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Information Processing Letters 91 (2004) 293–298

Information Processing Letters

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The super-connected property of recursive circulant graphs

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Received 23 October 2003; received in revised form 19 April 2004

Available online 24 June 2004

Communicated by M. Yamashita

Abstract

In a graph G, a k-container $C_k(u,v)$ is a set of k disjoint paths joining u and v. A k-container $C_k(u,v)$ is k^*-container if every vertex of G is passed by some path in $C_k(u,v)$. A graph G is k^*-connected if there exists a k^*-container between any two vertices. An m-regular graph G is super-connected if G is k^*-connected for any k with $1 \leqslant k \leqslant m$. In this paper, we prove that the recursive circulant graphs $G(2^m,4)$, proposed by Park and Chwa [Theoret. Comput. Sci. 244 (2000) 35–62], are super-connected if and only if $m \neq 2$.

Keywords: Super-connected; Container; Recursive circulant; Interconnection networks

1. Introduction

For the graph definitions and notations we follow Bondy and Murty [2]. $G=(V,E)$ is a graph if V is a finite set and E is a subset of $\{(a,b) \mid (a,b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices a and b are *adjacent* if $(a,b) \in E$. A *path* of length k from x to y is a finite set of distinct vertices $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, where $x = v_0$, $y = v_k$, $(v_{i-1}, v_i) \in E$ for all $1 \leqslant i \leqslant k$. For convenience, we use the sequence $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k\rangle$, where $Q = \langle v_i, v_{i+1}, \ldots, v_j\rangle$ to denote the path $\langle v_0, v_1, v_2, \ldots, v_k\rangle$. Note that we allow Q to be a path of length zero. Let P be the path $\langle v_0, v_1, \ldots, v_{k-1}, v_k\rangle$. We say that the vertex v_i, $0 \leqslant i \leqslant k$, is passed by the path P. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$. In particular, let $l(P)$ denote the length of the path P. The distance between u and v in G, denoted by $d(u,v)$, is the length of the shortest path joining u and v. A path is a *Hamiltonian path* if its vertices are distinct and span V. A graph, G, is *Hamiltonian connected* if there exists a Hamiltonian path joining any two vertices of G. A *cycle* is a path (except the first vertex is the same as the last vertex) that contains at least three vertices. A *Hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

A circulant graph can be defined as follows. Let n be a positive integer and let $S = \{k_1, k_2, \ldots, k_r\}$ with $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n/2$. Then the vertex set of

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0020-0190/$ – see front matter
doi:10.1016/j.ipl.2004.05.013

the circulant graph (G, S) is $\{0, 1, \ldots, n-1\}$ and the set of neighbors of the vertex u is $\{(u \pm k_j) \bmod n \mid j = 1, \ldots, r\}$. The graph we deal with here is the circulant graph $G(2^m, 4)$ proposed by Park and Chwa [7]. This family belongs to the family of circulant graphs denoted by $G(N, d)$ with $N, d \in \mathcal{N}$. The vertex set of $G(N, d)$ is $\{0, 1, \ldots, N-1\}$. Two vertices, u and v, are adjacent if and only if $u \pm d^i \equiv v \pmod N$ for some i with $0 \leqslant i \leqslant \lceil \log_d N \rceil - 1$. For example, $G(8, 4)$ and $G(16, 4)$, as shown in Fig. 1. Several interesting properties of $G(2^m, 4)$ have been studied in the literature [3,6–8]. For example, it was proved by Park and Chwa [7] that $G(2^m, 4)$ is an m connected and Hamiltonian graph. The embedding of meshes and hypercubes are studied in Park and Chwa [7]. The embedding of trees are studied by Lim et. al. [3]. The Hamiltonian decomposable property is studied by Micheneau [6].

A *k-container* $C_k(u, v)$ is a set of k disjoint paths joining u and v. The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. When G is a graph with $\kappa(G) \geqslant k$, it follows from Menger's theorem [5] that there is a k-container between any two different vertices of G. In this paper, we are interested in another type of container. A k-container $C_k(u, v)$ is a *k^*-container* if every vertex of G is passed by some path in $C_k(u, v)$. A graph G is *k^*-connected* if there exists a k^*-container between any two vertices. In particular, G is 1^*-connected if and only if G is Hamiltonian connected, and G is 2^*-connected if and only if G is Hamiltonian. Obviously, all 1^*-connected graphs, except K_1 and K_2, are 2^*-connected. The study of k^*-connected graphs is motivated by the globally 3^*-connected graphs proposed by Albert, Aldred and Holton [1]. We say a k-regular graph is *super-connected* if it is i^*-connected for all $1 \leqslant i \leqslant k$. Lin et al. [4] prove that the pancake graph P_n is super-connected if and only if $n > 3$. In this paper, we prove that $G(2^m, 4)$ is super-connected if and only if $m \neq 2$.

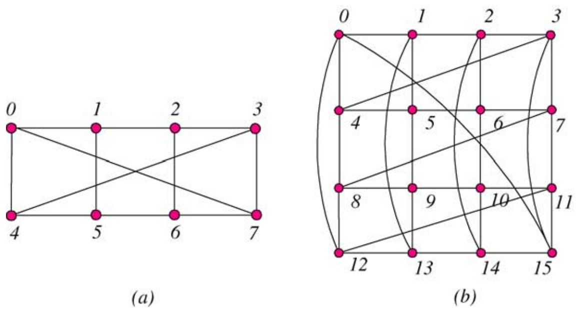

Fig. 1. Graphs $G(8, 4)$ and $G(16, 4)$.

Hypercubes are one of the most popular interconnection networks being used. A hypercube Q_m is a graph with 2^m vertices. Two vertices in Q_m are joined by an edge if and only if their binary representations differ in exactly one bit position. The number of vertices of $G(2^m, 4)$ is 2^m, which is equal to that of Q_m. The connectivity of $G(2^m, 4)$ is m, which is the best possible. The diameter of $G(2^m, 4)$ is less than that of Q_m. $G(2^m, 4)$ has good fault-tolerant Hamiltonian properties [8]. The super-connected property of $G(2^m, 4)$ is important in such a sense that it can be considered as a measure of the reliability of $G(2^m, 4)$.

In Section 2, we give some basic properties of $G(2^m, 4)$. Then in Section 3, we discuss the super-connected property of $G(2^m, 4)$. Finally, conclusions are given in Section 4.

2. Basic properties

For $0 \leqslant i < 2^m$, let f_i be the function from $V(G(2^m, 4))$ into itself defined by $f_i(x) \equiv (x + i) \pmod{2^m}$. It is easy to see that f_i is an automorphism of $G(2^m, 4)$. Similarly, let g be the function from $V(G(2^m, 4))$ into itself defined by $g(x) \equiv -x \pmod{2^m}$. Again, g is an automorphism of $G(2^m, 4)$. Let h_i be the function from $V(G(2^m, 4))$ into itself defined by $h_i(x) \equiv (x - i) \pmod{2^m}$ for all $0 \leqslant i < 2^m$. Similarly, h_i is an automorphism of $G(2^m, 4)$. Thus, $G(2^m, 4)$ is vertex transitive. Micheneau [6] also pointed out that $G(2^m, 4)$ has the following recursive property: For $0 \leqslant j \leqslant 3$, let G_j be the subgraph of $G(2^m, 4)$ induced by vertices $\{v \mid v \equiv j \pmod 4\}$. The edge set R in $E(G(2^m, 4))$, but not in $E(G_0) \cup E(G_1) \cup E(G_2) \cup E(G_3)$, is $\{(i, i + 1 \pmod{2^m})) \mid 0 \leqslant i \leqslant 2^m - 1\}$. Thus, R forms a Hamiltonian cycles of $G(2^m, 4)$. Moreover, each G_j is isomorphic to $G(2^{m-2}, 4)$. We have the following theorems.

Theorem 1 [7]. *The diameter of* $G(2^m, 4)$ *is* $\lceil (3m - 1)/4 \rceil$.

Theorem 2 [8]. *Assume that F is a subset of $V(G(2^m,4)) \cup E(G(2^m,4))$. Then $G(2^m,4) - F$ is Hamiltonian if $|F| \leqslant m-2$ and $G(2^m,4) - F$ is Hamiltonian connected if $|F| \leqslant m-3$, where $m \geqslant 3$.*

Therefore, we have the following corollary.

Corollary 1. *$G(2^m,4)$ are both 1^*-connected and 2^*-connected if $m \geqslant 3$.*

Corollary 2. *Let (x,y) be any edge of $G(2^m,4)$ with $m \geqslant 3$. Then there are two Hamiltonian cycles C_1 and C_2 of $G(2^m,4)$ such that $(x,y) \in E(C_1)$ and $(x,y) \notin E(C_2)$.*

Lemma 1. *Assume that x and y are any two different vertices of $G(2^m,4)$ with $m \geqslant 3$. Then there exists a 3^*-container $C_3(x,y) = \{P_1, P_2, P_3\}$ joining x and y such that P_1 is a shortest path between x and y. Hence, $G(2^m,4)$ is 3^*-connected if $m \geqslant 3$.*

Proof. Since $G(2^m,4)$ is vertex transitive, we only need to find a desired 3^*-container between vertex 0 and any vertex x of $G(2^m,4)$ with $x \neq 0$. Let P_1 be a shortest path joining 0 and x. By Theorem 1, $l(P_1) \leqslant \lceil (3m-1)/4 \rceil$. We may write P_1 as $\langle 0, x_1, x_2, \ldots, x_k, x \rangle$. Since $\lceil (3m-1)/4 \rceil \leqslant m-1$ for $m \geqslant 3$, $k \leqslant m-2$, therefore by Theorem 2, there exists a Hamiltonian cycle C of $G(2^m,4) - \{x_i \mid 1 \leqslant i \leqslant k\}$. Clearly, C can be written as $\langle 0, P_2, x, (P_3)^{-1}, 0 \rangle$. Accordingly, P_1, P_2 and P_3 form a 3^*-container joining 0 and x. Therefore, $G(2^m,4)$ is 3^*-connected. □

Lemma 2. *Let x and y be any two different vertices of $G(16,4)$. Then there exists a 4^*-container $C_4(x,y) = \{P_1, P_2, P_3, P_4\}$ joining x and y. In particular, $P_1 = \langle x, y \rangle$ if x and y are adjacent.*

Proof. Since $G(2^m,4)$ is vertex transitive, we only need to find a desired 4^*-container between vertex 0 to any vertex x of $G(2^m,4)$ with $x \neq 0$. We list this 4^*-container in Table 1.

The lemma is proved completely. □

3. Super-connected property

Lemma 3. *Let x and y be two adjacent vertices in $G(2^m,4)$ with $m \geqslant 3$ and k be an integer with $2 \leqslant k \leqslant m$. Then there exists a k^*-container $C_k(x,y) = \{P_1, P_2, \ldots, P_k\}$ of $G(2^m,4)$ such that $P_1 = \langle x, y \rangle$.*

Proof. We prove this lemma by induction on m. With Corollary 2, the lemma is true for any $m \geqslant 3$ and $k = 2$. With Lemma 1, the lemma is true for any $m \geqslant 3$ and $k = 3$. With Lemma 2, the lemma is true for $m = 4$ and $k = 4$. Assume that the lemma holds for any $G(2^t,4)$ with $t < m$. We only need to consider the case $m \geqslant 5$ and $4 \leqslant k \leqslant m$. Since $G(2^m,4)$ is vertex transitive, we only need to find a desired k^*-container of $G(2^m,4)$ between vertex 0 and any neighbor x for $4 \leqslant k \leqslant m$. Since the function g is an automorphism of $G(2^m,4)$, we have the following cases: (1) $x = 1$ and (2) $x \equiv 4^l \pmod{2^m}$ for all $1 \leqslant l \leqslant \lceil m/2 \rceil - 1$.

Case 1: $x = 1$. By induction, there is a $(k-2)^*$-container $\{Q_1, Q_2, \ldots, Q_{k-2}\}$ of G_0 between 0 and 4 such that $Q_1 = \langle 0, 4 \rangle$. Obviously, $l(Q_i) \geqslant 2$ for $2 \leqslant i \leqslant k-2$. Thus, we can write Q_i as $\langle 0, R_i, b_i, 4 \rangle$ with $b_i \notin \{0, 4\}$ for $2 \leqslant i \leqslant k-2$. Let $\{f_1(Q_1), f_1(Q_2), \ldots, f_1(Q_{k-2})\}$ be the image of $\{Q_1, Q_2, \ldots, Q_{k-2}\}$ under the function f_1. Thus, $\{f_1(Q_1), f_1(Q_2), \ldots, f_1(Q_{k-2})\}$ forms a $(k-2)^*$-container of G_1 between 1 and 5. Since there are 2^{m-2} vertices in G_2 and $m \geqslant 4$, $|V(G_2)| \geqslant 4$. Then there is a vertex y in G_2 such that $y \neq 2$ and $y \neq 2^m - 2$. By Theorem 2, there exists a Hamiltonian path S_2 of G_2 joining y to 2, and there exists a Hamiltonian path S_3 of G_3 joining $2^m - 1$ to $y + 1$. We set

$$P_i = \begin{cases} \langle 0, 1 \rangle & \text{for } i = 1, \\ \langle 0, R_i, b_i, b_i + 1, (f_1(R_i))^{-1}, 1 \rangle & \text{for } 2 \leqslant i \leqslant k-2, \\ \langle 0, 4, 5, 1 \rangle & \text{for } i = k-1, \\ \langle 0, 2^m - 1, S_3, y+1, y, S_2, 2, 1 \rangle & \text{for } i = k. \end{cases}$$

Thus, $\{P_1, P_2, \ldots, P_k\}$ forms a desired k^*-container of $G(2^m,4)$ between 0 and x.

Case 2: $x \equiv 4^l \pmod{2^m}$ for all $1 \leqslant l \leqslant \lceil m/2 \rceil - 1$. Thus $x \in V(G_0)$. By induction, there is a $(k-2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of G_0 between 0 and x such that $P_1 = \langle 0, x \rangle$. Since $x \neq 0$, $x + 1 \neq 1$ and $x - 1 \not\equiv 2^m - 1 \pmod{2^m}$. Since G_i is isomorphic to $G(2^{m-2},4)$ for all $0 \leqslant i \leqslant 3$, by Theorem 2, there exists a Hamiltonian path Q_1 of G_1, joining 1 to $x+1$; and there exists a Hamiltonian path Q_2 of G_3, joining $2^m - 1$ to $x - 1$. We rewrite Q_2 as $\langle 2^m - 1, S, t, x - 1 \rangle$. Therefore, $t - 1$ and $x - 2$ are two

Table 1

x	4^*-container $C_4(0,x)$
1	$\langle 0,1\rangle$, $\langle 0,4,3,2,1\rangle$, $\langle 0,15,14,13,1\rangle$, $\langle 0,12,11,10,9,8,7,6,5,1\rangle$
2	$\langle 0,1,2\rangle$, $\langle 0,15,3,2\rangle$, $\langle 0,4,5,6,2\rangle$, $\langle 0,12,8,7,11,10,9,13,14,2\rangle$
3	$\langle 0,4,3\rangle$, $\langle 0,15,3\rangle$, $\langle 0,1,13,14,2,3\rangle$, $\langle 0,12,11,10,6,5,9,8,7,3\rangle$
4	$\langle 0,4\rangle$, $\langle 0,1,5,4\rangle$, $\langle 0,15,3,4\rangle$, $\langle 0,12,11,10,9,13,14,2,6,7,8,4\rangle$
5	$\langle 0,4,5\rangle$, $\langle 0,1,5\rangle$, $\langle 0,15,3,2,6,5\rangle$, $\langle 0,12,13,14,10,11,7,8,9,5\rangle$
6	$\langle 0,1,5,6\rangle$, $\langle 0,4,3,2,6\rangle$, $\langle 0,12,11,10,6\rangle$, $\langle 0,15,14,13,9,8,7,6\rangle$
7	$\langle 0,15,11,7\rangle$, $\langle 0,12,8,7\rangle$, $\langle 0,1,2,3,7\rangle$, $\langle 0,4,5,9,13,14,10,6,7\rangle$
8	$\langle 0,4,8\rangle$, $\langle 0,12,8\rangle$, $\langle 0,15,11,10,6,5,9,8\rangle$, $\langle 0,1,13,14,2,3,7,8\rangle$
9	$\langle 0,1,13,9\rangle$, $\langle 0,12,8,9\rangle$, $\langle 0,4,3,7,11,10,9\rangle$, $\langle 0,15,14,2,6,5,9\rangle$
10	$\langle 0,15,11,10\rangle$, $\langle 0,4,5,6,10\rangle$, $\langle 0,12,13,14,10\rangle$, $\langle 0,1,2,3,7,8,9,10\rangle$
11	$\langle 0,12,11\rangle$, $\langle 0,15,11\rangle$, $\langle 0,1,13,14,10,11\rangle$, $\langle 0,4,3,2,6,5,9,8,7,11\rangle$
12	$\langle 0,12\rangle$, $\langle 0,4,8,12\rangle$, $\langle 0,15,14,13,12\rangle$, $\langle 0,1,2,3,7,6,5,9,10,11,12\rangle$
13	$\langle 0,12,13\rangle$, $\langle 0,1,13\rangle$, $\langle 0,15,11,10,14,13\rangle$, $\langle 0,4,5,6,2,3,7,8,9,13\rangle$
14	$\langle 0,15,14\rangle$, $\langle 0,12,11,10,14\rangle$, $\langle 0,4,3,2,14\rangle$, $\langle 0,1,5,6,7,8,9,13,14\rangle$
15	$\langle 0,15\rangle$, $\langle 0,1,2,3,15\rangle$, $\langle 0,12,13,14,15\rangle$, $\langle 0,4,5,6,7,8,9,10,11,15\rangle$

distinct vertices in G_2. By Theorem 2, there exists a Hamiltonian path Q_3 of G_2, joining $t-1$ to $x-2$. Consequently, we set P_{k-1} as $\langle 0,1,Q_1,x+1,x\rangle$ and P_k as $\langle 0,2^m-1,S,t,t-1,Q_3,x-2,x-1,x\rangle$. Thus, $\{P_1,P_2,\ldots,P_k\}$ forms a k^*-container of $G(2^m,4)$ between 0 and x. □

Theorem 3. *$G(2^m,4)$ is super-connected if and only if $m\neq 2$.*

Proof. It is easy to see that $G(2^m,4)$ is isomorphic to K_2 if $m=1$ and $G(2^m,4)$ is isomorphic to C_4 if $m=2$. Clearly, $G(2^1,4)$ is super-connected. However, C_4 is not Hamiltonian connected. Hence, $G(2^2,4)$ is not super-connected. Now, by induction we prove that $G(2^m,4)$ is super-connected for $m\geqslant 3$. With Corollary 1 and Lemma 1, $G(2^3,4)$ is super-connected. With Corollary 1, Lemma 1, and Lemma 2, $G(2^4,4)$ is super-connected. Assume that $G(2^n,4)$ is super-connected for any n with $3\leqslant n<m$ with $m\geqslant 5$. By Corollary 1 and Lemma 1, $G(2^m,4)$ is k^*-connected with $k=1,2,$ and 3. Assume that $4\leqslant k\leqslant m$. By Lemma 3, if x and y are adjacent then there exists a k^*-container $C_k(x,y)=\{P_1,P_2,\ldots,P_k\}$ of $G(2^m,4)$. Consequently, we need to find a k^*-container between any two nonadjacent vertices of $G(2^m,4)$ for $4\leqslant k\leqslant m$.

Since $G(2^m,4)$ is vertex transitive, we only need to find a k^*-container between 0 and x with $x\neq 0$, x is not adjacent to 0, and $4\leqslant k\leqslant m$. We have the following five cases: (1) $x\equiv 0 \pmod 4$ and $x\not\equiv \pm 4^l \pmod{2^m}$ for all $1\leqslant l\leqslant \lceil m/2\rceil$, (2) $x\equiv \pm 1 \pmod 4$, $x\neq 1$, and $x\neq 2^m-1$, (3) $x=2$ or $x=2^m-2$, (4) $x\equiv 2\pm 4^l \pmod{2^m}$ and $x\neq 2^m-2$ for all $1\leqslant l\leqslant \lceil m/2\rceil-1$, and (5) $x\equiv 2 \pmod 4$ and $x\not\equiv 2\pm 4^l \pmod{2^m}$ for all $1\leqslant l\leqslant \lceil m/2\rceil$.

Case 1: $x\equiv 0 \pmod 4$ and $x\not\equiv \pm 4^l \pmod{2^m}$ for all $1\leqslant l\leqslant \lceil m/2\rceil$. Thus $x\in V(G_0)$. By induction, there is a $(k-2)^*$-container $\{P_1,P_2,\ldots,P_{k-2}\}$ of G_0 between 0 and x. Since $x\neq 0$, $x+1\neq 1$ and $x-1\not\equiv 2^m-1 \pmod{2^m}$. Note that G_i is isomorphic to $G(2^{m-2},4)$ for all $0\leqslant i\leqslant 3$. By Theorem 2, there exists a Hamiltonian path Q_1 of G_1 joining 1 to $x+1$ and there exists a Hamiltonian path Q_2 of G_3 joining 2^m-1 to $x-1$. We write Q_2 as $\langle 2^m-1,S,t,x-1\rangle$. Therefore, $t-1$ and $x-2$ are two distinct vertices in G_2. By Theorem 2, there exists a Hamiltonian path Q_3 of G_2 joining $t-1$ to $x-2$. We set P_{k-1} as $\langle 0,1,Q_1,x+1,x\rangle$ and P_k as $\langle 0,2^m-1,S,t,t-1,Q_3,x-2,x-1,x\rangle$. Thus, $\{P_1,P_2,\ldots,P_k\}$ forms a k^*-container of $G(2^m,4)$ between 0 and x.

Case 2: $x\equiv \pm 1 \pmod 4$, $x\neq 1$, and $x\neq 2^m-1$. Thus, $x\in V(G_1)$ or $x\in V(G_3)$. Since the function g is an automorphism of $G(2^m,4)$, we may assume that $x\in V(G_1)$. Thus, $x-1\neq 0$. By induction, there exists a $(k-2)^*$-container $\{P_1,P_2,\ldots,P_{k-2}\}$ of G_0 between 0 and $x-1$. Without loss of generality, we assume that $l(P_1)\leqslant l(P_i)$ for all $2\leqslant i\leqslant k-2$. Hence, $l(P_i)\geqslant 2$ for $2\leqslant i\leqslant k-2$. Thus, we can write P_i as $\langle 0,R_i,b_i,x-1\rangle$ for $1\leqslant i\leqslant k-2$. Note that $l(R_1)=0$ if $b_1=0$.

Obviously, $b_i + 1$ is a neighborhood of x for $1 \leqslant i \leqslant k-2$. Let $B = \{(x, x \pm 4^i \pmod{2^m}) \mid 1 \leqslant i \leqslant \lceil m/2 \rceil - 1$ and $x \pm 4^i \not\equiv b_j + 1 \pmod{2^m}$ for all $1 \leqslant j \leqslant k-2\}$. We set F_1 to be the union of B and the set $\{a_i \mid 3 \leqslant i \leqslant k-2$ and $a_i = b_i + 1\}$. Clearly, $|F_1| = m - 4$ and the only neighbors of x in $G_1 - F_1$ are a_1 and a_2. By Theorem 2, there exists a Hamiltonian cycle C of $G_1 - F_1$. We can write C as $\langle x, a_1, S_1, 1, S_2, a_2, x\rangle$. Without loss of generality, we may assume that $l(S_1) \leqslant l(S_2)$. Since the number of vertices in $G_1 - F_1$ are $2^{m-2} - k + 4$ with $k \leqslant m$, $l(C) \geqslant 7$. Thus, $l(S_2) \geqslant 3$. We can rewrite S_2 as $\langle 1, v, T, u, a_2\rangle$ with $l(T) \geqslant 0$.

Clearly, $u + 1$ and $v + 1$ are two distinct vertices in G_2. By Theorem 2, there exists a Hamiltonian path S of G_2 joining $u + 1$ and $v + 1$. We write S as $\langle u+1, S_3, w, x+1, t, S_4, v+1\rangle$. Thus, one of vertices w and t is not $2^m - 2$. Without loss of generality, we assume that $t \neq 2^m - 2$. Again, we can write S as $\langle u+1, S_5, x+1, t, S_4, v+1\rangle$. Since G_3 is Hamiltonian connected, there exists a Hamiltonian path S_6 of G_3 joining $2^m - 1$ and $t + 1$. We set

$$Q_i = \begin{cases} \langle 0, P_1, x-1, x\rangle & \text{for } i = 1,\\ \langle 0, R_i, b_i, b_i+1, x\rangle & \text{for } 2 \leqslant i \leqslant k-2,\\ \langle 0, 1, S_1, a_1, x\rangle & \text{for } i = k-1,\\ \langle 0, 2^m-1, S_6, t+1, t, S_4, v+1, v, T, u, \\ \quad u+1, S_5, x+1, x\rangle & \text{for } i = k. \end{cases}$$

Apparently, $\{Q_1, Q_2, \ldots, Q_k\}$ forms a k^*-container of $G(2^m, 4)$ between vertices 0 and x, as shown by Fig. 2.

Case 3: $x = 2$ or $x = 2^m - 2$. Since g is an automorphism of $G(2^m, 4)$, we consider only the case $x = 2$. Note that 0 and 4 are adjacent in G_0. By Lemma 3, there exists a $(k-2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of G_0 between 0 and 4 such that $P_1 = \langle 0, 4\rangle$. Hence $l(P_i) \geqslant 2$ for $2 \leqslant i \leqslant k-2$. Without loss of generality, we assume that $l(P_{k-2}) \geqslant l(P_i)$ for $1 \leqslant i \leqslant k-3$. Therefore, we can write P_i as $\langle 0, a_i, R_i, b_i, 4\rangle$ for $2 \leqslant i \leqslant k-3$. Note that $a_i = b_i$ if $l(R_i) = 0$. Obviously, $l(P_{k-2}) \geqslant \lceil (2^{m-2} - 2)/(k-3)\rceil + 1$. Since $4 \leqslant k \leqslant m$ and $m \geqslant 5$, $l(P_{k-2}) \geqslant 4$. We can write P_{k-2} as $\langle 0, a_{k-2}, R_{k-2}, y, z, 4\rangle$. Note that $l(R_{k-2}) = 0$ if $l(P_{k-2}) = 4$. Therefore, $z \neq 0$. Suppose that $m \geqslant 6$. By Theorem 2, there exists a Hamiltonian path S of $G_1 - \{1\}$ joining $y - 3$ and $z - 3$. Suppose that $m = 5$. Then $V(G_0) = \{0, 4, 8, 12, 16, 20, 24, 28\}$ and G_0 is isomorphic to $G(8, 4)$. There are three vertices 0, 8, and 20 in G_0 adjacent to vertex 4. Since $z \neq 0$, $z \in \{8, 20\}$. Hence $z - 3 \in \{5, 17\}$. Consequently, $(y-3, z-3)$ is an edge of G_1 and hence $(y-3, z-3) \in \{(9,5), (21,5), (13,17), (21,17)\}$. We can find a Hamiltonian path S of $G_1 - \{1\}$ joining $y - 3$ and $z - 3$ in Table 2.

Now, we set

$$Q_i = \begin{cases} \langle 0, 4, 3, 2\rangle & \text{for } i = 1,\\ \langle 0, a_i, R_i, b_i, b_i-1, (h_1(R_i))^{-1}, a_i-1, \\ \quad a_i-2, h_2(R_i), b_i-2, 2\rangle & \text{for } 2 \leqslant i \leqslant k-3,\\ \langle 0, a_{k-2}, R_{k-2}, y, z, z-1, y-1, \\ \quad (h_1(R_{k-2}))^{-1}, a_{k-2}-1, a_{k-2}-2, \\ \quad h_2(R_{k-2}), y-2, y-3, S, z-3, z-2, 2\rangle & \text{for } i = k-2,\\ \langle 0, 1, 2\rangle & \text{for } i = k-1,\\ \langle 0, 2^m-1, 2^m-2, 2\rangle & \text{for } i = k. \end{cases}$$

Apparently, $\{Q_1, Q_2, \ldots, Q_k\}$ forms a k^*-container of $G(2^m, 4)$ between 0 and x.

Case 4: $x \equiv 2 \pm 4^l \pmod{2^m}$ and $x \neq 2^m - 2$ for all $1 \leqslant l \leqslant \lceil m/2\rceil - 1$. Clearly, x is in G_2. Therefore, $x - 2$ is adjacent to 0 in G_0. By Lemma 3, there exists a $(k-2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of G_0 between 0 and $x - 2$ such that $P_1 = \langle 0, x-2\rangle$. Hence $l(P_i) \geqslant 2$ for $2 \leqslant i \leqslant k-2$. We can write P_i as $\langle 0, a_i, R_i, b_i, x-2\rangle$ for $2 \leqslant i \leqslant k-2$. Since $x \neq 2^m - 2$, $x + 1$ and $2^m - 1$ are two distinct vertices of G_3. By Theorem 2,

Table 2

$(y-3, z-3)$	S
(9, 5)	$\langle 9, 25, 29, 13, 17, 21, 5\rangle$
(13, 17)	$\langle 13, 29, 25, 9, 5, 21, 17\rangle$
(21, 5)	$\langle 21, 17, 13, 29, 25, 9, 5\rangle$
(21, 17)	$\langle 21, 5, 9, 25, 29, 13, 17\rangle$

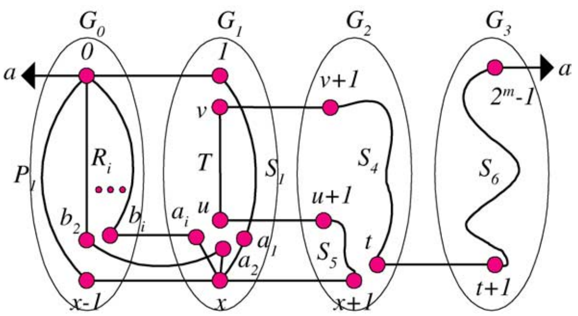

Fig. 2. Illustration of Theorem 3, Case 2.

there exists a Hamiltonian path T of G_3 joining $x+1$ and 2^m-1. We set

$$Q_i = \begin{cases} \langle 0, x-2, x-1, x\rangle & \text{for } i=1,\\ \langle 0, a_i, R_i, b_i, f_1(b_i), (f_1(R_i))^{-1}, f_1(a_i), \\ \quad f_2(a_i), f_2(R_i), f_2(b_i), x\rangle & \text{for } 2 \leqslant i \leqslant k-2,\\ \langle 0,1,2,x\rangle & \text{for } i=k-1,\\ \langle 0, 2^m-1, T, x+1, x\rangle & \text{for } i=k. \end{cases}$$

Thus, $\{Q_1, Q_2, \ldots, Q_k\}$ forms a k^*-container of $G(2^m, 4)$ between 0 and x.

Case 5: $x \equiv 2 \pmod 4$ and $x \not\equiv 2 \pm 4^l \bmod 2^m)$ for all $1 \leqslant l \leqslant \lceil m/2 \rceil$. By induction, there is a $(k-2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of G_0 between 0 and $x-2$. Since $x-2 \not\equiv \pm 4^l \pmod{2^m}$, $l(P_i) \geqslant 2$ for all $1 \leqslant i \leqslant k-2$. We can write P_i as $\langle 0, a_i, R_i, b_i, x-2\rangle$ for $1 \leqslant i \leqslant k-2$. We recursively define a sequence of vertices in G_3 as follows: Set $z_1 = 3$ and $z_i = z_{i-1} + 4$ for $2 \leqslant i \leqslant 2^{m-2}$. Clearly, $\langle 3 = z_1, z_2, \ldots, z_{2^{m-2}} = 2^m - 1, 3 = z_1\rangle$ forms a Hamiltonian cycle C of G_3. Since $x-2 \not\equiv \pm 4^l \pmod{2^m}$, $x-3$, $x+1$, 2^m-1, and 3 are four distinct vertices of G_3. We may write C as $\langle 3, S, x-3, x+1, T, 2^m-1, 3\rangle$. Now, we set

$$Q_i = \begin{cases} \langle 0, a_i, R_i, b_i, f_1(b_i), (f_1(R_i))^{-1}, f_1(a_i), \\ \quad f_2(a_i), f_2(R_i), f_2(b_i), x\rangle \\ \qquad\qquad\qquad \text{for } 1 \leqslant i \leqslant k-2,\\ \langle 0,1,2,3,S,x-3,x-2,x-1,x\rangle \\ \qquad\qquad\qquad \text{for } i=k-1,\\ \langle 0, 2^m-1, T^{-1}, x+1, x\rangle \\ \qquad\qquad\qquad \text{for } i=k. \end{cases}$$

Thus, $\{Q_1, Q_2, \ldots, Q_k\}$ forms a k^*-container of $G(2^m, 4)$ between 0 and x. □

4. Conclusions

Recursive circulant graphs $G(2^m, 4)$ are the major concern in this paper. $G(2^m, 4)$ has the connectivity m and the diameter $\lceil (3m-1)/4 \rceil$; which is less than m, the diameter of the hypercube Q_m. The main result of this paper is proving that the recursive circulant graphs $G(2^m, 4)$ have super-connected property if and only if $m \neq 2$. A k-container $C_k(u, v)$ between two distinct vertex u and v in G is a set of k disjoint paths between u and v. The length of a $C_k(u, v)$, written as $l(C_k(u, v))$, is the length of the longest path in $C_k(u, v)$. The k-wide distance between u and v is $d_k(u, v)$, which is the minimum length among all k-containers between u and v. Let κ be the connectivity of G. The wide diameter of G, denoted by $D_\kappa(G)$, is the maximum of κ-wide distances among all pairs of vertices u, v in G, $u \neq v$. Assume that G is k^*-connected. We may define the k^*-wide distance between any two vertices u and v, denoted by $d_k^*(u, v)$, which is the minimum length among all k^*-containers between u and v. Let $D_k^*(G) = \max\{d_k^*(u, v) \mid u$ and v are two different vertices of $G\}$. We say that $D_k^*(G)$ is the k^*-diameter of G. In our future work, we are interested to find $D_k^*(G(2^m, 4))$ for $2 \leqslant k \leqslant m$.

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