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SCIENCE DIRECT®
Information Sciences 165 (2004) 59–71
INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL
www.elsevier.com/locate/ins

# Hyper hamiltonian laceability on edge fault star graph

Tseng-Kuei Li [a,*], Jimmy J.M. Tan [b], Lih-Hsing Hsu [b]

[a] *Department of Computer Science and Information Engineering, Ching Yun University, JungLi 320, Taiwan, ROC*

[b] *Department of Computer and Information Science, National Chiao Tung University, Hsinchu 300, Taiwan, ROC*

Received 30 December 2002; received in revised form 17 March 2003; accepted 17 September 2003

**Abstract**

The star graph possess many nice topological properties. Edge fault tolerance is an important issue for a network since the edges in the network may fail sometimes. In this paper, we show that the $n$-dimensional star graph is $(n-3)$-edge fault tolerant hamiltonian laceable, $(n-3)$-edge fault tolerant strongly hamiltonian laceable, and $(n-4)$-edge fault tolerant hyper hamiltonian laceable. All these results are optimal in a sense described in this paper.


*Keywords:* Star graph; Hamiltonian laceable; Strongly hamiltonian laceable; Hyper hamiltonian laceable; Fault tolerant

## 1. Introduction

Network topology is a crucial factor for a network since it determines the performance of the network. For convenience of discussing their properties, networks are usually represented by graphs. In this paper, a network topology is represented by a simple undirected graph, which is loopless and without

* Corresponding author.
*E-mail address:* tealee@ms8.hinet.net (T.-K. Li).

0020-0255/$ - see front matter 
doi:10.1016/j.ins.2003.09.023

multiple edges. For the graph definition and notation we follow [5]. $G = (V, E)$ is a graph if $V$ is a finite set and $E$ is a subset of $\{(a, b)|a \neq b \in V\}$, where $(a, b)$ denotes an unordered pair. We call $V$ the *vertex set* and $E$ the *edge set*. $a$ and $b$ are adjacent if and only if $(a, b) \in E$. A path is a sequence of adjacent vertices, denoted by $\langle v_0, v_1, \ldots, v_k \rangle$, in which $v_0, v_1, \ldots, v_k$ are distinct except that possibly $v_0 = v_k$. The length of the path is $k$. For ease of description, we may use $P$ or $\langle v_0, P, v_k \rangle$ to denote the path. A hamiltonian path of $G$ is a path which crosses all vertices of $G$. A graph $G$ is hamiltonian connected if there exists a hamiltonian path joining any two vertices of $G$.

Hypercubes [12] and stars [1] are bipartite graphs. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(a, b)|a \in V_0 \text{ and } b \in V_1\}$. Given vertices $x$ and $y$, we say that $x$ and $y$ are in the same partite set if $x, y \in V_i$ or in different partite sets if $x \in V_i$ and $y \in V_{1-i}$ for $i \in \{0, 1\}$. However, the concept of hamiltonian connectivity does not apply to bipartite graphs because bipartite graphs are definitely not hamiltonian connected except for a few exceptions such as $K_2$ or $K_1$. As such a property is important, Wong [19] introduced the concept of *hamiltonian laceability* on bipartite graphs. A bipartite graph $G = (V_0 \cup V_1, E)$ is hamiltonian laceable if there is a hamiltonian path between any two vertices $x$ and $y$ which are in different partite sets. It is trivial that $|V_0|$ must be equal to $|V_1|$. On the condition of $|V_0| = |V_1|$, Hsieh et al. [10] extended this concept and proposed the concept of *strongly hamiltonian laceability*. $G$ is strongly hamiltonian laceable if it is hamiltonian laceable and there is a path of length $|V_0| + |V_1| - 2$ between any two vertices in the same partite set. Lewinter and Widulski [13] introduced another concept, *hyper hamiltonian laceability*. $G$ is hyper hamiltonian laceable if it is hamiltonian laceable and for any vertex $v \in V_i$, there is a hamiltonian path of $G - v$ between any two vertices in $V_{1-i}$. So hyper hamiltonian laceability is definitely also strongly hamiltonian laceability.

Fault tolerance is an important property of network performance. Hsieh, Chen, and Ho [9] proposed *the edge fault-tolerant hamiltonicity* to measure the performance of the hamiltonian property in the faulty networks. A graph $G$ is *k-edge-fault tolerant hamiltonian* if $G - F$ remains hamiltonian for every $F \subseteq E(G)$ with $|F| \leqslant k$. Extending this concept, we introduce the following indicators. *The edge fault tolerant hamiltonian laceability* of the graph $G$ is the integer value $f$ such that for any $F \subseteq E(G)$ with $|F| \leqslant f$, $G - F$ is still hamiltonian laceable and there exits $F' \subseteq E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not hamiltonian laceable. We use $\mathrm{eftHL}(G)$ to denote this capacity. Similarly, we can define *the edge fault tolerant strongly hamiltonian laceability* of $G$, denoted by $\mathrm{eftSHL}(G)$, and *the edge fault tolerant hyper hamiltonian laceability* of $G$, denoted by $\mathrm{eftHHL}(G)$. $\mathrm{eftSHL}(G)$ is the integer $f$ such that for any $F \subseteq E(G)$ with $|F| \leqslant f$, $G - F$ is still strongly hamiltonian laceable and there exits $F' \subseteq E(G)$ with $|F'| = f + 1$ such that $G - F'$ is not strongly hamiltonian laceable. $\mathrm{eftHHL}(G)$ is the integer $f$ such that for any $F \subseteq E(G)$ with $|F| \leqslant f$,

$G-F$ is still hyper hamiltonian laceable and there exits $F' \subseteq E(G)$ with $|F'| = f+1$ such that $G-F'$ is not hyper hamiltonian laceable. We say a graph $G$ is optimal with respect to eftHL (eftSHL, eftHHL, respectively) if for a fixed number of vertices, $G$ contains the least number of edges among all graphs $G'$ with eftHL($G'$) = eftHL($G$) (eftSHL($G'$) = eftSHL($G$), eftHHL($G'$) = eftHHL($G$)).

This paper is to study these three indicators of the star graphs. The star graphs [2] are Cayley graphs. They have many nice properties such as recursiveness, vertex and edge symmetry, maximal fault tolerance, sublogarithmic degree and diameter [2]. These properties are important for designing interconnection topologies for parallel and distributed systems. Star graphs are able to embed cycles [19], grids [11], trees [3], and hypercubes [16]. Many efficient communication algorithms for shortest-path routing [17], multiple-path routing [6], broadcasting [15], gossiping [4], and scattering [8] were proposed. And many efficient algorithms designed for sorting and merging [14], selection [17], Fourier transform [7], and computational geometry [18] have been proposed. As a result, star graphs are recognized as an attractive alternative to the hypercubes.

In this paper, we show that the $n$-dimensional star graphs are optimal with respect to the edge fault tolerant hamiltonian laceability, the edge fault tolerant strongly hamiltonian laceability, and the edge fault tolerant hyper hamiltonian laceability. In the next section, we introduce the definition of star graphs. And then in Section 3, we show our main result. Finally, we make our conclusion in Section 4.

## 2. Definition and basic properties

In this section, we introduce the definition and some properties of the star graph.

**Definition 1.** The $n$-dimensional star graph is denoted by $S_n$. The vertex set $V$ of $S_n$ is $\{a_1 \ldots a_n | a_1 \ldots a_n$ is a permutation of $1, 2, \ldots, n\}$ and the edge set $E$ is $\{(a_1 a_2 \ldots a_{i-1} a_i a_{i+1} \ldots a_n,\ a_i a_2 \ldots a_{i-1} a_1 a_{i+1} \ldots a_n) | a_1 \ldots a_n \in V$ and $2 \leqslant i \leqslant n\}$.

By definition, $S_n$ contains $n!$ vertices and each vertex is of degree $(n-1)$. For example, vertex 1234 in $S_4$ connects to 2134, 3214, and 4231. $S_1$, $S_2$, and $S_3$ are a vertex, an edge, and a cycle of length 6, respectively. We show $S_4$ in Fig. 1. It is easy to observe that there are four vertex-disjoint $S_3$'s embedded in $S_4$. The following proposition states this property.

**Proposition 1.** *There are $\frac{n!}{k!}$ vertex-disjoint $S_k$'s embedded in $S_n$ for $k \geqslant 1$.*

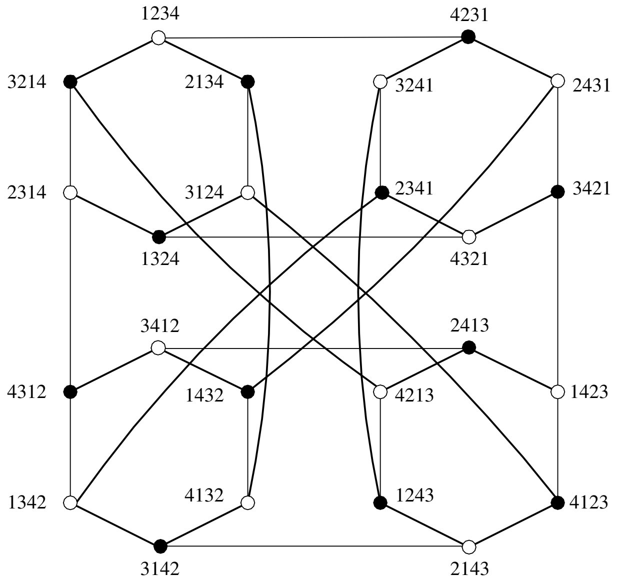


Fig. 1. Four-dimensional star graph.

**Proof.** Let $B = \{b_{k+1} \dots b_n | b_{k+1} \dots b_n$ is a permutation of any $(n-k)$ elements of $1, 2, \dots, n\}$. So $|B| = \frac{n!}{k!}$. For any $b_{k+1} \dots b_n \in B$, let $S_k^{b_{k+1} \dots b_n}$ denote the induced subgraph of $S_n$ with vertex set $\{a_1 \dots a_n | a_{k+1} \dots a_n = b_{k+1} \dots b_n\}$. Obviously, $S_k^{b_{k+1} \dots b_n}$ and $S_k^{b'_{k+1} \cdots b'_n}$ are vertex-disjoint for $b_{k+1} \cdots b_n \neq b'_{k+1} \cdots b'_n$ and $V(S_n) = \bigcup_{b_{k+1} \cdots b_n \in B} V(S_k^{b_{k+1} \cdots b_n})$.

Let $u = u_1 \dots u_k b_{k+1} \cdots b_n$ be some vertex in $S_k^{b_{k+1} \dots b_n}$. Define $f_u^k(a_1 \dots a_n) = i_1 \dots i_k$ for $a_j = u_{i_j}$ and $1 \leqslant j \leqslant k$. For example, let $u = 54\,123$ be a vertex in $S_5$. Then $f_u^2(54\,123) = 12$ and $f_u^2(45\,123) = 21$. We can easily check that $\{f_u^k(v) | v \in V(S_k^{b_{k+1} \dots b_n})\} = V(S_k)$ and $(f_u^k(v_1), f_u^k(v_2))$ is an edge if and only if $(v_1, v_2)$ is an edge. So $S_k^{b_{k+1} \dots b_n} \cong S_k$ and the proposition follows. $\square$

In the following discussion, we will frequently use the notation $S_k^{b_{k+1} \dots b_n}$ defined in the proof above. We call $S_k^{b_{k+1} \dots b_n}$ a *substar* of $S_n$ or specifically, a *k-dimensional substar* of $S_n$. Let $u$ be a vertex not in $S_k^{b_{k+1} \dots b_n}$. We say that $u$ is adjacent to $S_k^{b_{k+1} \dots b_n}$ if $u$ is adjacent to a vertex in $S_k^{b_{k+1} \dots b_n}$. And we call $S_k^{b_{k+1} \dots b_n}$ an *adjacent substar* of $u$. The following proposition and corollary are concerning adjacent substars:

**Proposition 2.** *Given $k$ with $1 \leqslant k \leqslant n-1$ and $b_{k+1} \dots b_n$, a vertex $u = u_1 \dots u_n$ is adjacent to $S_k^{b_{k+1} \dots b_n}$ if and only if $u_{k+1} \dots u_{i-1} u_1 u_{i+1} \dots u_n = b_{k+1} b_{k+2} \dots b_n$ for some $i$ with $k+1 \leqslant i \leqslant n$.*

**Corollary 1.** *There are* $(k-1)!$ *edges between* $S_k^{b_{k+1}\dots b_n}$ *and* $S_k^{b'_{k+1}\dots b'_n}$ *if there is exactly one different bit between* $b_{k+1}\dots b_n$ *and* $b'_{k+1}\dots b'_n$.

**Proof.** Without loss of generality, assume that $b_{k+1}\neq b'_{k+1}$ and $b_{k+2}\dots b_n = b'_{k+2}\dots b'_n$. The first bit of all vertices in $S_k^{b_{k+1}\dots b_n}$ being adjacent to $S_k^{b'_{k+1}\dots b'_n}$ must be $b'_{k+1}$. So the number of these vertices is $(k-1)!$. And the corollary follows. □

For example, there are $(n-2)!$-edges between $S^i_{n-1}$ and $S^j_{n-1}$ for $1\leqslant i\neq j\leqslant n$. We use $E^{i,j}(S_n)$ to denote the set of these edges. And we call these edges outgoing edges of $S^i_{n-1}$ (or $S^j_{n-1}$). Particularly, we say $(u,v)$ an outgoing edge of $u$ if $(u,v)\in E^{i,j}(S_n)$ for some $1\leqslant i\neq j\leqslant n$.

It has been shown that the star graphs are edge symmetric [2], i.e., for any two edges $(x,y),(u,v)\in E(S_n)$, there is an automorphism of $S_n$ mapping $x,y$ into $u,v$, respectively. For ease of description, we use $\pi(F)$ to denote the edge set $\{(\pi(u),\pi(v))|(u,v)\in F\}$ if $\pi$ is an automorphism of $S_n$ and $F\subseteq(S_n)$. Thus, we have following proposition.

**Proposition 3.** *Let* $F\subseteq E(S_n)$. *Then there is an edge set* $F'\subseteq E(S_n)$ *and an automorphism* $\pi$ *of* $S_n$ *such that* $\pi(F)=F'$ *and* $|F'\cap E(S^i_{n-1})|\leqslant|F|-1$ *for each* $1\leqslant i\leqslant n$.

**Proof.** If $|F\cap E(S^i_{n-1})|\leqslant|F|-1$ for each $1\leqslant i\leqslant n$, let $F'$ be $F$ and $\pi$ be the identity mapping. Then the statement follows. Otherwise, choose an arbitrary edge $(x,y)\in F$. With the edge symmetric property, there is an automorphism $\pi$ of $S_n$ such that $\pi(x)=123\dots(n-1)n$ and $\pi(y)=n23\dots(n-1)1$. Let $F'=\pi(F)$. So $(123\dots(n-1)n, n23\dots(n-1)1)\in F'$. But $(123\dots(n-1)n, n23\dots(n-1)1)\notin E(S^i_{n-1})$ for all $1\leqslant i\leqslant n$. Thus, $|F'\cap E(S^i_{n-1})|\leqslant|F|-1$ for each $1\leqslant i\leqslant n$. □

By this proposition, *given any edge set* $F\subseteq E(S_n)$, we may assume that $|F\cap E(S^i_{n-1})|\leqslant|F|-1$ for each $1\leqslant i\leqslant n$. This property will help us simplify the proof a lot.

## 3. Main result

In this section, we present our main result on the three indicators, which are the edge fault tolerant hamiltonian laceability (eftHL), edge fault tolerant strongly hamiltonian laceability (eftSHL), and edge fault tolerant hyper hamiltonian laceability (eftHHL) of the star graphs. We provide a lemma to give three upper bounds for the bipartite graphs and then three theorems to give the exact values for the three indicators on the star graphs. We will see that all the values match the upper bounds. So the star graphs are optimal with respect to these properties. Now we show the upper bounds.

**Lemma 1.** *Let* $G = (V_0 \cup V_1, E)$ *be a bipartite graph with* $|V_0| = |V_1|$ *and let* $\delta$ *be the minimum degree of G among all vertices. We have* $\text{eftHL}(G) \leqslant \delta - 2$, $\text{eftSHL}(G) \leqslant \delta - 2$ *for* $\delta \geqslant 2$, *and* $\text{eftHHL}(G) \leqslant \delta - 3$ *for* $\delta \geqslant 3$.

**Proof.** Assume that the degree of vertex $u$ is $\delta$. Removing $(\delta - 1)$-edges connecting to $u$. Suppose that $v$ is the remainder vertex connecting to $u$ and $v'$ is a neighbor of $v$ which is not $u$ (see Fig. 2). Then it is easy to check that there is no hamiltonian path from $v$ to $v'$. So $G$ is at most $(\delta - 2)$-edge fault tolerant hamiltonian laceable and obviously, at most $(\delta - 2)$-edge fault tolerant strongly hamiltonian laceable.

Then consider removing $(\delta - 2)$-edges which connect to $u$. Suppose that $v_1$ and $v_2$ are the remainder vertices connecting to $u$ and let $u'$ be a vertex connecting to $v_1$ which is not $u$ (see Fig. 3). Then it is easy to check that there is no hamiltonian path of $G - u'$ from $v_1$ to $v_2$. So $G$ is at most $(\delta - 3)$-edge fault tolerant hyper hamiltonian laceable.

Hence, the lemma follows. □

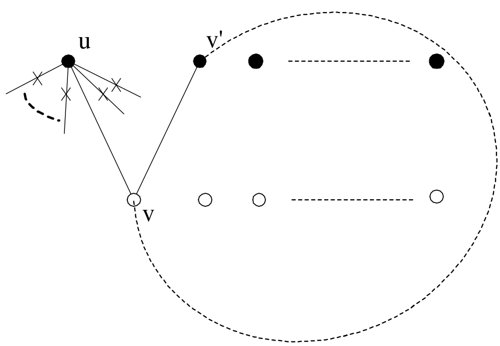


Fig. 2. Upper bound for eftHL($G$).

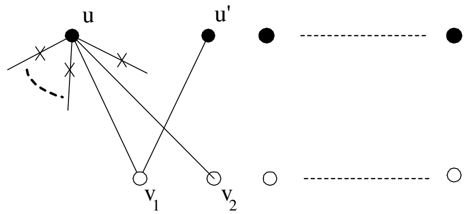


Fig. 3. Upper bound for eftHHL($G$).

Next, we show the capacity of star graphs on these three indicators. First, we use a computer program to check the base case $S_4$ (see Fig. 1) and the case indeed holds for $S_4$. So we state the results in the following lemma. Then we prove our results by induction.

**Lemma 2.** *$S_4$ is 1-edge fault tolerant hamiltonian laceable, 1-edge fault tolerant strongly hamiltonian laceable, and hyper hamiltonian laceable.*

To make the proofs clear, we introduce the following transform:

**Definition 2.** Given a fixed $n$, let $V \subseteq \{1, 2, \ldots, n\}$ and $F \subseteq E(S_n)$. Then $STG_n(V, F)$ is the graph $G(V, E)$ such that $E = \{(i, j) | i, j \in V$ and $E^{i,j}(S_n) \cap F < \frac{(n-2)!}{2}\}$. (*STG* means to transmit a star graph to another graph.)

In fact, $STG_n$ maps the substar $S^i_{n-1}$ in $(S_n - F)$ into the vertex $i$ in $G$ for all $i \in V$. And for $i \neq j \in V$, if $i$ and $j$ are adjacent in $G$, there is a vertex in each partite set of $S^i_{n-1}$ adjacent to $S^j_{n-1}$ in $(S_n - F)$. So we have following lemma:

**Lemma 3.** *Let $G = STG_n(V, F)$ for $V \subseteq \{1, 2, \ldots, n\}$ with $|V| \geqslant 2$ and $F \subseteq E(S_n)$. And let $x \in S^{j_1}_{n-1}$ and $y \in S^{j_2}_{n-1}$ with $j_1 \neq j_2 \in V$ such that $x, y$ are in different partite sets. Assume that $S^i_{n-1} - F$ is hamiltonian laceable for each $i \in V$. Then there is a path from $x$ to $y$ crossing all vertices in all $S^i_{n-1}$ for $i \in V$ without crossing edges in $F$ if there is a hamiltonian path from $j_1$ to $j_2$ in $G$.*

**Proof.** Let $|V| = h$. And let $\langle j_1, j_3, j_4, \ldots, j_h, j_2 \rangle$ be a hamiltonian path from $j_1$ to $j_2$ in $G$. Since $j_1$ and $j_3$ are adjacent in $G$, we can find a vertex $v^1 \in V(S^{j_1}_{n-1})$ adjacent to $S^{j_3}_{n-1}$ such that $v^1, x$ are in different partite sets and the outgoing edge, say $(v^1, u^3)$, of $v^1$ is not in $F$ (see Fig. 4). Similarly, we can find vertices $v^3 \in S^{j_3}_{n-1}, v^4 \in S^{j_4}_{n-1}, \ldots, v^h \in S^{j_h}_{n-1}$ adjacent to $S^{j_4}_{n-1}, S^{j_5}_{n-1}, \ldots, S^{j_2}_{n-1}$, respectively, such that $v^1, v^3, v^4, \ldots, v^h$ are in the same partite set and the outgoing edges of these vertices are not in $F$. Assume that the outgoing edges of these vertices are $(v^3, u^4), (v^4, u^5), \ldots, (v^h, v^2)$. Then by assumption that each $S^i_{n-1} - F$ is hamiltonian laceable, we can construct a path from $x$ to $y$ crossing all vertices in all $S^i_{n-1}$ for $i \in V$ as follows:

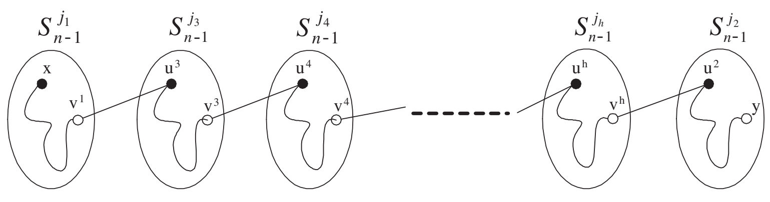


Fig. 4. Remaining path.

$\langle x, \ldots, v^1, u^3, \ldots, v^3, u^4, \ldots\ldots\ldots, v^h, u^2, \ldots, y \rangle$

Hence, the lemma follows. □

Now we can show our first result:

**Theorem 1.** *$S_n$ is $(n-3)$-edge fault tolerant hamiltonian laceable for $n \geqslant 4$.*

**Proof.** We prove it by induction. By Lemma 2, we know that $S_4$ is 1-edge fault tolerant hamiltonian laceable. In the induction step, we assume that $S_{n-1}$ is $(n-4)$-edge fault tolerant hamiltonian laceable for $n \geqslant 5$. Then consider $S_n$.

Let $F \subseteq E(S_n)$ be arbitrary faulty edge set such that $|F| \leqslant n-3$. By Proposition 3, we may assume that $|F \cap E(S^i_{n-1})| \leqslant n-4$ for each $1 \leqslant i \leqslant n$. So $S^i_{n-1} - F$ is still hamiltonian laceable for each $1 \leqslant i \leqslant n$. Let $x \in V(S^{j_1}_{n-1})$ and $y \in V(S^{j_2}_{n-1})$ such that $x, y$ are in different partite sets. We shall construct a fault-free hamiltonian path from $x$ to $y$. Consider the following two cases:

*Case* 1. $j_1 \neq j_2$. Let $V = \{1, 2, \ldots, n\}$. Since $|F| \leqslant n-3 < \frac{(n-2)!}{2}$ for $n \geqslant 5$, $E^{i,j}(S_n) \cap F < \frac{(n-2)!}{2}$ for any $i \neq j \in V$. So $G = STG_n(V, F)$ is a complete graph. It is easy to find a hamiltonian path of $G$ from $j_1$ to $j_2$. By Lemma 3, there is a hamiltonian path of $S_n$ from $x$ to $y$.

*Case* 2. $j_1 = j_2 = j$. There is a hamiltonian path $P$ of $S^j_{n-1}$ from $x$ to $y$. The length of $P$ is $(n-1)! - 1$. So we can find an edge, say $(u, v)$, on path $P$ such that the outgoing edges of $u$ and $v$ are fault-free. (If such $(u, v)$ does not exist, $|F| \geqslant \frac{(n-1)!-1}{2} > n-3$ for $n \geqslant 5$.) Let $P = \langle x, P_1, u, v, P_2, y \rangle$ and $(u, v'), (v, u')$ are the outgoing edges of $u$ and $v$, where $v' \in S^{j_3}_{n-1}$ and $u' \in S^{j_4}_{n-1}$ (see Fig. 5). $v'$ and $u'$ are in different partite sets of $S_n$ and $j_3 \neq j_4$. Let $V = \{1, 2, \ldots, n\} - j$. Then

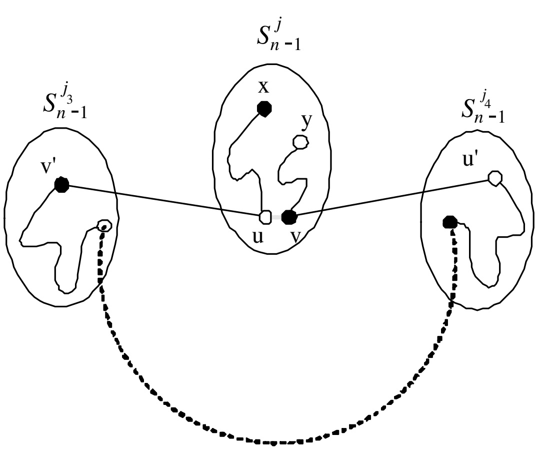


Fig. 5. $x$ and $y$ are in the same substar.

$G = STG_n(V, F)$ is a complete graph with $(n-1)$ vertices since $\frac{(n-2)!}{2} > |F|$ for $n \geqslant 5$. Thus, there is a hamiltonian path of $G$ from $j_3$ to $j_4$. By Lemma 3, there is a path $P_3$ crossing all vertices of all $S^i_{n-1}$ for $i \in V$ from $v'$ to $u'$. So a hamiltonian path of $S_n$ from $x$ to $y$ can be constructed as follows:

$$\langle x, P_1, u, v', P_3, u', v, P_2, y \rangle$$

Hence, the theorem follows. □

Since $S_n$ is $(n-1)$ regular, by Lemma 1, $S_n$ is optimal with respect to edge fault tolerant hamiltonian laceability and $\mathrm{eftHL}(S_n) = n-3$.

**Theorem 2.** *$S_n$ is $(n-3)$-edge fault tolerant strongly hamiltonian laceable for $n \geqslant 4$.*

**Proof.** We also prove it by induction. $S_4$ is shown to be 1-edge fault tolerant strongly hamiltonian laceable in Lemma 2. So we need only to consider the induction step. Assume that $S_{n-1}$ is $(n-4)$-edge fault tolerant strongly hamiltonian laceable for $n \geqslant 5$ and consider $S_n$.

Given any fault edge set $F$ in $S_n$ with $|F| \leqslant n-3$, by Proposition 3, we can assume that $|F \cap E(S^i_{n-1})| \leqslant n-4$ for each $1 \leqslant i \leqslant n$. So $S^i_{n-1} - F$ is still strongly hamiltonian laceable for each $1 \leqslant i \leqslant n$. Let $x \in V(S^{j_1}_{n-1})$ and $y \in V(S^{j_2}_{n-1})$ such that $x$ and $y$ are in the same partite set. Consider the following two cases:

*Case* 1. $j_1 \neq j_2$. Let $V_e$ be the number of vertices which are in the different partite set from $x$ and which are not adjacent to $S^{j_2}_{n-1}$. Then $V_e$ is equal to $\frac{(n-1)!}{2} - \frac{(n-2)!}{2}$ which is strictly greater than $|F|$. So there is a fault-free edge $(u^1, v^3)$, i.e., not in $F$, such that $u^1 \in V(S^{j_1}_{n-1})$, $v^3 \in V(S^{j_3}_{n-1})$ for $j_3 \notin \{j_1, j_2\}$, and $x, y, u^1$ are in the same partite set of $S_n$ (see Fig. 6). By the induction hypothesis, there is a path $P_1$ of length $(n-1)! - 2$ in $S^{j_1}_{n-1}$ from $x$ to $u^1$. Then consider the remainder subgraphs. Let $V = \{1, 2, \ldots, n\} - \{j_1\}$. Thus, $|V| \geqslant 2$ and $G = STG_n(V, F)$ is a complete graph. There is a hamiltonian path of $G$ from $j_3$ to $j_2$. Since $u^1$ and $y$ are in the same partite set, $y$ and $v^3$ are in different partite sets. So there is a path $P_2$ crossing all vertices of all $S^i_{n-1}$ for $i \in V$ from $v^3$ to $y$.

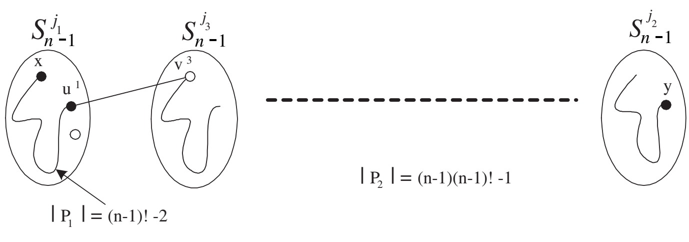


Fig. 6. $x$ and $y$ are in different substars.

The length of this path is $(n-1)(n-1)!-1$. We can construct a path from $x$ to $y$ as follows:

$$\langle x, P_1, u^1, v^3, P_2, y\rangle$$

The length of this path is

$$[(n-1)!-2]+1+[(n-1)(n-1)!-1] = n!-2$$

So the theorem follows in this case.

*Case* 2. $j_1 = j_2 = j$. The proof of this case is similar to that of case 2 in Theorem 1 except that the path in $S^j_{n-1}$ from $x$ to $y$ is of length $(n-1)!-2$.

Hence, the theorem follows. □

Since $S_n$ is $(n-1)$ regular, by Lemma 1, $S_n$ is also optimal with respect to the edge fault tolerant strongly hamiltonian laceability.

**Theorem 3.** *$S_n$ is $(n-4)$-edge fault tolerant hyper hamiltonian laceable for $n \geqslant 4$.*

**Proof.** The proof is a little more complex than the previous two theorems. Again, $S_4$ is hyper hamiltonian laceable by Lemma 2. So we show that the statement is true for $n \geqslant 5$. Assume that $S_{n-1}$ is $(n-5)$-edge fault tolerant hyper hamiltonian laceable for $n \geqslant 5$.

Let $F$ be a faulty edge set in $S_n$ with $|F| \leqslant n-4$. By Proposition 3, we may assume that $|F \cap E(S^i_{n-1})| \leqslant n-5$ for each $1 \leqslant i \leqslant n$. So $S^i_{n-1} - F$ is still hyper hamiltonian laceable and obviously, strongly hamiltonian laceable for each $1 \leqslant i \leqslant n$. Given a vertex $v$, in the following we will construct a hamiltonian path of $(S_n - F) - v$ between any two vertices in the partite set which $v$ is not in. Let $x$ and $y$ be two such vertices. Consider the following four cases:

*Case* 1. $v, x, y$ are in the same substar, say $S^{j_1}_{n-1}$ (see Fig. 7(a)). By the induction hypothesis, there is a hamiltonian path $P$ of $(S^{j_1}_{n-1} - F) - v$ from $x$ to $y$. The length of $P$ is $(n-1)!-2 > 2|F|$ for $n \geqslant 5$. So there is an edge $(u^1, v^1)$ on $P$ such that the outgoing edges of $u^1$ and $v^1$, say $(u^1, v^2)$ and $(v^1, u^3)$, are fault-free. ($x, u^1$ are not necessary in the same partite set.) Let $P = \langle x, P_1, u^1, v^2, P_2, y\rangle$. Clearly, $v^2$ and $u^3$ are in different partite sets of $S_n$. Assume that $v^2 \in S^{j_2}_{n-1}$ and $u^3 \in S^{j_3}_{n-1}$. So $j_2 \neq j_3$. Let $V = \{1, 2, \ldots, n\} - \{j_1\}$. Then $STG_n(V, F)$ is a complete graph. There is a hamiltonian path from $j_2$ to $j_3$ and so a path $P_3$ from $v^2$ to $u^3$ crossing all vertices of $S^i_{n-1}$ for all $i \in V$. Therefore, we can construct a hamiltonian path of $(S_n - F) - v$ as: $\langle x, P_1, u^1, v^2, P_3, u^3, v^1, P_2, y\rangle$.

*Case* 2. $v, x \in S^{j_1}_{n-1}$ and $y \in S^{j_2}_{n-1}$ with $j_1 \neq j_2$ (see Fig. 7(b)). Let $j_3 \neq j_2$. Since $\frac{(n-2)!}{2} - 1 > |F|$ for $n \geqslant 5$, we can easily find a vertex $u^1 \neq x \in S^{j_1}_{n-1}$ such that $u^1$ and $x$ are in the same partite set and the outgoing edge of $u^1$, say $(u^1, v^3)$, is fault-free. (Note that since $u^1 \neq x$, there are $\frac{(n-2)!}{2} - 1$ choices for $u^1$ in $S^{j_1}_{n-1}$.) By the induction hypothesis, there is a hamiltonian path $P_1$ of $(S^{j_1}_{n-1} - F) - v$ from $x$ to $u^1$. Let $V = \{1, 2 \ldots, n\} - \{j_1\}$. Then $STG_n(V, F)$ is a complete graph. Note

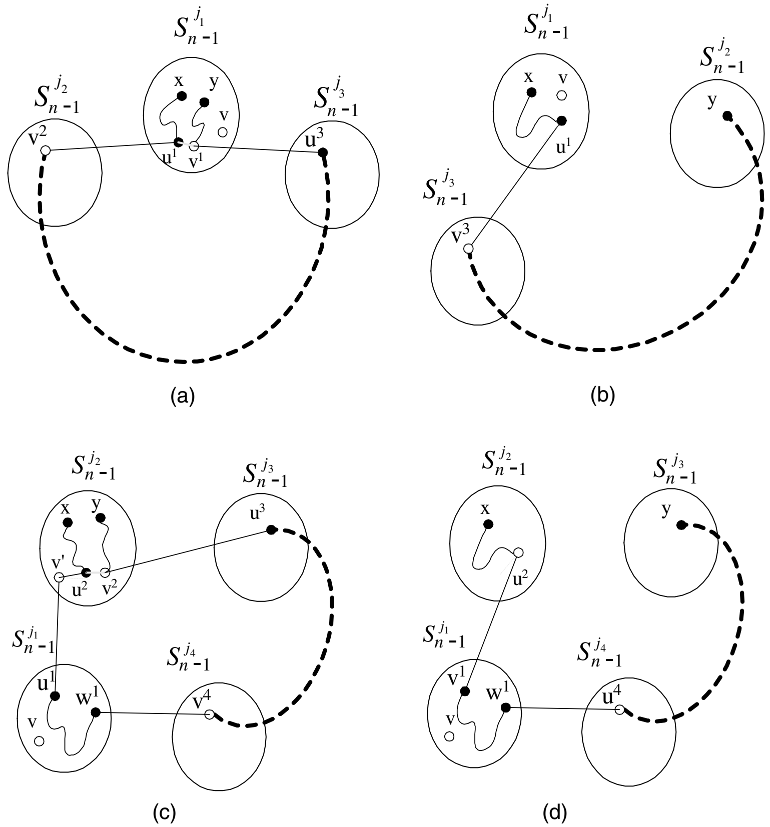


Fig. 7. Edge fault tolerant hyper hamiltonian laceability of the star.

that $v^3$ and $y$ are in different partite sets. So there is a hamiltonian path from $j_3$ to $j_2$ and a path $P_2$ from $v^3$ to $y$ crossing all vertices of all $S^i_{n-1}$ for $i \in V$. Hence, we have a hamiltonian path $\langle x, P_1, u^1, v^3, P_2, y\rangle$ of $(S_n - F) - v$.

*Case* 3. $v \in S^{j_1}_{n-1}$ and $x, y \in S^{j_2}_{n-1}$ with $j_1 \neq j_2$ (see Fig. 7(c)). Since $\frac{(n-2)!}{2} > |F|$, there is a vertex $v' \in V(S^{j_2}_{n-1})$ adjacent to $S^{j_1}_{n-1}$ such that the outgoing edge of $v'$, say $(v', u^1)$, is fault-free and $v', v$ are in the same partite set. By the induction hypothesis, there is a hamiltonian path $P$ of $(S^{j_2}_{n-1} - F) - v'$ from $x$ to $y$. Since there are $(n-2)$ neighbors of $v'$ in $S^{j_2}_{n-1}$ and $|F| < (n-2)$, there exists an edge $(u^2, v^2)$ on $P$ such that $u^2$ is adjacent to $v'$ and the outgoing edge of $v^2$, say $(v^2, u^3)$, is fault-free. Clearly, $j_3 \notin \{j_1, j_2\}$ since $v^2, v'$ are neighbors of $u^2$ but $v^2 \neq v'$. Let $P = \langle x, P_1, u^2, v^2, P_2, y\rangle$. (Note that $P$ may be $\langle x, P_1, v^2, u^2, P_2, y\rangle$ and the argument of this case is similar to the following discussion.) Let $j_4 \notin \{j_1, j_2, j_3\}$. Since $\frac{(n-2)!}{2} - 1 > |F|$ for $n \geqslant 5$, there is a vertex $w^1 \in S^{j_1}_{n-1}$ adjacent to $S^{j_4}_{n-1}$ such that $w^1, u^1$ are in the same partite set and the outgoing edge of $w^1$, say $(w^1, v^4)$, is fault-free. So $v^4 \in V(S^{j_4}_{n-1})$ and $v^4, u^3$ are in different partite sets. By the induction hypothesis, there is a hamiltonian path $P_3$ of $(S^{j_1}_{n-1} - F) - v$ from $u^1$ to $w^1$. Let $V = \{1, 2, \ldots, n\} - \{j_1, j_2\}$. Then $STG_n(V, F)$ is a complete graph. There is a hamiltonian path from $j_4$ to $j_3$ and so a path $P_4$ crossing all vertices of $S^i_{n-1}$ for all $i \in V$ from $v^4$ to $u^3$. Thus, we have a hamiltonian path of $(S_n - F) - v$ as follows:

$$\langle x, P_1, u^2, v', u^1, P_3, w^1, v^4, P_4, u^3, v^2, P_2, y \rangle.$$

*Case* 4. $v \in S^{j_1}_{n-1}$, $x \in S^{j_2}_{n-1}$, and $y \in S^{j_3}_{n-1}$ for distinct $j_1$, $j_2$, and $j_3$ (see Fig. 7(d)). Since $\frac{(n-2)!}{2} > |F|$, there is a vertex $u^2 \in V(S^{j_2}_{n-1})$ adjacent to $S^{j_1}_{n-1}$ such that $u^2, x$ are in different partite sets and the outgoing edge of $u^2$, say $(u^2, v^1)$, is fault-free. By the induction hypothesis, there is a hamiltonian path $P_1$ of $(S^{j_2}_{n-1} - F)$ from $x$ to $u^2$. Let $j_4 \notin \{j_1, j_2, j_3\}$. In $S^{j_1}_{n-1}$, since $\frac{(n-2)!}{2} - 1 > |F|$, there is a vertex $w^1 \neq v^1$ adjacent to $S^{j_4}_{n-1}$ such that $w^1, v^1$ are in the same partite set and the outgoing edge of $w^1$, say $(w^1, u^4)$, is fault-free. By the induction hypothesis, there is also a hamiltonian path $P_2$ of $(S^{j_1}_{n-1} - F) - v$ from $v^1$ to $w^1$. For the remaining substars, let $V = \{1, 2, \ldots, n\} - \{j_1, j_2\}$. Then $G = STG_n(V, F)$ is a complete graph. So there is a hamiltonian path of $G$ from $j_4$ to $j_3$ and thus, a path $P_3$ from $u^4$ to $y$ crossing all vertices of $S^i_{n-1}$ for all $i \in V$. Finally, we have a hamiltonian path $\langle x, P_1, u^2, v^1, P_2, w^1, u^4, P_3, y \rangle$ of $(S_n - F) - v$.

Hence, the theorem follows. □

Since $S_n$ is $(n-1)$ regular, by Lemma 1, $S_n$ is optimal with respect to the edge fault tolerant hyper hamiltonian laceability.

## 4. Conclusion

Fault tolerance is an important research subject of the multi-process computer systems. Graphs are usually used to represent the interconnection architecture of these systems, where vertices represent processors and edges represent links between processors. Many researches concerned the vertex-fault tolerant or edge-fault tolerant properties of some specific graphs. In this paper, we study some fault tolerant results of the star graphs. We show that the $n$-dimensional star graph is $(n-3)$-edge fault tolerant hamiltonian laceable, $(n-3)$-edge fault tolerant strongly hamiltonian laceable, and $(n-4)$-edge fault tolerant hyper hamiltonian laceable.

In particular, we use computer programs to check the base cases. It not only gives us some preliminary intuition but also simplifies our proof. If we did such check by theoretical proof, we would have spent too much effort since there would have been too many subcases to deal with. Apparently, such a method may be applied in other cases nowadays, especially, for those facts which can be proved by induction.

## Acknowledgements

This work was supported in part by the National Science Council of the Republic of China under Contract NSC 91-2218-E-231-002.

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