\sin^2(\frac{\theta}{2}))^{1/2}}{2}\,,\\ \sin(2x) &= \frac{4\beta\sin(\frac{\theta}{2})}{(\delta^2+16\beta^2\sin^2(\frac{\theta}{2}))^{1/2}}\,.\end{aligned}$$

The probability P , equation (9), then has the approximation

$$\begin{aligned}P &\approx 1-\cos^2(mw)\cos^2(\beta)-\sin^2(mw)\cos^2(2x)\\ &= \sin^2(mw)\sin^2(2x)\,,\end{aligned} \tag{11}$$

with a maximum value, by letting $\sin^2(mw)=1$,

$$P_{\max}\approx\sin^2(2x)=\frac{16\beta^2\sin^2(\frac{\theta}{2})}{\delta^2+16\beta^2\sin^2(\frac{\theta}{2})}\,. \tag{12}$$

The function (12) for two values, $\delta=0.01$ and $\delta=0.001$, are depicted in Fig. 1 and Fig. 2 respectively.

Considering Fig. 1 and Fig. 2, one realizes that the function (12) depicted by the solid line coincides with the exact value, obtained by Eq. (9) and Eq. (10), shown by cross marks. On the contrary, the result of Long *et al.*,

$$P_{\max}\approx\frac{4\beta^2\sin^2(\frac{\theta}{2})}{\delta^2+4\beta^2\sin^2(\frac{\theta}{2})}\,, \tag{13}$$

is an underestimation depicted by the dash lines.

To summarize, under the inevitable gate imperfections, we have derived the generalized and improved criterion (12), from the exact formulation of the Grover kernel after m iterations and the approximation of small values of the involved parameters, for tolerated error and its corresponding size of the quantum database. Moreover, the minimum number of iterations for obtaining the maximum probability, $m_{\min}(\beta,\phi,\theta)$, is also presented. By considering Fig. 1 and Fig. 2, one can realize that the improved criterion (12) is nearly exact. Besides, utilizing condition (12), one can realize that the value of tolerated error decreases with the growth of the database, in other words, it is important to have a good control over the tolerated error if we have a large quantum database. Therefore, quantum search machines should avoid gate imperfections as much as possible. If we cannot get rid of these errors, we must limit the size of the quantum database precisely. The result of this

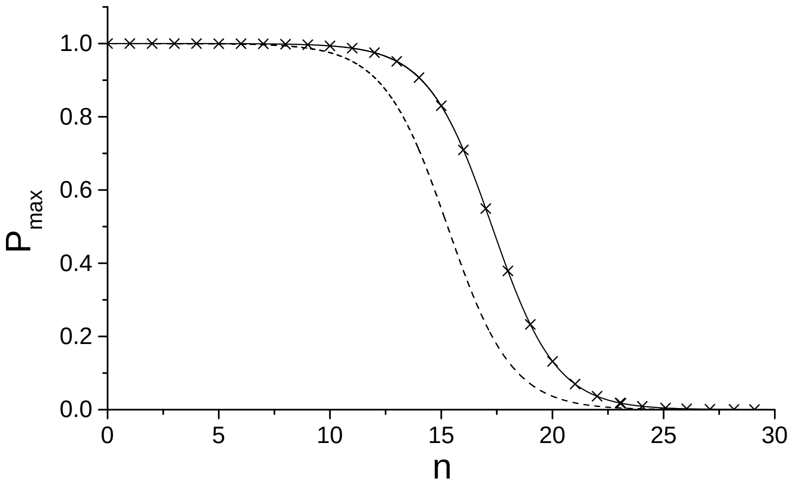

FIG. 1: Variations of the exact value of $P_{\max}(n)$ (cross marks), $16\beta^2\sin^2\left(\frac{\theta}{2}\right)/(\delta^2+16\beta^2\sin^2\left(\frac{\theta}{2}\right))$ (solid), and $4\beta^2/(\delta^2+4\beta^2)$ (dash), for $\theta=\pi$, $\delta=0.01$, where $\beta=\sin^{-1}\left(2^{-n/2}\right)$.

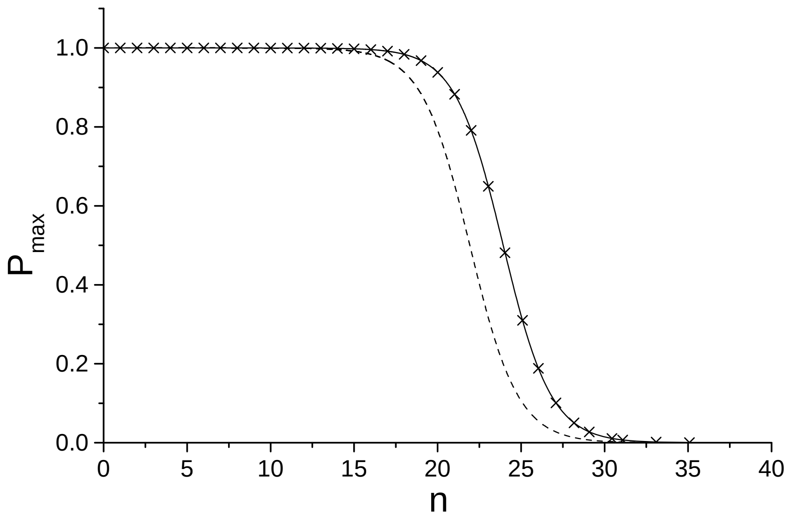

FIG. 2: Variations of the exact value of $P_{\max}(n)$ (cross marks), $16\beta^2\sin^2\left(\frac{\theta}{2}\right)/(\delta^2+16\beta^2\sin^2\left(\frac{\theta}{2}\right))$ (solid), and $4\beta^2/(\delta^2+4\beta^2)$ (dash), for $\theta=\pi$, $\delta=0.001$, where $\beta=\sin^{-1}\left(2^{-n/2}\right)$.

study presents a more accurate characterization of the relation between systematic errors and the size of a quantum database. A nearly exact criterion (12) can be utilized in order to achieve the practical equilibrium between the actual gate imperfection and the size of the quantum database.

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